The Optimum Target Value for a Process Based on a Quadratic Loss Function

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Abstract

The purpose of this paper is to determine the most economic target value of a process. Consider a production process where products are produced continuously and whose specification limits are specified for screening inspection. There are many instances where the losses due to producing a product below the lower specification limit and above the upper specification limit are not equal. Moreover, a loss is always incurred between the specification limits no matter how small the deviation from the target for the process is when a quadratic quality loss function is used. Due to these losses, the total loss to the customer will greatly depend on the target value. In this paper, an optimum target value for the process whose quality characteristics is normally distributed with a known mean and variance is derived.

Keywords
Optimum target value; quadratic loss function; screening inspection

1. INTRODUCTION

In the traditional concept of screening inspection, a product is classified as nonconforming if it fails to meet predetermined specification limits; otherwise, it is classified conforming. In other words, products are evaluated as “good” or “bad” on a go/no-go basis. For many quality characteristics, however, the quality difference for a product that is just within the specifications or just outside may be very slight, but one product is called “conforming” and the other is called “nonconforming.” Consequently, this step loss function does not adequately reflect customer’s perception of quality. Typically, the exact form of the loss function is unknown. Various loss functions (DeGroot 1970, Berger, 1985) have been discussed in the literature of statistical decision theory. However, there is a general consensus that quadratic quality loss function based on Taylor series expansion is a reasonable means to transfer the deviation of a product’s quality characteristic from a target value to a monetary scale (see Chernoff and Moses 1959; Taguchi, 1986, 1987; Phadke, 1989).

In dealing with the problem of selecting the best manufacturing process, the usual assumption is the one or both of the specification limits is given, and the problem is then to determine the mean for the process or jointly determine the process and one of the specification limits (see, for example, Springer, 1951; Bettes, 1962; Hunter and Kartha 1977; Carlsson, 1984; Bisgaard et al., 1984; Gohlar, 1987; Gohlar and Pollock, 1988). It is noted that all of these studies have been based on the step-loss function. Several researchers have studied the problem concerned with how to determine the most economics specification limits(s) based on the quadratic loss function so that total loss is minimized. Tang (1988) discussed the economic model for selecting the most profitable specification limits in a complete inspection plan for the case where the inspection cost is constant and the inspection cost is a linear function of the width of the specification limits. Fathi (1990) presented a graphical procedure for the development of producer’s specification limits, assuming that a quality characteristic is normally distributed and its mean is equal to the target. Kapur (1988) considered a lognormal and a normal distribution for a quality characteristic and presented the optimization models for the economic specification limits by truncating the distribution for screening inspection. Further, Kapur and Cho (1994) considered a Weibull distribution for a quality characteristic, which is more flexible in terms of shape and thus can model more realistic situations.
This paper, in contrast, addresses the issue of determining the most economic target value for the process. When the quadratic quality loss function is used for the quality characteristic, the expected quality loss function for the quality characteristic can be evaluated by the squared bias (squared value of the difference between the process mean and the target value for the quality characteristic) plus the variance for the characteristic (see Taguchi, 1986, 1987; Phadke, 1989). Hence, the bias as well as the variance should be reduced in order to decrease the quality loss for a product. For many industrial processes, the most desired values of the distribution parameters for the quality characteristic of interest may be known and very expensive to change. In this case, one important issue for quality improvement is to identify the optimum target value for the quality characteristics in terms of the process parameters, such as mean, variance, and specification limits for the characteristic of interest. In this paper, it is assumed that the products are produced continuously and all the outgoing products are subject to screening inspection based on the predetermined specification limits. Further, it is assumed that the quality characteristic of interest is normally distributed with a known mean and variance. When both the lower and upper specification limits are considered, it is more realistic that the loss due to a product below the lower specification limit and above the upper specification limit is not always equal. For instance, suppose that the minimum of a certain product is fixed by government legislation. Then a product, which falls below the lower specification limit, must be scrapped. In contrast, a scrapped product, which falls above the upper specification limit, may be reworked or sold at the secondary market. In this case, two losses are unequal. Due to the unequal losses outside the specification limits and the loss inside the specification limits, the total loss will depend greatly on the target value. In this paper, an optimization method is presented for the problem of determining the optimum target value for the quality characteristic, which minimizes the total expected loss, within the framework of the quadratic loss function.

