A Polynomial Rooting Approach for Analysis of Competition among Secondary Users in Cognitive Radio Networks

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***Abstract*—The competitive coexistence of a large number of secondary users in heterogeneous cognitive radio networks (CRN) is a challenging issue. In this paper, we address this issue by developing an efficient approach to analyze the competition among the secondary users. Based on the Markov model bank that we developed recently, we can change the evaluation of the secondary user throughput into studying a special root of a polynomial that is much more simplified. Under this framework, we study the competition among the secondary users in three special cases: homogeneous CRN with identical secondary users, heterogeneous CRN with a group of identical secondary users plus an outlier, and heterogeneous CRN with two different groups of secondary users. In each case, we derive the polynomial, prove the uniqueness of the root, and derive the throughput of each secondary user based on the root. Furthermore, we derive closed-form approximate solutions to the root, based on which constrained optimization can be formulated to study the optimal competition strategies for the secondary users. The optimization results are in water-filling principles and can be derived in closed-form expressions. Simulations are conducted to verify the analysis results.**

Keywords*—cognitive radio network, Markov chain, throughput, dynamic spectrum access, polynomial root

I. Introduction

Cognitive radio networks (CRN) allow secondary users (SU) to reuse the spectrum currently occupied exclusively by primary users (PU) with dynamic spectrum access (DSA). The SUs conduct spectrum sensing, and access the spectrum that is not occupied by the PUs. The SUs must vacate the spectrum whenever the PUs become active.

Due to the inherent flexibility of cognitive radios, different SUs may adopt different transmission parameters and different spectrum sensing and spectrum access strategies. For example, since the PU signals are different temporarily and geographically, the SUs may choose to adopt either more conservative or more aggressive polices toward sensing errors. Some SUs may exploit handshaking or control channels more than others. In particular, there are many different spectrum access strategies for the SUs to adopt. This not only makes the CRN heterogeneous, but also raises the issue of competition among the (possibly selfish) secondary users when they are trying to gain the access of the limited spectrum resource.

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Although there have been extensive research published in CRN, including spectrum sensing, Physical-layer modulation, MAC/Network layer protocols, hardware/testbed development, etc, very limited study has been conducted on the competitive coexistence of a large number of different SUs in heterogeneous CRNs. There is a class of work on the CRN transmission power/capacity analysis based on the physical-layer interference analysis, but they are conducted under certain simplified communication models [1]. There is another class of work on CRN MAC protocol design and analysis, such as the performance analysis of the CSMA-type protocols [2]-[4]. Nevertheless, the homogeneity assumption overlooks most of the complex competitions. Under the selfish and decentralized operations assumption, various game-theoretic spectrum sensing and spectrum access strategies have been developed to address the competition. However, the performance was usually too complicated to analysis. More importantly, all the secondary users are still assumed to have the same strategy space, which inherently makes the CRN homogeneous rather than heterogeneous. As a result, the equilibrium of the CRN in these works may still be very different from the competitive coexistence of heterogeneous CRNs.

The large number of dissimilar users makes the modeling and analysis of heterogeneous CRN challenging. Most of the existing works on CRN analysis that consider more or less the heterogeneous nature of CRN have to make substantial simplifications [5]-[12]. The SU's competition for spectrum access is skipped in [5] and [6] because only simultaneous transmissions are considered, where spectrum access becomes just the constant arrival/departing rate. In addition, most of these works focus on the simple scenario of two or three secondary users only to simplify the Markov chain modeling. The general case becomes prohibitively complex due to the rapidly increased number of Markov states and transitional probabilities.

In [13], we developed an innovative Markov Model Bank (MMB) for modeling and analyzing the heterogeneous CRN with multiple dissimilar SUs and multiple dissimilar channels. We focused on evaluating the SU throughput under certain simple spectrum access strategies. Our basic idea was to decompose the complex Markov model into a bank of relatively separated and thus simplified Markov chains, one for each SU. This makes the Markov chains easy to analyze. In addition, MMB allows us to exploit spatial, channel and user de-correlations to drastically simplify the interference analysis, which makes it tractable to analyze large heterogeneous CRNs.

In this paper, we focus on analyzing the competing behavior of the SUs using different transmission parameters and different spectrum access strategies. We change the performance evaluation of the SUs into studying special roots of some polynomials. By studying the properties and by deriving approximate closed-form solutions to the roots, the competition among the secondary users and their throughput can be conveniently analyzed.

The organization of this paper is as follows. In Section II, we give the system model and the MMB analysis framework. In Section III, we adopt appropriate network decomposition techniques to change the network throughput evaluation into polynomial rooting. Then in Section IV, we study the competition among the secondary users under three typical CRN scenarios. Simulations are conducted in Section V, and conclusions are given in Section VI.

II. System model

The CRN we consider in this paper consists of a set of N randomly distributed SUs and some PUs. Each SU is a transmitting-receiving pair of nodes, where the transmitting node transmits to the receiving node. While transmitting to the receiving node, the transmitter creates interference to other users. We assume there are M channels available for the N SUs to choose.

For each SU, we consider a cognitive radio transmission model that includes four states: spectrum sensing, data transmission, waiting and channel switching. If the spectrum sensing indicates the channel is available for secondary access, then the cognitive radio transmits a data packet, and the model shifts into the data transmission state. If the spectrum sensing indicates the channel is not available (due to the PU activity, interference from other SUs activity, or the lack of offered load from this SU), then the cognitive radio enters the waiting state, which will be followed by the channel switching state. We assume that the cognitive radio always shifts back to the spectrum sensing state after either the channel switching state or the data transmission state.

