Wavelet Transform Theory

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What is a Wavelet Transform?

- Decomposition of a signal into constituent parts
- Note that there are many ways to do this. Some are:
  - Fourier series: harmonic sinusoids; single integer index
  - Fourier transform (FT): nonharmonic sinusoids; single real index
  - Walsh decomposition: “harmonic” square waves; single integer index
  - Karhunen-Loeve decomp: eigenfunctions of covariance; single real index
  - Short-Time FT (STFT): windowed, nonharmonic sinusoids; double index
    - provides time-frequency viewpoint
  - Wavelet Transform: time-compacted waves; double index
- Wavelet transform also provides time-frequency view
  - Decomposes signal in terms of duration-limited, band-pass components
    - high-frequency components are short-duration, wide-band
    - low-frequency components are longer-duration, narrow-band
  - Can provide combo of good time-frequency localization and orthogonality
    - the STFT can’t do this
  - More precisely, wavelets give time-scale viewpoint
    - this is connected to the multi-resolution viewpoint of wavelets
General Characteristics of Wavelet Systems

• Signal decomposition: build signals from “building blocks”, where the
  building blocks (i.e. basis functions) are doubly indexed.
• The components of the decomposition (i.e. the basis functions) are
  localized in time-frequency
  – ON can be achieved w/o sacrificing t-f localization
• The coefficients of the decomposition can be computed efficiently (e.g.,
  using $O(N)$ operations).

Specific Characteristics of Wavelet Systems

• Basis functions are generated from a single wavelet or scaling function
  by scaling and translation
• Exhibit multiresolution characteristics: dilating the scaling functions
  provides a higher resolution space that includes the original
• Lower resolution coefficients can be computed from higher resolution
  coefficients through a filter bank structure
Fourier Development vs. Wavelet Development

- Fourier and others:
  - expansion functions are chosen, then properties of transform are found
- Wavelets
  - desired properties are mathematical imposed
  - the needed expansion functions are then derived
- Why are there so many different wavelets
  - the basic desired property constraints don’t use all the degrees of freedom
  - remaining degrees of freedom are used to achieve secondary properties
    - these secondary properties are usually application-specific
    - the primary properties are generally application-nonspecific
- What kinds of signals are wavelets and Fourier good for?
  - Wavelets are good for transients
    - localization property allows wavelets to give efficient rep. of transients
  - Fourier is good for periodic or stationary signals
Why are Wavelets Effective?

• Provide unconditional basis for large signal class
  – wavelet coefficients drop-off rapidly
  – thus, good for compression, denoising, detection/recognition
  – goal of any expansion is
    • have the coefficients provide more info about signal than time-domain
    • have most of the coefficients be very small (sparse representation)
  – FT is not sparse for transients
• Accurate local description and separation of signal characteristics
  – Fourier puts localization info in the phase in a complicated way
  – STFT can’t give localization and orthogonality
• Wavelets can be adjusted or adapted to application
  – remaining degrees of freedom are used to achieve goals
• Computation of wavelet coefficient is well-suited to computer
  – no derivatives of integrals needed
  – turns out to be a digital filter bank
Multiresolution Viewpoint
Multiresolution Approach

• Stems from image processing field
  – consider finer and finer approximations to an image
• Define a nested set of signal spaces

\[ \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2 \]

• We build these spaces as follows:
• Let \( V_0 \) be the space spanned by the integer translations of a fundamental signal \( \phi(t) \), called the scaling function:

that is, \( f(t) \) is in \( V_0 \) \textbf{then} it can be represented by:

\[ f(t) = \sum_{k} a_k \phi(t - k) \]

• So far we can use just about any function \( \phi(t) \), but we’ll see that to get the nesting only certain scaling functions can be used.
Multiresolution Analysis (MRA) Equation