2. THE MODEL

Let $L_1(y)$ be a measure of losses associated with the quality characteristic $y$. Let us assume that $L_1(y)$ is differentiable function in the neighborhood of $t$, where $t$ is a target value. The equation can be expanded using Taylor’s series around $t$ as given by equation (1).

$$L_1(y) = L_1(t) + (L_1'(t) / 1!)(y - t) + (L_1''(t) / 2!)(y - t)^2 + \ldots.$$  \hspace{1cm} (1)

Note that the target must be developed so that quality loss is minimum at the target and thus $L_1'(t) = 0$. By evaluating loss due to variation from $t$ only, $L(y)$ may be defined as:

$$L(y) = L_1(y) - L_1(t) = (L_1''(t) / 2!)(y - t)^2 + \ldots.$$  \hspace{1cm} (2)

If terms higher than the second order are ignored, equation (2) reduces to

$$L(y) \approx (L_1''(t) / 2)(y - t)^2 \text{ or } L_1(y) \approx k(y - t)^2,$$  \hspace{1cm} (3)

where $k$ is the loss coefficient. Actually, the loss function is nothing but a means to transfer the deviation of the characteristic from a target value to a monetary scale. For any product or process, variability always occurs due to noise factors. That makes $y$ a random variable. Let $f(y)$ be the probability density function for the random variable $Y$. Then the expected loss can be evaluated using equation (4).

$$E[L(Y)] = \int L(y)f(y)dy.$$  \hspace{1cm} (4)

Hence the expected loss is based on all possible values of $y$ and not just values of $y$ which are outside the specification limits. Consider the production process where the products are produced continuously. Assume that the process is operating in a state of statistical control. Letting the lower and upper specification limit denoted by $L$ and $U$, it is noted that the losses incurred due to producing a product below $L$ and above $U$ are $k(y-L)^2$ and $k(U-y)^2$, respectively, and the loss incurred due to the deviation from the target between $L$ and $U$ is $k(y-t)^2$. Then the Expected Total Loss ($ETL$) for all possible values of $y$ is given by

$$ETL = k(t - L)^2 \int_{-\infty}^{L} f(y)dy + k \int_{L}^{U} (y - t)^2 f(y)dy + k(U - t)^2 \int_{U}^{\infty} f(y)dy.$$  \hspace{1cm} (5)
The objective then is to determine optimum target value \( t^* \) so that the ETL is minimized, assuming that \( Y \) is normally distributed with a known mean \( \mu \) and variance \( \sigma^2 \).

### 3. THE OPTIMAL SOLUTION

For a quality characteristic \( Y \), let \( \phi(\cdot) \) and \( \Phi(\cdot) \) represent the normal density function and the cumulative normal distribution function, respectively. The following results can be obtained:

\[
\int_{L}^{U} yf(y)dy = \mu \left[ \Phi\left( \frac{U - \mu}{\sigma} \right) - \Phi\left( \frac{L - \mu}{\sigma} \right) \right] + \sigma \left[ \Phi\left( \frac{U - \mu}{\sigma} \right) - \Phi\left( \frac{L - \mu}{\sigma} \right) \right]. \tag{6}
\]

\[
\int_{L}^{U} y^2 f(y)dy = \sigma^2 \left[ \Phi\left( \frac{U - \mu}{\sigma} \right) - \Phi\left( \frac{L - \mu}{\sigma} \right) + \left( \frac{L - \mu}{\sigma} \right) \Phi\left( \frac{L - \mu}{\sigma} \right) - \left( \frac{U - \mu}{\sigma} \right) \Phi\left( \frac{U - \mu}{\sigma} \right) \right]. \tag{7}
\]

