12.5 Multistage Rate Change
Motivation for Multi-Stage Schemes

Consider Decimation:

When $M$ is large (typically $> 10$ or so) it is usually inefficient to implement decimation in a single step (i.e., in a single stage).

The Culprit: Large $M$ requires the LPF to have a stopband edge of $\theta_s = \pi/M$, which is small for large $M$

$\Rightarrow$ Need a LPF with a very narrow passband

$\Rightarrow$ Requires a long FIR filter

$\Rightarrow$ Inefficient since long filters require a large # of multiplies

Solution: If $M$ can be factored into a product of integers $(M = M_1 M_2 M_3 \ldots M_p)$. Then decimation by $M$ can be done by:
Trick to Get Efficiency from Multi-Stage

The design of $H_1(z)$ (& other “front-end” stages) can be relaxed from what you would use for a single-stage design.

Certainly, you need $H_1(z)$ to have $\theta_s = \pi/M_1 > \pi/M$ so no aliasing occurs after $\downarrow M_1$….

But… it is even better than that.

Can let $\theta_s > \pi/M_1$ … which lets some aliasing occur

But… only so much aliasing – such that the aliasing that occurs gets suppressed by the next filter, $H_2(z)$

Higher Stopband Edge ➔ Shorter Filter ➔ More Efficient
Let’s See Why for a 2-Stage Case

Say that the signal $x[n]$ has “spectral content of worth” only up to frequency $\theta = \theta_p < \pi/M \ldots$ with $M = M_1M_2$.

**Single-Stage Method**

Suppose we decimate using a single-stage scheme:

\[
\begin{align*}
x[n] & \xrightarrow{H(z)} \downarrow M \quad y[n]
\end{align*}
\]

Then we need

**Sharp Transition** \(\rightarrow\) **Long Filter**
Let’s See Why for a 2-Stage Case (cont.)

2-Stage Method

\[ x[n] \xrightarrow{} H_1(z) \xrightarrow{} M_1 \xrightarrow{} H_2(z) \xrightarrow{} M_2 \xrightarrow{} y[n] \]

After \( H_1(z) \) but before \( \downarrow M_1 \) we need:

\[
\theta_{s,1} = \frac{(2M_2 - 1)\pi}{M}
\]

Same Passband as Single-Stage

Slow Transition \( \Rightarrow \) Short Filter
Let’s See Why for a 2-Stage Case (cont.)

Let’s see the impact of this slower transition on aliasing:

After 1\text{st} Filter

\[ H_1(z) \]

\[ \theta_p \quad (2M_2 - 1)\pi / M \]

\[ H_2(z) \]

\[ M_1 \theta_p \]

\[ 2\pi - M_1(2M_2 - 1)\pi / M = \pi / M_2 \]

\[ M_1(2M_2 - 1)\pi / M = 2\pi - \pi / M_2 \]

Stopband Suppresses Aliased Replica
Design Requirements
So… say you want to design a 2-stage multirate scheme instead of a 1-stage multirate scheme:

If single stage, say the specs need to be:
- Passband Cutoff = $\theta_p$
- Stopband Cutoff = $\theta_s$
- Passband Ripple = $\delta_p$
- Stopband Level = $\delta_s$

For a 2-stage scheme, our above results say we need:

- **1\textsuperscript{st} Stage**
  - $\theta_{p,1} = \theta_p$
  - $\theta_{s,1} = (2M_2-1)\pi/M > \theta_s$
  - $\delta_{p,1} = \delta_p/2$
  - $\delta_{s,1} = \delta_s$

- **2\textsuperscript{nd} Stage**
  - $\theta_{p,2} = M_1\theta_p$
  - $\theta_{s,2} = \pi/M_2$
  - $\delta_{p,2} = \delta_p/2$
  - $\delta_{s,2} = \delta_s$

Passband Ripple is split between 2 filters
Ex. 12.6: How 2-Stage Reduces Computation