- Now that we have $V_0$ how do we make the others and ensure that they are nested?
- If we let $V_1$ be the space spanned by integer translates of $\phi(2t)$ we get the desired property that $V_1$ is indeed a space of functions having higher resolution.
- Now how do we get the nesting?
- We need that any function in $V_0$ also be in $V_1$; in particular we need that the scaling function (which is in $V_0$) be in $V_1$, which the requires that

$$
\phi(t) = \sum_{n} h(n) \sqrt{2} \phi(2t - n)
$$

where the expansion coefficient is $h(n)\sqrt{2}$
- This is the requirement on the scaling function to ensure nesting: it must satisfy this equation
  - called the multiresolution analysis (MRA) equation
  - this is like a differential equation that the scaling function is the solution to
The h(n) Specify the Scaling Function

- Thus, the coefficients h(n) determine the scaling function
  - for a given set of h(n), φ(t)
    - may or may not exist
    - may or may not be unique
- Want to find conditions on h(n) for φ(t) to exist and be unique, and also:
  - to be orthogonal (because that leads to an ON wavelet expansion)
  - to give wavelets that have desirable properties

h(n) must satisfy conditions

MRA Equation

φ(t)
**Whence the Wavelets?**

- The spaces $V_j$ represent increasingly higher resolution spaces
- To go from $V_j$ to higher resolution $V_{j+1}$ requires the addition of “details”
  - These details are the part of $V_{j+1}$ not able to be represented in $V_j$
  - This can be captured through the “orthogonal complement of $V_j$ w.r.t $V_{j+1}$
- Call this orthogonal complement space $W_j$
  - all functions in $W_j$ are orthogonal to all functions in $V_j$
  - That is:
    $$<\phi_{j,k}(t),\psi_{j,l}(t)> = \int \phi_{j,k}(t)\psi_{j,l}(t)dt = 0 \quad \forall j,k,l \in \mathbb{Z}$$

- Consider that $V_0$ is the lowest resolution of interest
- How do we characterize the space $W_0$?
  - we need to find an ON basis for $W_0$, say $\{\psi_{0,k}(t)\}$ where the basis functions arise from translating a single function (we’ll worry about the scaling part later):
    $$\psi_{0,k}(t) = \psi(t - k)$$
Finding the Wavelets

- The wavelets are the basis functions for the $W_j$ spaces
  - thus, they lie in $V_{j+1}$
- In particular, the function $\psi(t)$ lies in the space $V_1$ so it can be expanded as
  \[ \psi(t) = \sum_{n} h_1(n)\sqrt{2}\phi(2t - n), \quad n \in \mathbb{Z} \]
- This is a fundamental result linking the scaling function and the wavelet
  - the $h_1(n)$ specify the wavelet, via the specified scaling function
Wavelet-Scaling Function Connection

- There is a fundamental connection between the scaling function and its coefficients $h(n)$, the wavelet function and its coefficients $h_1(n)$:

$$h_1(n)$$

$\psi(t)$

$h(n)$

$\phi(t)$

How are $h_1(n)$ and $h(n)$ related?

MR Equation (MRE)

Wavelet Equation (WE)
Relationship Between $h_1(n)$ and $h(n)$

- We state here the conditions for the important special case of
  - finite number $N$ of nonzero $h(n)$
  - ON within $V_0$: \[ \int \phi(t)\phi(t-k)dt = \delta(k) \]
  - ON between $V_0$ and $W_0$: \[ \int \psi(t)\phi(t-k)dt = \delta(k) \]
- Given the $h(n)$ that define the desired scaling function, then the $h_1(n)$ that define the wavelet function are given by
  \[ h_1(n) = (-1)^n h(N - 1 - n) \]

- Much of wavelet theory addresses the origin, characteristics, and ramifications of this relationship between $h_1(n)$ and $h(n)$
  - requirements on $h(n)$ and $h_1(n)$ to achieve ON expansions
  - how the MRE and WE lead to a filter bank structure
  - requirements on $h(n)$ and $h_1(n)$ to achieve other desired properties
  - extensions beyond the ON case
The Resulting Expansions

- Let $f(t)$ be in $L^2(\mathbb{R})$
- There are three ways of interest that we can expand $f(t)$

1. We can give an limited resolution approximation to $f(t)$ via

$$f_j(t) = \sum_k a_k 2^{j/2} \phi(2^j t - k)$$

   - increasing $j$ gives a better (i.e., higher resolution) approximation

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2$$

   - this is in general not the most useful expansion
The Resulting Expansions (cont.)