We assume that the SU can successfully detect the activity of the PUs. Therefore, we can denote the activity of the PUs in each channel k by a random variable θ_k, which denotes the probability that the channel k is not used by any PU. This makes us to focus on analyzing the coexistence of SUs via their mutual interference.

In the spectrum sensing state, the signal received by the SU i's receiver in the channel k is

$$u_{i,k}(n) = \sum_{j=1,j\neq i}^{N} f_{j,k}\sqrt{P_{j,k}}h_{ji,k}s_j(n) + v_{i,k}(n) \quad (1)$$

where $P_{j,k}$ is the transmission power that the SU j spends in the channel k, $s_j(n)$ is the signal transmitted by the SU j, $h_{ji,k}$ is the flat fading channel from the SU j's transmitter to the SU i's receiver, and $v_{i,k}(n)$ is the AWGN with zero-mean and power $\sigma^2_{i,k}$. The variable $f_{j,k} = 1$ or 0 denotes whether the SU j is currently using the channel k. The instantaneous

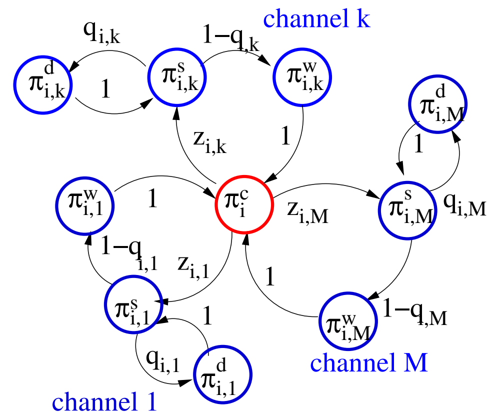

Fig. 1. Markov Model Bank (MMB) for multi-user heterogenous CRN, where only the Markov chain for the SU i is shown. Each of the SUs has a similar Markov chain. Each Markov chain has $3M+1$ states: 3 states in each channel k, plus a single channel switching state for all the channels. The transitional probabilities $q_{i,k}$ and $z_{i,k}$ are determined by the spectrum sensing and the spectrum access strategies.

signal-to-noise ratio (SNR) of the SU i in the channel k can thus be obtained from (1) as

$$\gamma^s_{i,k} = \frac{\sum_{j=1,j\neq i}^{N} f_{j,k}P_{j,k}|h_{ji,k}|^2}{\sigma^2_{i,k}} \quad (2)$$

If SNR $\gamma^s_{i,k}$ is larger than the detection threshold $\Gamma_{i,k}$, then the SU i makes a decision that the channel is not available for transmission. It turns to the waiting state. Note that due to the asynchronous and decentralized operation of the competing secondary users in heterogeneous CRNs, we do not consider specifically any common MAC-layer protocol where the secondary users compete for channel access in a slotted signaling structure. In our case, different secondary users may use different slot/packet durations, and such transmission parameters are competition factors.

As [13], we can use a Markov Model Bank (MMB) to model the competitive spectrum access among the N CRN SUs. The MMB consists of a Markov chain for each SU, as shown in Fig. 1, where $\pi^s_{i,k}$,$\pi^d_{i,k}$,$\pi^w_{i,k}$ are the probabilities of the SU i staying in the channel sensing, data packet transmission and waiting state in the channel k. The probability π^c_i refers to the probability that the SU i staying in the channel switching state. The transitional probability $q_{i,k}$ denotes the probability that the channel k can be used by the SU i to transmit a data packet. It is a composite result of PU detection, SU carrier sensing, as well as the offered load. The transitional probability $z_{i,k}$ denotes the probability for the SU i to choose the channel k to sense and access. It is determined by the channel access strategy. We assume $0 \leq z_{i,k} \leq 1$ and

$$\sum_{k=1}^{M} z_{i,k} = 1, \quad i = 1, \cdots, N, \quad (3)$$

which means that a SU just uses one channel each time.

Note that although different SUs' Markov chains look separated from each other, their transitional probabilities $q_{i,k}$ are inter-related. In addition, depending on the strategies used in channel selection, the transitional probabilities $z_{i,k}$ for different secondary users may also be inter-related. As a matter of fact, the MMB approach simplifies the Markov model, but increases the complexity of $q_{i,k}$ and possibly $z_{i,k}$. Fortunately, MMB can bring some useful network decomposition properties to reduce the complexity.

For each SU i, the durations of the spectrum sensing, data transmission, waiting, and channel switching are $T^s_{i,k}$, $T^d_{i,k}$, $T^w_{i,k}$ and T^c_i, respectively. Note that we have defined the SU's offered load as a random variable α_i, but we will not use the SU's exiting probability. Such exiting probability, if needed, can be integrated into the data packet/slot length $T^d_{i,k}$. The packet/slot lengths are variables to be studied in this paper.

From [13], the steady state probabilities of the Markov model of Fig. 1 is given as

$$\pi^s_{i,k} = \frac{z_{i,k}}{\left(1 + 2\sum_{\ell=1}^{M} \frac{z_{i,\ell}}{1-q_{i,\ell}}\right)(1-q_{i,k})} \tag{4}$$

and

$$\pi^d_{i,k} = q_{i,k}\pi^s_{i,k},\ \pi^w_{i,k} = (1-q_{i,k})\pi^s_{i,k},\ \pi^c_i = \frac{1-q_{i,k}}{z_{i,k}}\pi^s_{i,k}. \tag{5}$$

In this paper, we focus on analyzing the throughput of each secondary user. Each SU i has a throughput $R_{i,k}$ obtained in the channel k, where $R_{i,k}$ is defined as the fraction of time when the SU i is using the channel k for data packet transmissions. To simplify analysis, we use the throughput definition that is more simplified than those used in [13]. Specifically, we do not consider the transmission channel capacity in the throughput definition. Note that the data transmission can be assumed free of collisions due to the perfect PU and SU sensing assumption.

