The Wavelet Transform:
What It Is and What It’s Good For

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The Forest...

The Wavelet Transform (WT) is a signal processing tool that is replacing the Fourier Transform (FT) in many applications.

Physics
- Quantum Mech.

Mathematics
- Harm. Analysis
- Group Theory

Signal Processing
- Signal Decomp.
- Filter Banks
- Image Analysis

Wavelets

Applications
- Data Compression.
- Computer Vision
- Denoising Signals
- Sonar/Radar
- Biomedical Proc.
- Communications
- Turbulence
- Geophysical Proc.
- Music Processing

Etc. Etc. Etc.

We’ll focus on the signal processing aspects:

Signal Decompositions

... And
Now
Some
Trees:
**So, What's Wrong With The FT?**

First, recall the FT:

\[ X(f) = \int_{-\infty}^{\infty} x(t) \ e^{-j2\pi ft} \ dt \]

\[ x(t) = \int_{-\infty}^{\infty} X(f) \ e^{j2\pi ft} \ df \]

Remember: an integral is like summation; so, the second equation says we are decomposing \( x(t) \) into a weighted "sum" of complex exponentials, and the first equation tells what each weight should be:

weight @ \( f \) is \( X(f) \)

component @ \( f \) is \( e^{j2\pi ft} \)

Note: these components exist for *ALL* time!

⇒ This is *not necessarily* a good model for real-life signals.

<<<< Think of Musical Notation Analogy >>>
EXAMPLE

Frequency Hopping Chirped Pulses
But What About The Short-Time FT (STFT)?

\[ X(f,t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) e^{-j2\pi ft} d\tau \]
Problems with the STFT

- Uncertainty Principle: \((\Delta t)(\Delta f) \geq \frac{1}{4\pi}\)

Improved Time Resolution  \(\Rightarrow\) Degraded Frequency Resolution
Improved Frequency Resolution  \(\Rightarrow\) Degraded Time Resolution

Problem: Same  \(\Delta t\) and  \(\Delta f\) throughout the entire time-frequency plane

- STFT is a Redundant Representation  \(\Rightarrow\) Not Good For Compression

For the STFT: nonredundancy implies poor time-frequency localization

Main Advantage of the STFT

- Intuitive Display of T-F characteristics  \(\Rightarrow\) Good for applications where humans view the results:

  - bioacoustics (birds, whales, etc. at Cornell Lab of Bioacoustics)
  - medical diagnoses (TMJ and other joint problems)

But, generalizations of Wavelets (Wave Packets, Matching Pusuits) are making inroads here, too.

The Wavelet Transform Can Help!!
So, What IS the WT?

Recall the STFT:

\[ X(f, t) = \int_{-\infty}^{\infty} x(\tau) [h(t - \tau) e^{-j2\pi f \tau}] d\tau \]

This is “comparing” \( x(\tau) \) to the basis functions.

\( h(\tau) = \) the prototype basis function.

Other basis functions: time shift and frequency shift the prototype.

This leads to the “uniform tiling” seen before.

★★★ There is an alternative to frequency shifting: **scaling** ★★★

\[ h(\tau/s) \leftrightarrow H(sf) \]

**increasing** \( s \) “scrunches” the (positive) spectrum to the **left**

**decreasing** \( s \) “stretches” the (positive) spectrum to the **right**

**large** \( s = \) **low** frequency = **coarse** scale

**small** \( s = \) **high** frequency = **fine** scale
The Concept of Scale

Think of a coastline:

• If you view it while flying in a jet at 20,000 ft,
  ▶ you see the coarse scale variations

• If you view it while floating in a hot air balloon,
  ▶ you see the medium scale variations

• If you view it while walking the beach,
  ▶ you see the fine scale variations
Each line shows $h(t-\tau s)$ as a function of $\tau$ for a specific $(t, s)$ pair.

Note: these #s are just convenient labels, NOT scale values.

Some S8 Symmlets at Various Scales and Locations

Increasing Time Shift
Each line shows the FT of \( h(t - \gamma s) \) for a specific \((t, s)\) pair.

Note: these \#s are just convenient labels, NOT scale values.
Representative T-F cells for Symmlets 3, 5, and 8 Shown Above

\[
\text{Frequency}
\]

\[
\text{Time}
\]

We still have to obey the uncertainty principle: 
\[
(\Delta t)(\Delta f) \geq \frac{1}{4\pi}
\]

But now \( \Delta t \) and \( \Delta f \) are adjusted depending on what region of frequency is being “probed”.

