Lecture #1
Review of DT Signal & System Concepts
Notational Conventions (P-1.2)

Sets of Numbers:  
\[ R = \text{real numbers} \]
\[ C = \text{complex numbers} \]
\[ Z = \text{integers (…, -3, -2, -1, 0, 1, 2, 3, …)} \]

Signals:  
\[ x(t), y(t) = \text{continuous-time (CT) signals} \]
\[ x[n], y[n] = \text{discrete-time (DT) signals} \]

Modulo Operator – “Clock Math”:  
\[ n \mod N \]
\[ n: \ldots -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \ldots \]
\[ n \mod N: \ldots 2 \quad 0 \quad 1 \quad 2 \quad 0 \quad 1 \quad 2 \quad 0 \quad 1 \quad \ldots \]

“15 o’clock” = 15 mod 12 = “3 o’clock”

DT Convolution:  
\[ \{ x \ast y \}[n] = \sum_{m=\infty}^{\infty} x[m] y[n-m] \]

DT Circular Convolution:  
\[ \{ x \bigoplus y \}[n] = \sum_{m=0}^{N-1} x[m] y[(n-m) \mod N] \quad 0 \leq n \leq N - 1 \]

Recall: We don’t want Circular convolution – it is what we get if we try to implement Convolution in the frequency domain by multiplying DFT’s (but not DFT’s).

But… we can fix it so that it works…. We use proper zero-padding!!!
Transforms (P-2)

Fourier Transform for CT Signals

\[ X^F(\omega) = \{Fx\}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \]
\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^F(\omega)e^{j\omega t} \, d\omega \]

Fourier Transform for DT Signals

\[ X^f(\theta) = \{Fx\}(\theta) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n} \]
\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^f(\theta)e^{j\theta n} \, d\theta \]

Set \( z = e^{j\theta} \)

Z Transform for DT Signals

\[ X^z(z) = \{Zx\}(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

Discrete Fourier Transform for DT Signals

Discrete Fourier Transform

\[ X^d[k] = \{Dx\}[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \quad k = 0, 1, 2, ..., N - 1 \]
\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X^d[k]e^{j2\pi kn/N} \quad n = 0, 1, 2, ..., N - 1 \]

Inverse ZT done using partial fractions & a ZT table
Fourier Transforms for DT Signals (P-2.7,P-4)

**DTFT** = “In Head/On Paper” tool for analysis & design

**DFT** = “In Computer” tool for Implementation/Analysis & Design
   (In both cases you generally strive to make the DFT behave like DTFT)

- DFT for Implementation:
  ▪ Compute DFT of signal to be processed and manipulate DFT values
  ▪ May or may not need zero-padding (depends on application)
- DFT for Analysis & Design:
  ▪ Compute DFT of given filter’s TF to see its frequency response
  ▪ Generally use zero-padding to get fine grid to get approximate DTFT

DFT/DTFT Properties you **Must Know**:

- Periodicity ................... (DFT&DTFT both periodic in FD)
  ......................... (IDFT gives periodic TD signal)
- Time Shift ..................... (Time Shift in TD ⇔ Linear Phase Shift in FD)
- Frequency Shift ................ (Mult. by complex sine in TD ⇔ Freq Shift in FD)
- Convolution Theorem .......... (Conv. in Time Domain ⇔ Mult. in Freq Domain)
- Multiplication Theorem ........ (Mult. in Time Domain ⇔ Conv. in Freq Domain)
- Parseval’s Theorem ........... (Sum of Squares in TD = Sum of Squares in FD)
Summation Rules (P-1.3)
Make sure you are familiar with rules for dealing with summations, as shown in Section 1.3 of Porat.