Based on each single SU's throughput $R_{i,k}$, we can derive the overall throughput for a group of secondary users or even the entire CRN. We can also define the channel usage as the probability of a channel being occupied by all the SUs, which can also be derived from $R_{i,k}$.

III. Network decomposition and the polynomial rooting approach

The most complex task is perhaps to analyze $q_{i,k}$ which requires us to analyze the mutual interference among all the SUs. The probability $q_{i,k}$ is closely related to the SU's spectrum sensing strategy. In our case, it actually depends on the PUs' activity, the other SUs' activity, as well as this SU's offered load. We define

$$q_{i,k} = \theta_k\alpha_i\mathbb{P}\left[\frac{f_{j,k}P_{j,k}|h_{ji,k}|^2}{\sigma^2_{i,k}} < \Gamma_{i,k}, \forall j = 1,\cdots,N, j\neq i\right], \tag{6}$$

which means that the SU i can use the channel k to transmit a data packet only if there is no PU activity in the channel k, the SU i has data packet to transmit, and the interference from any other SU using the channel k simultaneously is low enough.

The probability that another SU j is also using the channel k can be modeled as a Bernoulli random variable with probability

$$\begin{aligned}\beta_{j,k} &= \mathbb{P}[f_{j,k}=1] \\ &= \frac{\pi^d_{j,k}T^d_{j,k}}{\pi^c_j T^c_j + \sum_{\ell=1}^{M}\left(\pi^s_{j,\ell}T^s_{j,\ell} + \pi^d_{j,\ell}T^d_{j,\ell} + \pi^w_{j,\ell}T^w_{j,\ell}\right)}.\end{aligned} \tag{7}$$

From (4) we can easily see

$$\frac{\pi^s_{i,k}}{\pi^s_{i,\ell}} = \frac{z_{i,k}(1-q_{i,\ell})}{z_{i,\ell}(1-q_{i,k})}. \tag{8}$$

Applying (4) and (8), we can derive from (7)

$$\begin{aligned}\beta_{j,k} = &\frac{z_{j,k}q_{j,k}T^d_{j,k}}{1-q_{j,k}} \\ &\times\frac{1}{T^c_j + \sum_{\ell=1}^{M}\left[T^s_{j,\ell} + q_{j,\ell}T^d_{j,\ell} + (1-q_{j,\ell})T^w_{j,\ell}\right]\frac{z_{j,\ell}}{1-q_{j,\ell}}}.\end{aligned} \tag{9}$$

In other words, $f_{j,k}$ is a Bernoulli random variable with probability $\beta_{j,k}$.

According to (6), $q_{i,k} = 0$ whenever there is a SU j with $f_{j,k} = 1$ and $P_{j,k}|h_{ji,k}|^2/\sigma^2_{i,k} \geq \Gamma_{i,k}$. In this paper, we assume that the SUs use very low detection SNR thresholds $\Gamma_{i,k}$ so that (6) can be simplified to

$$q_{i,k} = \theta_k\alpha_i\prod_{j=1,j\neq i}^{N}(1 - c_{i,j}\beta_{j,k}). \tag{10}$$

where

$$c_{i,j} = -2 + \frac{T^s_{i,k}}{T^d_{j,k}}. \tag{11}$$

Note that the probability that the SU j is transmitting data packets while the SU i is conducting spectrum sensing is

$$\frac{\pi^d_{j,k}(2T^d_{j,k} + T^s_{i,k})}{\pi^c_j T^c_j + \sum_{\ell=1}^{M}\left(\pi^s_{j,\ell}T^s_{j,\ell} + \pi^d_{j,\ell}T^d_{j,\ell} + \pi^w_{j,\ell}T^w_{j,\ell}\right)}$$

because when the SU i begins spectrum sensing, it can detect the SU j's transmission when the latter is transmitting from $T^d_{j,k}$ seconds before the sensing to $T^d_{j,k} + T^s_{i,k}$ after. Following (9) we can derive the probability as $c_{i,j}\beta_{j,k}$.

The essential idea of (10) is that we set the threshold $\Gamma_{i,k}$ small enough so as to forbid simultaneous transmissions among the secondary users. This is similar to the practical MAC-layer protocols such as Carrier Sense Multiple Access (CSMA) that prevent collisions (i.e., simultaneous transmissions) among the users. This simplification is reasonable in practice when the SUs have relatively strong interference among each other, such as in closely distributed networks. It was called *spatial de-correlation* in [13] because it effectively subdivides a large CRN into many smaller un-correlated clusters. In our case, it makes us focus better on the competition analysis. Generally speaking, weak mutual interference has negligible competition effects, and can thus be omitted.

In this paper, the throughput of the SU i achieved in the channel k is defined as

$$R_{i,k} = \beta_{i,k}, \tag{12}$$

which is the duty-cycle when the SU i is staying in the data transmission state. Due to the assumption of perfect PU sensing and SU sensing, data transmission can be considered as free of collisions. Therefore, (12) is the probability of the successful data transmission in the channel k.