Therefore, based on the above results, equation (5) becomes:

\[
ETL = k \left( (U^2 - 2Ut + t^2) + k \right)
\]

\[
\Phi\left( \frac{L - \mu}{\sigma} \right) \left( L^2 - \sigma^2 + 2\mu t - 2Lt - \mu^2 \right) + \Phi\left( \frac{U - \mu}{\sigma} \right) \left( \sigma^2 - U^2 + 2\mu t - 2U + U^2 \right) +
\]

\[
\Phi\left( \frac{L - \mu}{\sigma} \right) \left( \sigma^2 L - \mu \right) + 2\sigma^2 t - 2\mu \sigma + \Phi\left( \frac{U - \mu}{\sigma} \right) \left( \sigma^2 U - \mu \right) + 2\sigma^2 t - 2\mu \sigma \right]. \tag{8}
\]

Furthermore, if \( g(\cdot) \) is differentiable function, the following fundamental results are noted:

\[
\frac{\partial}{\partial y} \left[ \Phi(g(y)) \right] = \left[ - \frac{\partial}{\partial y} \left( g(y) \right) \right] g(y) \Phi(g(y)). \tag{9a}
\]

\[
\frac{\partial}{\partial y} \left[ \Phi(g(y)) \right] = \frac{\partial}{\partial y} \left[ g(y) \right] \Phi(g(y)). \tag{9b}
\]

Using the above results, and differentiating equation (8) with respect to \( t \), the following is obtained:

\[
\frac{\partial ETL}{\partial t} = 2k \left[ (\mu - L) \Phi\left( \frac{L - \mu}{\sigma} \right) + (U - \mu) \Phi\left( \frac{U - \mu}{\sigma} \right) \right] -
\]

\[
2k\sigma \left[ \Phi\left( \frac{U - \mu}{\sigma} \right) - \Phi\left( \frac{L - \mu}{\sigma} \right) \right] + 2k(t - U). \tag{10}
\]

Taking the second derivative of \( ETL \) with respect to \( t \) yields:

\[
\frac{\partial^2 ETL}{\partial t^2} = 2k. \tag{11}
\]

Since \( k \) is a positive constant, equation (6) is a convex function of \( t \). The optimal value for \( t \) is found by the first derivative of equation (10) to zero and solving for \( t \).

\[
t^* = \sigma \left[ \Phi\left( \frac{L - \mu}{\sigma} \right) - \Phi\left( \frac{U - \mu}{\sigma} \right) \right] + (L - \mu) \Phi\left( \frac{L - \mu}{\sigma} \right) - (U - \mu) \Phi\left( \frac{U - \mu}{\sigma} \right) + U. \tag{12}
\]
4. NUMERICAL EXAMPLE

In this section, an example is presented and then, based on this example, sensitivity analyses are performed to study the effects of the changes in $\mu$, $\sigma$, $L$, and $U$ on the optimal solution of $t$. A chemical company wishes to know the best target value for the concentration of a certain chemical in its finished products. If the concentration is too low or too high, the product does not perform its intended function very well. Hence, they want to produce products whose performance is as close to the target value as possible. It is relatively easy to measure the concentration of the chemical. Based on the statistical studies for the concentration, it was found that the mean of 2.0 ounces and the standard deviation of 1.0 ounce can be adequately described by a normal distribution. The lower and upper specification limits are set to 1.0 ounce and 4.0 ounces, respectively.

It is easily found from equation (9) that $t^* = 2.6041$ ounces. The results of sensitivity analysis by varying the value of process mean, process standard deviation, lower specification limit, and upper specification limit are shown in Figures 1-4.

![Figure 1. Sensitivity of the optimum process target to process mean](image1.png)

![Figure 2. Sensitivity of the optimum process target to lower specification limit](image2.png)
5. DISCUSSION AND CONCLUSIONS
For most industrial processes, the losses due to producing a product below in the lower specification limit and above the upper specification limit differ. When the quadratic quality loss function is used, a loss is always incurred no matter how small the deviation from the target is. Based on this classification of the losses, in this paper, the most economical target value is derived, assuming that the quality characteristic of interest is normally distributed with a known mean and variance. Furthermore, sensitivity analyses for the effects of process parameters on the optimal target value are performed to provide more insight.
Biographical Sketch

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