2 A low-resolution approximation plus its wavelet details

\[ f(t) = \sum_{k} c_{j_0}(k)2^{j_0/2} \phi(2^{j_0} t - k) + \sum_{j=j_0}^{\infty} \sum_{k} d_{j}(k)2^{j/2} \psi(2^{j} t - k) \]

- Low-Resolution Approximation
- Wavelet Details

- Choosing \( j_0 \) sets the level of the coarse approximation

\[ L^2 = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus W_{j_0+2} \oplus \cdots \]

- This is most useful in practice: \( j_0 \) is usually chosen according to application
  - Also in practice, the upper value of \( j \) is chosen to be finite
The Resulting Expansions (cont.)

3 Only the wavelet details

\[ f(t) = \sum_{k} \sum_{j=-\infty}^{\infty} d_j(k) 2^{j/2} \psi(2^j t - k) \]

- Choosing \( j_0 = -\infty \) eliminates the coarse approximation leaving only details

\[ L^2 = \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots \]

- This is most similar to the “true” wavelet decomposition as it was originally developed
- This is not that useful in practice: \( j_0 \) is usually chosen to be finite according to application
The Expansion Coefficients $c_{j0}(k)$ and $d_j(k)$

- We consider here only the simple, but important, case of ON expansion
  - i.e., the $\phi$’s are ON, the $\psi$’s are ON, \textit{and} the $\phi$’s are ON to the $\psi$’s
- Then we can use standard ON expansion theory:

  $c_{j0}(k) = \left\langle f(t), \varphi_{j0,k}(t) \right\rangle = \int f(t)\varphi_{j0,k}(t) \, dt$

  $d_j(k) = \left\langle f(t), \psi_{j,k}(t) \right\rangle = \int f(t)\psi_{j,k}(t) \, dt$

- We will see how to compute these without resorting to computing inner products
  - we will use the coefficients $h_1(n)$ and $h(n)$ instead of the wavelet and scaling function, respectively
  - we look at a relationship between the expansion coefficients at one level and those at the next level of resolution
Summary of Multiresolution View

- Nested Resolution spaces:
  \[ \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2 \]

- Wavelet Spaces provide orthogonal complement between resolutions
  \[ L^2 = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus W_{j_0+2} \oplus \cdots \]

- Wavelet Series Expansion of a continuous-time signal \( f(t) \):
  \[
  f(t) = \sum_{k} c_{j_0}(k)2^{j_0/2}\phi(2^{j_0}t - k) + \sum_{k} \sum_{j=j_0}^{\infty} d_{j}(k)2^{j/2}\psi(2^{j}t - k)
  \]

- MR equation (MRE) provides link between the scaling functions at successive levels of resolution:
  \[
  \phi(t) = \sum_{n} h(n)\sqrt{2}\phi(2t - n), \quad n \in \mathbb{Z}
  \]

- Wavelet equation (WE) provides link between a resolution level and its complement
  \[
  \psi(t) = \sum_{n} h_1(n)\sqrt{2}\phi(2t - n), \quad n \in \mathbb{Z}
  \]
Summary of Multiresolution View (cont.)

- There is a fundamental connection between the scaling function and its coefficients $h(n)$, the wavelet function and its coefficients $h_1(n)$:

$$h_1(n) \leftrightarrow \phi(t)$$

MR Equation (MRE)

How are $h_1(n)$ and $h(n)$ related?

$$h(n) \leftrightarrow \psi(t)$$

Wavelet Equation (WE)
Filter Banks and DWT
Generalizing the MRE and WE

- Here again are the MRE and the WE:

\[ \phi(t) = \sum_{n} h(n)\sqrt{2}\phi(2t - n) \]

\[ \psi(t) = \sum_{n} h_1(n)\sqrt{2}\phi(2t - n) \]

scale & translate: replace \( t \rightarrow 2^j t - k \)