The equations (9)-(12) are general expressions that can be used to derive numerically the throughput performance of SUs in general heterogeneous CRNs. However, they are not convenient for more insightful analysis due to their complexity and the interconnection among all the SUs and all the channels. For analysis purpose, we need to adopt some reasonable assumptions to further simplify the expressions.

First, in (9), we find that there is a special connection between $T_{i,\ell}^d$ and $T_{i,\ell}^w$ which are the slot lengths of the data transmission slot and the waiting slot, respectively. Without loss of generality, we assume that $T_{i,k}^d = T_{i,k}^w$, i.e., the duration of waiting state is the same as the data transmission state. In practice, when a channel is detected as unavailable, the SUs usually wait for just one data packet transmission slot duration before trying another channel. Therefore, this assumption is highly practical and reasonable. Then (9) can be reduced to

$$\beta_{j,k} = \frac{z_{j,k} q_{j,k} T_{j,k}^d}{(1-q_{j,k})[T_j^c + \sum_{\ell=1}^M (T_{j,\ell}^s + T_{j,\ell}^d)\frac{z_{j,\ell}}{1-q_{j,\ell}}]}. \tag{13}$$

To further simplify the expressions, we assume that each SU uses the same slot length parameters when accessing all the channels. Therefore, $T_{j,k}^s = T_j^s$, $T_{j,k}^d = T_j^d$, $T_{j,k}^w = T_j^w$, for all $k = 1, \cdots, M$. In addition, we assume that $T_j^c = 0$ because channel selection is just a decision making without the need of any transmission or receiving. The information for the decision making can be gathered and processed during the waiting state. Then, we can simplify (13) into

$$\beta_{j,k} = \frac{z_{j,k} q_{j,k} T_j^d}{(1-q_{j,k})(T_j^s + T_j^d)\sum_{\ell=1}^M \frac{z_{i,\ell}}{1-q_{i,\ell}}}. \tag{14}$$

In (14), each $\beta_{j,k}$ still depends explicitly on the same SU's activities in all the k channels. Fortunately, as shown in [13], we can take *channel de-correlation* to remove the explicit mutual dependence among all the channels ℓ. The channel de-correlation is conducted by replacing $z_{i,k}$ with $x_{i,k}$ where

$$x_{i,k} = \frac{z_{i,k}}{(1-q_{i,k})\sum_{\ell=1}^M \frac{z_{i,\ell}}{1-q_{i,\ell}}}. \tag{15}$$

From the properties $0 \le q_{i,k} \le 1$ and $\sum_{k=1}^M z_{i,k} = 1$, we can see that

$$0 \le x_{i,k} \le 1, \quad \sum_{k=1}^M x_{i,k} = 1. \tag{16}$$

In addition, given $x_{i,k}$, we can uniquely determine $z_{i,k}$ via

$$z_{i,k} = \frac{x_{i,k}(1-q_{i,k})}{\sum_{\ell=1}^M x_{i,\ell}(1-q_{i,\ell})}. \tag{17}$$

Note that $z_{i,k}$ is the channel selection probability, while $x_{i,k}$ is a new random variable describing the same channel selection probability, i.e., the probability for the SU i to select the channel k. For example, with uniformly-distributed random channel selection strategy, we have both $z_{i,k} = \frac{1}{M}$ and $x_{i,k} = \frac{1}{M}$. Nevertheless, $x_{i,k}$ and $z_{i,k}$ are different in general.

With the channel de-correlation, we can reduce the CRN throughput evaluation expressions (9)-(12) into

$$\begin{cases} R_{i,k} = b_i x_{i,k} q_{i,k}, \\ q_{i,k} = a_{i,k}\prod_{j=1,j\neq i}^N (1 - c_{i,j} b_j x_{j,k} q_{j,k}), \end{cases} \tag{18}$$

where $b_i = T_i^d/(T_i^s + T_i^d)$ is the transmission protocol parameter and $a_{i,k} = \alpha_i \theta_k$ includes both the offered load and the PU channel activity. Note that the throughput in channel k, i.e., $R_{i,k}$, relies on the parameters in the channel k only. Based on (18), we can calculate the throughput of each SU numerically by first finding N probabilities $q_{i,k}$, $i = 1, \cdots, N$, which are the roots of N nonlinear equations. However, in general, it is possible to change it into a simpler form, i.e., finding a special root of one polynomial, as outlined below.

Considering two SUs i and j. From (18), we find that

$$a_{i,k}^{-1} q_{i,k}(1 - c_{i,i} b_i x_{i,k} q_{i,k}) = a_{j,k}^{-1} q_{j,k}(1 - c_{j,j} b_j x_{j,k} q_{j,k}). \tag{19}$$

Define

$$D_k = a_{i,k}^{-1} q_{i,k}(1 - c_{i,i} b_i x_{i,k} q_{i,k}), \tag{20}$$

which is a special parameter shared by all the SUs. The value D_k satisfies some equations such as

$$D_k = \prod_{j=1}^N (1 - c_{i,j} b_j x_{j,k} q_{j,k}), \tag{21}$$

Conditioned on D_k, different SU's $q_{i,k}$ and $R_{i,k}$ can be evaluated separately, which is called *user de-correlation* in [13]. Since D_k is a common parameter for all the SUs, it is possible to compare different SUs' performance even without knowing the value of D_k. This provides a promising way for efficient and incisive CRN performance evaluation and analysis.

