12.2 Transform View (Frequency Domain)

Need to look at:
- Z- Transform
- DTFT
M-Fold Decimation – Frequency-Domain

Notation: \[ \{ Z_x(\downarrow_M) \}(z) = X_z(\downarrow_M)(z) = \{ X^z(z) \}(\downarrow_M) \]

- Similar for DTFT
- Similar for Expansion

Q: What is \( X_{\downarrow M}(z) \) in terms of \( X(z) \)?

What do we expect?!!!

Lower \( F_s \) causes Spectral Replicas to Move to Lower Frequencies
Should look exactly like sampling at a lower \( F_s \)
Thus… increased aliasing is possible!!!

To answer this we need to define a useful function ("comb" function):

\[ c_M[n] = \sum_{k=-\infty}^{\infty} \delta[n - kM] = \delta[n \mod M] = \frac{1}{M} \sum_{m=0}^{M} W_M^{mn} \]

See Eq. (4.6)
Call this (★)
**M-Fold Decimation – Frequency-Domain (cont.)**

Now… use the comb function to write decimation:

\[
x_{(\downarrow M)}[n] = x[nM]
= x[nM]c_M[nM]
\]

Now… take Z-Transform, using this form:

\[
X_{(\downarrow M)}(z) = \sum_{n=-\infty}^{\infty} x[nM]c_M[nM]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]c_M[n]z^{-n/M}
\]

Now… take Z-Transform, using this form:

\[
X_{(\downarrow M)}(z) = \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{M}\sum_{m=0}^{M-1} W_M^{mn}\right]z^{-n/M}
\]

Doesn’t Really Do Anything Here… But Later it Will!!

Action of \(C_M[n]\)
Now… just manipulate:

\[
X_{\downarrow M}^z(z) = \sum_{n=-\infty}^{\infty} x[n] \left( \frac{1}{M} \sum_{m=0}^{M-1} W_m^{mn} \right) z^{-n/M}
\]

\[
= \frac{1}{M} \sum_{m=0}^{M-1} \left( \sum_{n=-\infty}^{\infty} x[n] \left( W_M^{-m} z^{1/M} \right)^{-n} \right)
\]

\[
= \frac{1}{M} \sum_{m=0}^{M-1} X^z \left( W_M^{-m} z^{1/M} \right)
\]

ZT of Decimated Signal is…

\[
X_{\downarrow M}^z(z) = \frac{1}{M} \sum_{m=0}^{M-1} X^z \left( W_M^{-m} z^{1/M} \right)
\]
$M$-Fold Decimation – Frequency-Domain (cont.)

Now to see a little better what this says… convert $ZT$ to DTFT. **Recall**: DTFT is the $ZT$ evaluated on the unit circle:

$$ z = e^{j\theta} \implies z^{1/M} = e^{j\theta/M} $$

Also, by definition: $W_{-m}^M = e^{-j2\pi m/M}$

Then we get….

**DTFT of Decimated Signal is…**

$$ X_{(\downarrow M)}(\theta) = \frac{1}{M} \sum_{m=0}^{M-1} X\left(\frac{\theta - 2\pi m}{M}\right) $$

1. **Axis-Scale** $X^f(\theta)$ to get $X^f(\theta/M)$ – a stretch
2. Then shift by $2\pi m$ to get new replicas

⇒ Decimation Adds Spectral Replicas of Scaled DTFT
Example: DTFT for $M$-Fold Decimation $\textbf{M} = 3$

Original

$X^f(\theta)$

$\pm \pi/3$

$m = 0$

$X^f(\theta/3)$

$m = 1$

$X^f((\theta-2\pi)/3)$

$m = 2$

$X^f((\theta-4\pi)/3)$

$\theta$

$\theta$

$\theta$

$\theta$

$\theta$

$\theta$

$\theta$
Example: Continued

\[ M = 3 \]

\[ X^f(\theta/3) \]

\[ m = 0 \]

\[ X^f((\theta-2\pi)/3) \]

\[ m = 1 \]

\[ X^f((\theta-4\pi)/3) \]

\[ m = 2 \]

\[ M \times \text{DTFT of Decimated Signal} \]

No Aliasing!!!
Example: Insights

1. The $M$-decimated signal will have no aliasing… only if the signal being decimated has: $X^f(\theta) = 0$ for $|\theta| > \pi / M$

This makes complete sense from an “ordinary” sampling theorem view point!!

Such a signal is called an “$M^{th}$-Band Signal”

2. After $M$-decimating an $M^{th}$-band signal, the spectrum of the decimated signal will fill the $[-\pi, \pi]$ band.
Effect on “Physical” Frequency

Although decimation changes the digital frequency of the signal, the corresponding “physical” frequency is not changed… as the following example shows:

\[
x[n] \quad \downarrow 3 \quad x_{\downarrow 3}[n]
\]

\[F_{s_1} = 60 \text{ MSPS} \quad \downarrow 3 \quad F_{s_2} = 60/3 = 20 \text{ MSPS}\]

\[X_f(\theta) \quad \rightarrow \quad X_{\downarrow 3}(\theta)\]

Signal Still Occupies **Same** Physical Frequency
L-Fold Expansion – Frequency-Domain

Q: What is \( X_{(\uparrow L)}(z) \) in terms of \( X(z) \)???
What do we expect????!!!! 
Certainly **NOT** the same as *really* sampling at a higher rate 
because of the inserted zeros!!!

**Frequency Domain analysis answers this!!!**

\[
X_{(\uparrow L)}(z) = \sum_{n=-\infty}^{\infty} x_{(\uparrow L)}[n] z^{-n} \\
= + \cdots + x[0] z^0 + \underbrace{0 + \cdots + 0}_{\text{L-1 zeros}} + x[1] z^{-L} + \underbrace{0 + \cdots + 0}_{\text{L-1 zeros}} + x[2] z^{-2L} \\
= \sum_{n=-\infty}^{\infty} x[n] z^{-Ln} = X^z(z^L)
\]

**ZT of Expanded Signal is…**

\[
X_{(\uparrow L)}(z) = X^z(z^L)
\]
**L-Fold Expansion – Frequency-Domain (cont.)**

Now to see a little better what this says… convert ZT to DTFT. **Recall:** DTFT is the ZT evaluated on the unit circle:

\[ z = e^{j\theta} \Rightarrow z^L = e^{jL\theta} \]

DTFT of Decimated Signal is…

\[ X^f_{(\uparrow L)}(\theta) = X^f(L\theta) \]
Example: DTFT for \( L \)-Fold Expansion

\[ L = 3 \]

Expansion Causes Images to Appear in the \([-\pi, \pi]\) Range

Here’s what we’d have if we **REALLY** sampled 3 times as fast… **No Images!!!**