- We get:

\[ \phi(2^j t - k) = \sum_{m} h(m - 2k)\sqrt{2}\phi(2^{j+1} t - m) \]

Connects \( V_j \) to \( V_{j+1} \)

\[ \psi(2^j t - k) = \sum_{m} h_1(m - 2k)\sqrt{2}\phi(2^{j+1} t - m) \]

Connects \( W_j \) to \( V_{j+1} \)
Linking Expansion Coefficients Between Scales

- Start with the Generalized MRA and WE:

\[
\phi(2^j t - k) = \sum_{m} h(m - 2k)\sqrt{2}\phi(2^{j+1} t - m) \quad \psi(2^j t - k) = \sum_{m} h_1(m - 2k)\sqrt{2}\phi(2^{j+1} t - m)
\]

\[
c_j(k) = \langle f(t), \phi_{j,k}(t) \rangle \quad d_j(k) = \langle f(t), \psi_{j,k}(t) \rangle
\]

\[
c_j(k) = \sum_{m} h(m - 2k)\langle f(t), 2^{(j+1)/2} \phi(2^{j+1} t - m) \rangle \quad d_j(k) = \sum_{m} h_1(m - 2k)\langle f(t), 2^{(j+1)/2} \phi(2^{j+1} t - m) \rangle
\]

\[
c_{j+1}(m) = \sum_{m} h(m - 2k)c_{j+1}(m) \quad d_{j+1}(m) = \sum_{m} h_1(m - 2k)c_{j+1}(m)
\]
Convolution-Decimation Structure

New Notation For Convenience: $h(n) \rightarrow h_0(n)$

$$c_j(k) = \sum_m h_0(m - 2k)c_{j+1}(m)$$

$$d_j(k) = \sum_m h_1(m - 2k)c_{j+1}(m)$$

Convolution

$$y_0(n) = c_{j+1}(n) * h_0(-n) = \sum_m h_0(m - n)c_{j+1}(m)$$

Decimation

n = 2k = 0 2 4 6 8

k = 0 1 2 3 4
Summary of Progression to Convolution-Decimation Structure

\[
\phi(2^j t - k) = \sum_m h(m - 2k)\sqrt{2}\phi(2^{j+1} t - m)
\]

\[
c_j(k) = \sum_m h(m - 2k)c_{j+1}(m)
\]

\[
\psi(2^j t - k) = \sum_m h_1(m - 2k)\sqrt{2}\phi(2^{j+1} t - m)
\]

\[
d_j(k) = \sum_m h_1(m - 2k)c_{j+1}(m)
\]
Computing The Expansion Coefficients

- The above structure can be cascaded:
  - given the scaling function coefficients at a specified level all the lower resolution c’s and d’s can be computed using the filter structure
Filter Bank Generation of the Spaces

\[ V_{j-1} \downarrow 2 \rightarrow V_j \downarrow 2 \rightarrow V_{j+1} \]

\[ V_{j-2} \downarrow 2 \rightarrow W_{j-2} \downarrow 2 \rightarrow W_{j-1} \downarrow 2 \rightarrow W_j \]

\[ \frac{\pi}{8} \quad \frac{\pi}{4} \quad \frac{\pi}{2} \quad \pi \]

\[ \text{LPF } h_0(-n) \quad \text{HPF } h_1(-n) \quad \downarrow 2 \quad \text{LPF } h_0(-n) \quad \downarrow 2 \quad \text{HPF } h_1(-n) \quad \downarrow 2 \quad \text{LPF } h_0(-n) \quad \downarrow 2 \quad \text{HPF } h_1(-n) \quad \downarrow 2 \quad \text{LPF } h_0(-n) \quad \downarrow 2 \quad V_{j-2} \]

\[ W_j \quad W_{j-1} \quad W_{j-2} \quad V_j \quad V_{j-2} \]
Discrete Fourier Transform
WAVELET TRANSFORM

Freq

Time

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WT-Based Compression Example

- Bits allocated to quantizers to minimize MSE
- Then allocations less than $B_{min}$ are set to zero
  - Eliminates negligible cells
- Side info sent to describe allocations