We can formulate some polynomials of D_k in order to find its values or the properties. Specifically, from (20), $q_{i,k}$ has solution

$$q_{i,k} = \frac{1 \pm \sqrt{1 - 4c_{i,i} b_i x_{i,k} a_{i,k} D_k}}{2c_{i,i} b_i x_{i,k}}. \tag{22}$$

We usually just need the "minus" sign in (22) because $x_{i,k}$ is small. If $x_{i,k}$ is small enough, we can even use approximately $q_{i,k} \approx a_{i,k} D_k$. Then D_k is the root to a polynomial obtained by substituting (22) back into (21).

In summary, for CRN performance analysis, we can first find the polynomial roots as D_k, and then use the value D_k to calculate $q_{i,k}$ as well as the throughput $R_{i,k}$. This is the general outline of the *polynomial rooting approach* for CRN performance analysis. The details of this approach may change when considering specific CRN scenarios, which will be shown in Section IV.

Given each SU's throughput $R_{i,k}$, the overall throughput of a group of SUs obtained in a set of channels can be defined as

$$R^+ = \sum_k \sum_i R_{i,k}. \tag{23}$$

Unfortunately, such trivial additions may accumulate the errors in $R_{i,k}$, especially the inevitable systematic modeling errors (in addition to the numerical and noise-induced errors). Even though the systematic error in each $R_{i,k}$ is negligibly small, the accumulated error over a large number of SUs and channels may become significant. To avoid this problem, for large CRN, we may use the following throughput definition

$$R^* = 1 - \prod_k \prod_i (1 - R_{i,k}), \tag{24}$$

which can suppress systematic errors more effectively. For example, if the estimated throughput for each SU is $\hat{R}_{i,k} = R + \epsilon$, where ϵ is the systematic error that are identical for all the SUs, then (23) gives $\hat{R}^+ = NMR + NM\epsilon$, where the error increases linearly with NM. In contrast, (24) gives $\hat{R}^* = 1 - \prod_i (1 - R - \epsilon)^{NM} \approx 1 - (1-R)^{NM} + NM\epsilon(1-R)^{NM-1}$, where the accumulated error is attenuated by the factor $(1-R)^{NM-1}$ which becomes very small for large NM.

Nevertheless, since (23) is easier to analyze and is accurate in CRN with moderate size, we still prefer it for analysis.

Another performance metric based on $R_{i,k}$ is the probability that a channel k is used by all the SUs. It measures how the SUs changes the spectral environment, which may be called *spectral mutation*. This probability can be defined as

$$U_k^+ = \sum_{i=1}^{N} R_{i,k}, \quad \text{or} \quad U_k^* = 1 - \prod_{i=1}^{N} (1 - R_{i,k}). \tag{25}$$

There are some interesting questions with respect to (25), such as whether a group of SUs can change arbitrarily U_k^+, or, for an arbitrary U_k^+ whether a group of SUs can achieve it by adjusting their transmission parameters and strategies. All these are highly relative to the competition study of the SUs.

IV. Competition analysis of three special CRNs

The framework outlined in Section III models the general CRN where each secondary user can adopt transmission parameters or strategies that are different from others. This obviously makes the expressions and analysis unnecessarily complex. In many practical situations, we may have a group of SUs that adopt similar transmission parameters and strategies. We call them *homogeneous SUs*. In addition, there are some other SUs that adopt another set of transmission parameters and strategies, whom we call *outliers*. These two groups of SUs may compete with each other within the same CRN. In this section, we focus on analyzing their competition. We consider three cases: CRN with N homogeneous SUs, CRN with $N-1$ homogeneous SUs plus one outlier, and CRN with two different groups of homogeneous SUs. In each case, we will develop in details the polynomial rooting approach.

A. CRN with N homogeneous SUs

We first consider the CRN where all the N SUs have the same transmission parameters and spectrum access strategies. Specifically, in (18) we have $c_{i,j} = c$, $b_i = b$, $a_{i,k} = a_k$, and $x_{i,k} = x_k$, for $i = 1, \cdots, N$. Using these parameters in (18), we can see that the SUs have the same $q_{i,k}$ and $R_{i,k}$.

Denote $q_{i,k} = q_k$ and $R_{i,k} = R_k$. To find the throughput R_k is to solve

$$\begin{cases} R_k = bx_k q_k, \\ q_k = a_k(1 - cbx_k q_k)^{N-1}, \end{cases} \tag{26}$$

Note that each SU's throughput in the channel k can be evaluated independently from other channels, given the probability x_k of channel selection.

Proposition 1. Considering the constraint $0 \le q_k \le 1$, the equation (26) has one and only one valid solution $R_k = bx_k y^{N-1}$ and $q_k = y^{N-1}$, where y is the root of the polynomial

$$cbx_k y^{N-1} + a_k^{-\frac{1}{N-1}} y - 1 = 0, \tag{27}$$

that satisfies $0 \le y \le 1$.

Proof. From the second equation in (26), we have

$$q_k^{\frac{1}{N-1}} = a_k^{\frac{1}{N-1}} (1 - cbx_k q_k). \tag{28}$$

Define $y = q_k^{\frac{1}{N-1}}$, we can change (28) into (27). Obviously, for $0 \le q_k \le 1$, we have $0 \le y \le 1$. In addition, $q_k = y^{N-1}$.