**Goal:** Decimation by $M = 12$

**Filter Requirements**
- $\delta_p = 0.01 \quad \leftarrow$ to give some desired fidelity (application specific)
- $\delta_s = 0.001 \quad \leftarrow$ to limit aliasing to desired level (application specific)

$$\theta_p = \frac{\pi}{16} \quad \leftarrow \text{to pass desired band (application specific)}$$

$$\theta_s = \frac{\pi}{12} = \frac{\pi}{M} \quad \leftarrow \text{to prevent aliasing for desired decimation rate}$$

**Single-Stage Method**
Length of filter determines the # of computations

⇒ Use (9.79) to estimate filter order needed:

$$N = \frac{\frac{-\text{20log}_{10}\sqrt{\delta_p \delta_s}}{\text{13}}}{2.32\left[\theta_s - \theta_p\right]}$$

transition width
Ex. 12.6: Single-Stage Method (cont.)

Using this order estimate for the given filter requirements gives:

\[ N = 244 \quad \Rightarrow \quad \text{Length:} \quad L = N+1 = 245 \]

Note: #'s given below differ slightly from book because it (wrongly) uses \( N \) instead of \( L \) in its computation estimates.

Our chosen complexity measure: \# Multiplies/Input Sample

Each output sample (after decimation): \( L \) Multiplies/Output Sample

There are \( M \) Input Samples/Output Sample (due to decimation)

\[ \Rightarrow \quad (\# \text{ Multiplies})/(\text{Input Sample}) = \frac{L \text{ mult/output}}{M \text{ input/output}} = \frac{L}{M} = \frac{245}{12} \approx 20.4 \]

Single-Stage Complexity = 20.4 multiplies/input
Ex. 12.6: Double-Stage Method \((M = M_1 M_2: 12 = 3\times4)\)

**1st Stage**
- \(\theta_{p,1} = \theta_p = \pi/16\)
- \(\delta_{p,1} = \delta_p/2 = 0.005\)
- \(\theta_{s,1} = (2M_2-1)\pi/M = 7\pi/12\)
- \(\delta_{s,1} = \delta_s = 0.001\)

Estimated filter order gives: \(N_1 = 11\) \(L_1 = 12\)

So… Mult/Input = \(L_1/M_1 = 12/3 = 4\)

**2nd Stage**
- \(\theta_{p,2} = M_1\theta_p = 3\pi/16\)
- \(\delta_{p,2} = \delta_p/2 = 0.005\)
- \(\theta_{s,2} = \pi/M_2 = \pi/4\)
- \(\delta_{s,2} = \delta_s = 0.001\)

Estimated filter order gives: \(N_2 = 88\) \(L_2 = 89\)

So… Mult/Input = \(L_2/(M_1 M_2) = 89/12 = 7.4\)

Double-Stage Complexity = \(4 + 7.4 = 11.4\) multiplies/input

2-Stage has \(\approx \frac{1}{2}\) Complexity of 1-Stage

referenced back to input of whole system
Comments on Multistage Method

Q: What happens in this example when order of stages is switched? i.e., $M_1 = 4$ and $M_2 = 3$ (Left as Exercise!!!)

These 2-Stage design ideas can be extended to p-stage designs:

$$M = M_1 M_2 M_3 \ldots M_p$$

The order of these multiple stages matters

See textbook for discussion of multistage interpolation
Application: Multistage Rate Change

Convert Digital Audio Tape (DAT) format to Compact Disk (CD)

DAT uses $F_s = 48$ kHz

CD uses $F_s = 44.1$ kHz

\[
\begin{align*}
\frac{44.1 \times 10^3}{48 \times 10^3} &= \frac{441}{480} = \frac{3 \times 147}{3 \times 160} = \frac{147}{160}
\end{align*}
\]

Rate Change Ratio $= \frac{L}{M} = \frac{147}{160}$

Single-Stage Approach:

\[
\begin{align*}
\uparrow 147 & \quad H(z) \quad \downarrow 160 \\
F_s &= 48 \text{ kHz} & F_s &= 7.056 \text{ MHz}!!!!
\end{align*}
\]

Multiple-Stage Approach:

$L = 147 = 7 \times 7 \times 3$

$M = 160 = 10 \times 8 \times 2$