All this leads to... **The Wavelet Transform:**

\[
X(s,\tau) = \int_{-\infty}^{\infty} x(t) \left[ \frac{1}{\sqrt{s}} h \left( \frac{t-\tau}{s} \right) \right] dt, \quad s>0
\]

\[s = \text{scale} \quad \tau = \text{time shift}\]

**Requirements are:**

Finite Energy:

\[
h(\tau) \in L^2(\mathbb{R}) \quad \int_{-\infty}^{\infty} |h(\tau)|^2 d\tau < \infty
\]

Admissibility Condition:

\[
\int_{0}^{\infty} \frac{|H(\omega)|^2}{|\omega|} d\omega < \infty
\]

\[\Rightarrow \quad \text{must go to zero fast enough as } \omega \to 0\]
\[\Rightarrow \quad \text{must go to zero fast enough as } \omega \to \infty\]

\[\Rightarrow h(\tau) \text{ must be a bandpass signal}\]

**Reconstruction Formula:**

\[
x(t) = \frac{1}{C_\psi} \int_{0}^{\infty} \int_{-\infty}^{\infty} X(s,\tau) \left[ \frac{1}{\sqrt{s}} h \left( \frac{t-\tau}{s} \right) \right] \frac{ds d\tau}{s^2}
\]
The prototype basis function, \( h(\tau) \), is called the **mother wavelet**.

There are many choices of mother wavelet:
- each gives rise to a slightly different WT...
- with slightly different characteristics...
- suited to different applications.
Non-Redundant Form of the WT

It turns out... (don’t you just love those phrases!!!)

We only need to know the WT at discrete values of \( s \) and \( t \) in order to be able to uniquely represent the signal (i.e., in order to invert the WT):

\[
s = 2^m \quad \tau = n2^m \quad m = \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \\
n = \ldots -3, -2, -1, 0, 1, 2, 3, \ldots
\]

**Lower** Values of \( m \)  \( \Rightarrow \) **Smaller** Values of Scale  \( \Rightarrow \) **Higher** Frequency

Then the WT becomes a (countably infinite) collection of numbers (recall the Fourier Series vs. the Fourier Transform):

\[
X_{mn} = \int_{-\infty}^{\infty} x(t) \left[ 2^{-m/2} h(2^{-m} t - n) \right] dt
\]

In practice, you truncate the range of \( m \) and \( n \) to some finite region.

Sometimes it is more convenient to work in terms of \( 1/s \) instead of \( s \), because it is more closely associated with frequency.

This leads to the sampling of the (Time)-(Inverse Scale) plane that is shown on the next page:
(Time)-(Inverse Scale) Sampling Grid for Wavelet Transform
Advantages of this approach:

- Non-Redundant Representation (orthogonal representation)
  - good for data compression
  - leads to numerical stability
  - simple inversion formula:

\[
x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} X_{mn} [2^{-m/2} h(2^{-m} t - n)]
\]

- Leads to simple, efficient discrete-time implementation
The Discrete-Time WT

Recall the formula for the WT coefficients:

\[ X_{mn} = \int_{-\infty}^{\infty} x(t) \left[ 2^{-m/2} h(2^{-m} t - n) \right] dt \]

If the signal \( x(t) \) is bandlimited to \( B \) Hz, we can represent it by its samples taken every \( T_s = 1/2B \) seconds: \( x(kT_s) \rightarrow x(k) \).

**Our Goal:** since the \( x(k) \) uniquely and completely describe \( x(t) \), they should also uniquely and completely describe the \( X_{mn} \); **HOW DO WE DO IT??**

Consider the pedagogical case of choosing the mother wavelet to be a modulated sinc function. It has a spectrum as shown below:
Now consider the relationship between the signal spectrum and spectra of scaled versions of the mother wavelet:
Note that we don’t need to go to finer scale (i.e., smaller s, more negative m, higher frequency) because the signal doesn’t have any power there.

So in a sense, the finest scale that we need is on the order of the sampling interval $T_s$.

Note the recursive nature of this process:
- We split out the upper half of the signal, “leaving” the lower half
- Of this lower half, we split out the upper half, “leaving” the lower half
- Of this lower half, we split out the upper half, “leaving” the lower half etc.
  .
  etc.
  etc.

This procedure can be extended indefinitely in theory, but eventually we reach a point where it would be ridiculous to continue.

This leads to the following structure for computing the WT coefficients $X_{mn}$:
Cascaded Filter Structure for Computing WT Coefficients

\[
\begin{align*}
\text{x(n)} & \rightarrow \text{HPF} & \downarrow 2 & \rightarrow \text{HPF} & \downarrow 2 & \rightarrow \text{LPF} & \downarrow 2 & \rightarrow \text{HPF} & \downarrow 2 & \rightarrow \text{LPF} & \downarrow 2 & \rightarrow \text{LPF} & \downarrow 2 & \rightarrow \text{LPF} \\
& & h_1(k_1) & h_2(k_2) & h_3(k_3) & h_4(k_4) & l_4(k_4)
\end{align*}
\]

\(<\text{HPF Filter Coeffs = flipped LPF times 1 -1 1 -1 ..}>\)
Reverse Cascaded Filter Structure for Inverting the DWT

Filter Coefficients for the IDWT are flipped versions of those for the DWT.
Frequency Response of Filter Pair at **First** Stage:

![Graph showing frequency response](image1)

Frequency Response of Filter Pair at **Second** Stage:

![Graph showing frequency response](image2)

**ETC., ETC., ETC.**
Practical Orthogonal Filter Pair (Daubechies' $D_8$, length 8)
Another Way of Viewing the WT: The Coefficient Pyramid

Keep The Numbers That are Inside The Heavy Boxes

They Are The WT Coefficients
Another Way of Viewing the WT: Time-Frequency Tiling
Computational Complexity of DTWT

For a signal of length N:

- The number of multiplies and adds is $O(N)$

Lower order than the FFT, which is $O(N \log N)$
But watch out for the multiplicative constant.