The solutions to (26) can be obtained from the roots of the polynomial $f(y) = cbx_k y^{N-1} + a_k^{-\frac{1}{N-1}} y - 1$. We find that $f(0) = -1 < 0$, and $f(1) = cbx_k + a_k^{-\frac{1}{N-1}} - 1 > 0$ because $0 \le a_k \le 1$. Note that $a_k = \theta_k \alpha$ where α is the offered load for the SU which is assumed identical among all the SU. In addition, $\partial f(y)/\partial y = (N-1)cbx_k y^{N-2} + a_k^{-\frac{1}{N-1}} > 0$ since all the variables are non-negative. This means that $f(y)$ is monotone increasing over $0 \le y \le 1$. Therefore, the function $f(y)$ has one and only one root that satisfies $0 \le y \le 1$. This proves the existence of a unique solution to (26). □

Therefore, each SU can achieve a unique throughput R_k in the channel k. The throughput is identical for all the SUs. The throughput depends on the transmission parameters b, a_k and the channel selection probability x_k. It can be calculated from the unique valid root of (27).

One of the interesting yet challenging issue is how the SUs can maximize their throughput by selecting the optimal transmission parameters b and channel selection probabilities x_k. From the throughput expression and (27), we have $R_k = bx_k y^{N-1} = c^{-1}(1 - a_k^{-\frac{1}{N-1}} y)$. Theoretically, the throughput can be optimized by looking for the smallest root

$$\min_{\{b, x_k\}} \; y, \tag{29}$$

with the constraints $\sum_{k=1}^{M} x_k = 1$, $0 \le y \le 1$, and (27). Unfortunately, this optimization is difficult to solve, mainly because the root of the high-order polynomial (27) is elusive. We turn to an approximately but informative analysis instead.

Notice from (27) that $y \approx (a_k/c)^{\frac{1}{N-1}}$ for large N and M. Therefore, we assume $y = (a_k/c)^{\frac{1}{N-1}} + \epsilon$, where $|\epsilon| \ll 1$. Substituting y into (27) and applying the approximation $((a_k/c)^{\frac{1}{N-1}} + \epsilon)^{N-1} \approx a_k/c + (N-1)(a_k/c)^{\frac{N-2}{N-1}} \epsilon$, we can get an approximate solution to (27) as

$$y \approx (a_k/c)^{\frac{1}{N-1}} - a^{\frac{1}{N-1}} \frac{ba_k x_k + c^{-\frac{1}{N-1}} - 1}{1 + (N-1)bc^{\frac{1}{N-1}} a_k x_k}. \tag{30}$$

Substituting (30) back into $R_k = (1 - a^{-\frac{1}{N-1}} y)/c$ we can obtain

$$R_k \approx c^{-1} - c^{-\frac{N}{N-1}} + \frac{c^{-1} ba_k x_k + c^{-\frac{N}{N-1}} - c^{-1}}{1 + (N-1)bc^{\frac{1}{N-1}} a_k x_k}. \tag{31}$$

From (31) we can see that larger x_k or b gives higher throughput. Each SU get approximately $1/(N-1)$ throughput from a channel. For a single SU, it can choose $x_k = 1$ to maximize its throughput in the channel k.

Let us consider the problem of looking for the optimal channel selection strategy $\{x_k\}$ to maximize an SU's throughput in all channels. Note that it also maximizes the overall CRN throughput. Using (31), the optimization problem

$$\max_{\{x_k\}} \sum_{k=1}^{M} R_k, \quad s.t., \sum_{k=1}^{M} x_k = 1 \tag{32}$$

becomes a special case outlined in the Appendix, where the optimal solution follows the water-filling principle. Specifically, we have

$$\tilde{\alpha}_k = \frac{1}{c}\left(\frac{N-2}{N-1}c^{-\frac{1}{N-1}} - 1\right), \ \tilde{\beta}_k = (N-1)bc^{\frac{1}{N-1}}a_k \tag{33}$$

Then we can follow (58) to find λ^* with all the $j = 1, \cdots, M$ that satisfies (57). Then the optimal channel selection probabilities can be found from (59).

In particular, consider the special case with $a_k = 1$ and $b \approx 1$, i.e., channels are all available, and sensing slot duration is negligibly small compared with the data transmission duration. Then we can find that $x_k^* = 1/M$, which means uniformly distributed random channel selection is optimal. In this special case with $x_k^* = 1/M$, from (31), we find $R_k \approx \frac{1}{2(N-1)} + \frac{1}{2}(1 - 2^{-\frac{1}{N-1}})$ per SU per channel, which is between $1/(2N)$ to $1/N$. Note that $1/N$ is the ideal throughput. Therefore, the approximation method tends to under-estimate the throughput in large CRN.

From the water-filling principle (59), we can readily see that each SU should increase the probability x_k on channels with the stronger a_k (i.e., larger θ_k). In other words, SUs should utilize more the channels that are used less by the PUs. In addition, throughput increases with larger b and a_k.

From the x_k^* derived from (59), we can find q_k^*, and then calculate the practical channel selection probabilities (that is actually used directly by the SUs) z_k^* according to (17). In particular, for identical $a_k = a$, it is easy to see that the optimal uniformly distributed random channel selection means $x_k^* = z_k^* = \frac{1}{M}$.