In addition, a particular summation formula that shows up often is the "Geometric Summation":

\[
\sum_{n=N_1}^{N_2-1} \alpha^n = \begin{cases} 
\frac{\alpha^{N_1} - \alpha^{N_2}}{1-\alpha}, & \text{if } \alpha \neq 1 \\
N_2 - N_1, & \text{if } \alpha = 1
\end{cases}
\]

Special Case: \( N_1 = 0 \)

\[
\sum_{n=0}^{N_2-1} \alpha^n = \begin{cases} 
\frac{1-\alpha^{N_2}}{1-\alpha}, & \text{if } \alpha \neq 1 \\
N_2, & \text{if } \alpha = 1
\end{cases}
\]
Euler’s Formulas

\[ e^{j\theta} = \cos(\theta) + j \sin(\theta) \]
\[ e^{-j\theta} = \cos(\theta) - j \sin(\theta) \]

\[
\begin{align*}
\cos(\theta) &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\
\sin(\theta) &= \frac{e^{j\theta} - e^{-j\theta}}{2j}
\end{align*}
\]
Discrete-Time Processing (P-2, P-7)

\[ h[n] = \text{system’s “Impulse Response”} \]
It is response of system to input of \( \delta[n] \)

\[ H^f(\theta) = \text{system’s “Frequency Response”} \]
It is DTFT of impulse response

\[ H^z(z) = \text{system’s “Transfer Function”} \]
It is ZT of impulse response

\[
\begin{align*}
x[n] & \quad \rightarrow \quad h[n] \quad \rightarrow \quad y[n] = h[n] \ast x[n] \\
X^f(\theta) & \quad \rightarrow \quad H^f(\theta) \quad \rightarrow \quad Y^f(\theta) = H^f(\theta) X^f(\theta) \\
X^z(z) & \quad \rightarrow \quad H^z(z) \quad \rightarrow \quad Y^z(z) = H^z(z) X^z(z)
\end{align*}
\]
Discrete-Time System Relationships

**Time Domain**
- Difference Equation
- Impulse Response

**Z / Freq Domain**
- Transfer Function
- Pole/Zero Diagram
- Frequency Response

**Block Diagram**
- ZT (Theory)
- DTFT

**Inspection**
- Inspect
- Roots
- Unit Circle

**Equations**
- Impulse Response: $h[k]$ = $y[k] = -\sum_{i=1}^{p} a_i y[n-i] + \sum_{i=0}^{q} b_i x[n-i]$
- Transfer Function: $H^z(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}}$
- Frequency Response: $H^f(\theta) = H^z(z) \big|_{z=e^{j\theta}}$
Poles and Zeros of Transfer Function

Consider Here Only $p>q$

$$H^z(z) = \frac{b_0 + b_1z^{-1} + \cdots + b_qz^{-q}}{1 + a_1z^{-1} + \cdots + a_pz^{-p}}$$

$$= \frac{z^{p-q}z^q(b_0 + b_1z^{-1} + \cdots + b_qz^{-q})}{z^p(1 + a_1z^{-1} + \cdots + a_pz^{-p})}$$

$$= \frac{z^{p-q}(b_0z^q + b_1z^{q-1} + \cdots + b_q)}{z^p + a_1z^{p-1} + \cdots + a_p}$$

$$= \frac{z^{p-q}(z - \beta_1)(z - \beta_2)\cdots(z - \beta_q)}{(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_p)}$$

Numerator Roots define “zeros”
Denominator Roots define “poles”

In this case
$p-q$ zeros at $z=0$

Poles must be inside unit circle for stability

Poles & zeros must occur in Complex Conjugate pairs
to give real-valued TF coefficients
**IIR and FIR Filters**

\[
H^z(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}}
\]

**FIR**: Finite Impulse Response – \( h[n] \) has only finite many nonzero values

**Filter is FIR**: If \( a_i = 0 \) for \( i = 1, 2, \ldots, p \) \hspace{1cm} (It is IIR otherwise)

\[
H^z(z) = b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}
\]

\[
h[n] = \begin{cases} 
  b_n, & n = 0,1,\ldots,q \\
  0, & \text{otherwise}
\end{cases}
\]

TF coefficients are the impulse response values!!
Example System

Time Domain

- Difference Equation
- Input-Output Form: \( y(n) - \alpha y(n-1) = \beta x(n) \)
- Recursion Form: \( y(n) = \beta x(n) + \alpha y(n-1) \)

Z / Freq Domain

- Transfer Function
- \[ Y(z) \left[ 1 - \alpha z^{-1} \right] = \beta X(z) \]
- \[ H(z) = \frac{Y(z)}{X(z)} = \frac{\beta}{1 - \alpha z^{-1}} \]
Example System (cont.)

Time Domain

\[ y(n) = \beta x(n) + \alpha y(n-1) \]

Z / Freq Domain

\[ H(z) = \frac{\beta}{1 - \alpha z^{-1}} \]
Example System (cont.)