Each Filter has length $L << N$

For first stage:

- LPF: compute $N/2$ output points, each requiring $L$ multiplies
- HPF: compute $N/2$ output points, each requiring $L$ multiplies
- # Multiplies for first stage $\approx NL/2 + NL/2 = NL$

For second stage:

- LPF: compute $N/4$ output points, each requiring $L$ multiplies
- HPF: compute $N/4$ output points, each requiring $L$ multiplies
- # Multiplies for first stage $\approx NL/4 + NL/4 = NL/2$

For third stage:

- LPF: compute $N/8$ output points, each requiring $L$ multiplies
- HPF: compute $N/8$ output points, each requiring $L$ multiplies
- # Multiplies for first stage $\approx NL/8 + NL/8 = NL/4$

Numer of Mult. $= NL \sum_{m=0}^{\infty} 2^{-m}$

$= 2NL$
Examples of Computed WT’s

The next two pages show two examples; they each show a different way of displaying the wavelet coefficients.

Example #1:

- A Synthetic Chirp Signal: frequency decreases w/ time
- WT coefficients are displayed as spikes along lines of constant scale
- Notice that:
  - high frequency components dominate at early times
  - low frequency components dominate at later times
  - spikes more closely spaced for the high frequency components

Example #2:

- A Real-Life SSB Radio Signal
- WT coefficients are displayed as gray-scale “blocks”
  - dark = most-negative values
  - medium = near-zero values
  - light = most-positive values
- Notice that:
  - signal has significant components only in certain T-F regions
  - blocks narrow and closely spaced for the high frequencies
  - blocks wide and widely spaced for the low frequencies
Real Part of a SSB Complex Baseband Signal

WT Coefficients for Real Part of a SSB Complex Baseband Signal
So, What's It Good For?

- signal/image data compression
- computer vision
- enhancing noisy signals
- sonar/radar processing
- biomedical signal processing
- digital communications
- studying turbulence
- geophysical signal processing
- music synthesis

etc., etc., etc.!!!
Example of WT Compression Applied to Seismic Signal:

An Older Way (using standard FT methods):

A New Way (using WT methods):
Comparison of Compression Techniques for the Seismic Signal

DCT = Discrete Cosine Transform (a standard FT method)

NOTE: Best Basis Wave Packet and Best Basis Cosine Packet are extensions of the standard wavelet transform.
Image Compression

- Images are 2-D signals
- Can extend WT to a 2-D form

The WT is VERY good at efficiently representing lines and edges

One application is to the compression of fingerprint images:

- FBI uses the WT for compression of its fingerprint images
- Can achieve a 26:1 compression ratio with little degradation
- Avoids the "blocking effects" of JPEG
**Noise Reduction**

Makes use of the fact that the signal occupies only a subregion(s) of the entire T-F plane, but the noise is everywhere.

- Compute the WT of the noisy signal
- Discard WT coefficients below a threshold
- Compute the inverse WT to get de-noised signal

The next three pages show an example of denoising:

- four noise-free signals
- the four signals w/ noise added
- the four noisy signals after WT-based denoising
3 (a) VisuShrink[yBlocks]

3 (b) VisuShrink[yBumps]

3 (c) VisuShrink[yHeaviSine]

3 (d) VisuShrink[yDoppler]
Summary

The Wavelet transform provides a means to “see” the time-frequency structure of a signal:

- The WT consists of the coefficients of a signal expansion
  - the basis functions correspond to t-f cells

- The t-f cells adjust their shape to cover the same number of cycles
  - short and wide at low frequencies
  - tall and narrow at high frequencies

- The representation can be easily computed from signal samples
  - simple cascaded filter bank
  - computational complexity is $O(N)$; lower order than the FFT

- The representation is non-redundant (orthogonal)
  - good for compression

- Statistical methods have been developed for de-noising
  - work best when signal is concentrated in WT domain
What if I Want to Know More?

Papers:

Overview Tutorials


< Code Listing on pp. 100 - 101>


Technical Tutorials


Books:

The *first book* listed gives a nice, gentle overview of wavelets; it is good for technical folks who want to know more but don’t have the time to slog through more technical tomes.

The *second book* is intended for statisticians, but gives one of the nicest concise treatments I’ve seen of the mathematical theory of wavelets; it also covers denoising.

The *other books* assume a background in standard DSP topics.


Web Sites:

There are many, but most (if not all!) can be reached through links found at: http://www.amara.com/current/wavelet.html

Software:

See the above referenced web site for a list available software:

- some free, some not