B. CRN with $N-1$ homogeneous SUs and one outlier

If there is one outlier SU that adopts transmission parameters and spectrum access strategies that are different from the rest of the $N-1$ SUs, we would like to see how this outlier can compete with the rest of the SUs. Denote the parameters for each of the $N-1$ homogeneous SUs as b_1, a_{1k}, q_{1k}, x_{1k} and R_{1k}, while the corresponding parameters for the outlier are b_2, a_{2k}, q_{2k}, x_{2k} and R_{2k}. Because $T_{i,k}^s$ is usually much less than $T_{j,k}^d$, without loss of generality, we use a constant c for all the SUs. The equation (18) becomes

$$\begin{cases} R_{1k} = b_1 x_{1k} q_{1k} \\ R_{2k} = b_2 x_{2k} q_{2k} \\ q_{1k} = a_{1k}(1 - cb_1 q_{1k} x_{1k})^{N-2}(1 - cb_2 q_{2k} x_{2k}) \\ q_{2k} = a_{2k}(1 - cb_1 q_{1k} x_{1k})^{N-1} \end{cases} \tag{34}$$

Similar to Section IV-A, we can formulate the polynomial rooting approach for (34), and find the solution as below.

Proposition 2. The equation (34) has one and only one valid solution which can be expressed as

$$\begin{cases} q_{1k} = \frac{1 - a_{2k}^{-\frac{1}{N-1}} y}{cb_1 x_{1k}}, \\ q_{2k} = y^{N-1}, \end{cases} \tag{35}$$

where y is the root of the polynomial

$$c^2 b_1 x_{1k} b_2 x_{2k} a_{1k} y^{2N-3} - cb_1 x_{1k} a_{1k} y^{N-2} - a_{2k}^{\frac{N-3}{N-1}} y + a_{2k}^{\frac{N-2}{N-1}} = 0, \tag{36}$$

that satisfies $0 \le y \le 1$.

Proof. From the second equation in (34), we have

$$1 - cb_1 q_{1k} x_{1k} = \frac{q_{2k}^{\frac{1}{N-1}}}{a_{2k}^{\frac{1}{N-1}}}. \tag{37}$$

which gives

$$q_{1k} = \left(1 - \frac{q_{2k}^{\frac{1}{N-1}}}{a_{2k}^{\frac{1}{N-1}}}\right) \frac{1}{cb_1 x_{1k}}. \tag{38}$$

Substituting it into the first equation in (34) we obtain

$$\left(1 - \frac{q_{2,k}^{\frac{1}{N-1}}}{a_{2k}^{\frac{1}{N-1}}}\right) \frac{1}{cb_1 x_{1k}} \frac{q_{2,k}^{\frac{1}{N-1}}}{a_{2k}^{\frac{1}{N-1}}} = a_{1k} \frac{q_{2k}}{a_{2k}} (1 - cb_2 q_{2k} x_{2k}). \tag{39}$$

Define $y = q_{2k}^{\frac{1}{N-1}}$, then we can revise (39) into (36). Equations in (35) can also be obtained from (38).

Next, to show the existence and the uniqueness of the root, consider the constraint $0 \le y \le 1$ for the equation (36). Following similarly Proposition 1, at $y = 0$, the left hand side of (36) is larger than 0. At $y = 1$, the left hand side of (36) is less than 0. From the derivative of the left hand side of (36) with respect to y, we can show that the derivative is always negative. Therefore, the left hand side polynomial is monotone decreasing. There is one and only one solution for $0 \le y \le 1$. □

For the CRN with $N-1$ homogeneous SUs and one outlier, every homogeneous SU has the same throughput R_k in the channel k as other homogeneous SUs. However, the outlier may have different R_k given the different transmission parameters b_2, a_{2k} and the channel selection probability x_{2k}. Their throughputs can be calculated after finding the unique valid root of (36).

From (34) and (35), the throughputs of each homogeneous SU and the outlier are connected as

$$(b_2 a_{2k} x_{2k})^{-\frac{1}{N-1}} R_{2k}^{\frac{1}{N-1}} + cR_{1k} = 1. \tag{40}$$

If counting the overall throughput of the homogeneous SUs, we have $(b_2 a_{2k} x_{2k})^{-\frac{1}{N-1}} R_{2k}^{\frac{1}{N-1}} + \frac{c}{N-1}[(N-1)R_{1k}] = 1$. This shows the throughputs can be adjusted by the outlier's transmission parameters.

The throughputs

$$R_{1k} = c^{-1}(1 - a_{2k}^{\frac{1}{N-1}} y), \quad R_{2k} = b_2 x_{2k} y^{N-1} \tag{41}$$

can be studied or optimized from the root y of (36). Obviously, R_{1k} decreases with y while R_{2k} increases with y. When $b_1 x_{1k}$ are determined, the outlier can choose b_2 and x_{2k} to adjust the root y in (36). Many questions can be investigated, such as how the outlier can maximize its throughput, or how it can reduce the throughput of the homogeneous SUs. Nevertheless, due to the lack of closed-form solution to (36), the analysis is not easy.

In the sequel, we drive an approximate closed-form solution to (36) to give more incisive analysis of the competition among the homogeneous SUs and the outlier. The procedure is similar to the derivation of the approximate solution in Section IV-A. Since the expressions are more complex, we assume that $a_{1k} = a_{2k} = 1$ to simplify the notation. The procedure will be similar for the general case though. This assumption means that the channel has no PU activity and the SU's offered load is 1. Then, let $y = 1 + \epsilon$, where $|\epsilon| \ll 1$. Applying the approximation $y^K \approx 1 + K\epsilon$ to (36), we can obtain

$$\epsilon = \frac{c^2 b_1 x_{1k} b_2 x_{2k} - 2cb_1 x_{1k}}{1 - (2N-3)c^2 b_1 x_{1k} b_2 x_{2k} + 2(N-2)cb_1 x_{1k}}. \quad (42)$$

Based on (42), we have the approximate solutions

$$\begin{cases} R_{1k} \approx c^{-1}(1-y) = -c^{-1}\epsilon \\ R_{2k} = b_2 x_{2k}(1+\epsilon)^{N-1} \approx b_2 x_{2k}[1 + (N-1)\epsilon] \end{cases} \quad (43)$$

As a special example, in case $c = 2$, $b_1 = b_2 = 1$, and $x_{1k} = 1/M$ (which is the optimal channel selection probability for the homogeneous SUs), we have

$$\epsilon \approx -\frac{1 - x_{2k}}{M/4 + N - 2 - (2N-3)x_{2k}}. \quad (44)$$

The outlier can use its x_{2k} to adjust both R_{1k} and R_{2k}. For example, it can maximize $|\sum_{k=1}^{M} R_{2k} - R_{1k}|$ over x_{2k}, where the Appendix suggests some water-filling results.