Z / Freq Domain

\[ H(z) = \frac{\beta}{1 - \alpha z^{-1}} \]

On Unit Circle

\[ z = e^{j\Omega} \]
\[ z^{-1} = e^{-j\Omega} \]

Frequency Response

\[ H(\Omega) = \frac{\beta}{1 - \alpha e^{-j\Omega}} \]
\[ \Omega \in (-\pi, \pi] \]
Example System (cont.)

Plotting Transfer Function

\[ H(\Omega) = \frac{\beta}{1 - \alpha e^{-j\Omega}} = \frac{\beta}{1 - [\alpha \cos(\Omega) - j\alpha \sin(\Omega)]} \]

\[ = \frac{\beta}{[1 - \alpha \cos(\Omega)] + j\alpha \sin(\Omega)} \]

\[ |H(\Omega)| = \frac{\beta}{\sqrt{[1 - \alpha \cos(\Omega)]^2 + [\alpha \sin(\Omega)]^2}} \]

\[ \angle H(\Omega) = -\tan^{-1}\left[\frac{\alpha \sin(\Omega)}{1 - \alpha \cos(\Omega)}\right] \]
Example System (cont.)

**Z/Freq Domain**

- **Transfer Function**
  \[ H(z) = \frac{\beta}{1 - \alpha z^{-1}} = \frac{\beta z}{z - \alpha} \]

- **Roots**
  - Zero
    Num. = 0
    When \( z = 0 \)
  - Pole
    Den. = 0
    When \( z = \alpha \)

- **Pole/Zero Diagram**
  - Unit Circle
  - If \( \alpha < 1 \): Inside UC
  - If \( \alpha > 1 \): Outside UC
Example System (cont.)

Time Domain

$h(n) = \beta \alpha^n$

Impulse Response

Inv. DTFT

$h(n) = \int_{-\pi}^{\pi} \frac{\beta}{1 - \alpha e^{-j\Omega}} d\Omega$

Z / Freq Domain

$H(z) = \frac{\beta}{1 - \alpha z^{-1}}$

Transfer Function

Unit Circle

$H(\Omega) = \frac{\beta}{1 - \alpha e^{-j\Omega}}$

$\Omega \in (-\pi, \pi]$
Review of Standard Sampling Theory
(P-3.1-3.2)
Practical Sampling Set-Up

$x(t)$

ADC

Sample at $t = nT$

$T = $ Sampling Interval

$F_s = 1/T = $ Sampling Rate

"Hold"

$x[n] = x(nT)$

Pulse Gen

$\tilde{x}_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \bar{p}(t - nT)$

DAC

$\tilde{x}_r(t)$

$T = $ Sampling Interval

$F_s = 1/T = $ Sampling Rate

$Practical Sampling Set-Up$
Sampling Analysis (#1)

Goal = Determine Under What Conditions We Have:
Reconstructed CT Signal = Original CT Signal \( \tilde{x}(t) = x(t) \)

Simplify to Develop Theory: Use \( \tilde{p}(t) = \delta(t) \)

Why????

1. Because delta functions are \textbf{EASY} to analyze!!!
2. Because it leads to the best possible case

\[
\tilde{x}_p(t) \Rightarrow x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)
\]

\( x_p(t) \) is called the “Impulse Sampled” signal

Note: \( x_p(t) \) shows up during Reconstruction, not Sampling!!
**Sampling Analysis (#2)**

Now, the impulse sampled signal $x_p(t)$ is filtered to give the reconstructed signal $x_r(t)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)$$

$$= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$\xrightarrow{CT \ LPF} \quad H(f)$$

$$x_r(t) = ??? \quad \Updownarrow$$

$$X_r(f) = H(f) X_p(f)$$

Need to find $X_p(f)$…. But First a Trick!!!

$$x_p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= x(t) \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j2\pi k t / T}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} x(t) e^{j2\pi k F_s t}$$

$$X_p(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F\{x(t) e^{j2\pi k F_s t}\}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - k F_s)$$
Sampling Analysis (#3)

Let's analyze the process step by step:

1. **Input Signal** ($x(t)$)
2. **Sampling**
   - Sample at $t = nT$
3. **Hold**
4. **Digital Signal** ($x[n] = x(nT)$)
5. **Impulse Gen**
6. **CT LPF**
7. **Output Signal** ($x_r(t)$)

Mathematically:

- **Input Signal:** $x(t)$
- **Sampling:**
  - $x[n] = x(nT)$
- **Impulse Response:** $H(f)$
- **Output Signal:** $x_r(f)$

Frequency Domain:

- **Input Spectrum:** $X(f)$
- **Impulse Response Spectrum:** $H(f)$
- **Output Spectrum:** $X_r(f)$

**Theorem:**

If $F_s \geq 2B$, then $X_r(f) = X(f)$.
Sampling Analysis (#4)

What this says: Samples of a bandlimited signal completely define it as long as they are taken at $F_s \geq 2B$

Impact: To extract the info from a bandlimited signal we only need to operate on its (properly taken) samples

→ Use computer to process signals
Sampling Analysis (#5)

FT of Impulse Sampled Signal gives view of Original FT &

FT of Impulse Sampled Signal = DTFT of Samples

⇒ DTFT gives view of FT of original signal

\[
F\{x_p(t)\} = F\left\{ \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \right\} = \sum_{n=-\infty}^{\infty} x(nT)F\{\delta(t - nT)\} e^{-jn\omega T}
\]

\[= \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T} = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta} \quad \theta = \omega T\]

\[
X_p^F(\theta/T) = X^f(\theta)
\]
Sampling Analysis (#6)

\[ \theta = \omega T = \left[ \frac{2\pi}{F_s} \right] f \]

Read notes on web on “Concept of Digital Frequency”
Introduction to EE521 Case Study
EE521 Case Study: Emitter Location

**Processing Tasks**
- Intercept RF Signal @ Rx’s
- Sample Signal Suitably for Processing
- Detect Presence of Emitter’s Signal
- Estimate Characteristics of Signal
- Use Est’d Char’s to Classify Emitter
- Share Data Between Rx’s
- Cross-Correlate Signals to Locate Tx

**Radio Freq Transmitter (Tx)**
- Communications
- Radar

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Diagram:

- Radio Freq Transmitter (Tx)
- Receiver #1 (Rx1)
- Receiver #2 (Rx2)
- Receiver #3 (Rx3)
- Data Links
Processing Tasks Overview (#1)

RF Front-End (Analog Circuitry): "RF" = "Radio Frequency"
- Selects reception band; may need to scan bands
- Amplifies signal to level suitable for ADC, etc.
- Frequency-shifts signal spectrum to range suitable for ADC
  - Technology trend is toward requiring less shifting
- Unfortunately: noise introduced by analog electronics (*Random Signals*)

**Topic We’ll Cover**

FT of Signal Before RF-FE

- 1 GHz
- 3 GHz
- 4 GHz

Antenna

FT of Signal After RF-FE

- 200 MHz
Processing Tasks Overview (#2)

Sampling Sub-System (Mixed Analog/Digital Circuitry):
- Converts analog CT signal into digital DT signal (*Bandpass Sampling*)
- Converts real-valued RF signal into complex LP signal (*Equiv. LP Signals*)

Digital Front-End (Digital Processing):
- Convert to complex LP signal (*Equiv. LP Signals*)
- Equalizes response of analog front-end (*DFT-Based Filtering*)
  - Magnitude and Phase
- Bank of digital filters splits signal into sub-bands for further processing (*Filter Banks*)
- Reduce sampling rate after filtering to sub-bands (*Multi-Rate Processing*)
Processing Tasks Overview (#3)

Detect Presence of Signal (Digital Processing):
• Has signal been intercepted or only noise in subband (DFT-Based Proc.)

Estimate Parameters of Signal (Digital Processing) (see also EE522):
• Estimate frequency of signal (DFT-Based Processing)

Classify/Model Signal (Digital Processing):
• What type of signal is it? (Spectral Analysis of Random Signals)

Compress/De-Compress Signal (Digital Processing) (see also EE523):
• Exploit signal structure to allow efficient transfer (DFT-Based Proc./ Filter Banks / Spectral Analysis)

Cross-Correlate Signals (Digital Processing):
• Compute relative delay and Doppler (DFT-Based Proc. & Multi-Rate Proc.)