In general, the homogeneous SUs can optimize their own throughput R_{1k} via optimal channel selection probabilities x_{1k}. This leads to the constrained optimization

$$\max \sum_{k=1}^{M} R_{1k}, \quad s.t., \ \sum_{k=1}^{M} x_{1k} = 1. \quad (45)$$

This optimization can be formulated into the form (54), and a water-filling principle on the solution x_{1k} can be obtained. In this case, the optimal x_{1k} is a function of x_{2k}.

For the outlier to enhance its competition, it may select its own x_{2k} to either solely maximize its own throughput R_{2k} or to minimize the homogeneous SU's throughput R_{1k}. The first approach is the optimization

$$\max \sum_{k=1}^{M} R_{2k}, \quad s.t., \ \sum_{k=1}^{M} x_{2k} = 1, \quad (46)$$

and the second approach is the optimization

$$\min \sum_{k=1}^{M} R_{1k}, \quad s.t., \ \sum_{k=1}^{M} x_{2k} = 1. \quad (47)$$

Both the optimization can be revised into the form (54) and find the optimal water-filling principle on x_{2k}. Details are skipped to save space.

C. CRN with two groups of homogeneous SUs

Now consider the case that there are two groups of SUs competing for spectrum access. There are N_1 SUs in the first group that have the parameters b_1, a_{1k}, x_{1k}, q_{1k} and R_{1k}. The rest $N_2 = N - N_1$ SUs form the second group that have the parameters b_2, a_{2k}, x_{2k}, q_{2k} and R_{2k}. They share the same constant c. From (18) we have

$$\begin{cases} R_{1k} = b_1 x_{1k} q_{1k} \\ R_{2k} = b_2 x_{2k} q_{2k} \\ q_{1k} = a_{1k}(1 - cb_1 q_{1k} x_{1k})^{N_1 - 1}(1 - cb_2 q_{2k} x_{2k})^{N_2} \\ q_{2k} = a_{2k}(1 - cb_1 q_{1k} x_{1k})^{N_1}(1 - cb_2 q_{2k} x_{2k})^{N_2 - 1}. \end{cases} \quad (48)$$

We can construct a polynomial that can be used to study (48).

Proposition 3. The equation (48) has one and only one valid solution that can be written as

$$\begin{cases} q_{1k} = \frac{1 \pm \sqrt{1 - 4cb_1 x_{1k} a_{1k} a_{2k}^{-1} y (1 - cb_2 x_{2k} y)}}{2cb_1 x_{1k}}, \\ q_{2k} = y, \end{cases} \quad (49)$$

where y is the root of the polynomial

$$\begin{aligned} & cb_1 x_{1k} a_{1k} a_{2k}^{-\frac{N_1-1}{N_1}} y^{\frac{N_1-1}{N_1}} (1 - cb_2 x_{2k} y)^{\frac{N-1}{N_1}} \\ & \quad + a_{2k}^{-\frac{1}{N_1}} y^{\frac{1}{N_1}} (1 - cb_2 x_{2k} y)^{-\frac{N_2-1}{N_1}} - 1 = 0. \end{aligned} \quad (50)$$

that satisfies $0 \le y \le 1$. Note that only one q_{1k} in (49) satisfies $0 \le q_{1k} \le 1$.

Proof. From (48) we have

$$a_{1k}^{-1} q_{1k}(1 - cb_1 x_{1k} q_{1k}) = a_{2k}^{-1} q_{2k}(1 - cb_2 x_{2k} q_{2k}) \quad (51)$$

Define $y = q_{2,k}$, then we have (49). From the second equation in (48) we can get

$$1 - cb_1 x_{1k} q_{1k} = a_{2k}^{-\frac{1}{N_1}} y^{\frac{1}{N_1}} (1 - cb_2 x_{2k} y)^{-\frac{N_2-1}{N_1}}, \quad (52)$$

which gives

$$\begin{aligned} & \pm\sqrt{1 - 4cb_1 x_{1k} a_{1k} a_{2k}^{-1} y (1 - cb_2 x_{2k} y)} \\ & \quad = 2a_{2k}^{-\frac{1}{N_1}} y^{\frac{1}{N_1}} (1 - cb_2 x_{2k} y)^{-\frac{N_2-1}{N_1}} - 1. \end{aligned} \quad (53)$$

Taking square of both hand-sides, we can get the polynomial equation (50).

To show the solution is unique, similar to Proposition 1, the left hand side of (50) is less than 0 at $y = 0$, and is larger than 0 at $y = 1$. From the derivative of the left hand side of (50) with respect to y, we can show that the derivative is always positive. Therefore, the left hand side of (50) is monotone increasing. There is one and only one solution satisfies $0 \le y \le 1$. $\square$

Even though $q_{1,k}$ has two possible values in (49), only one is valid, and the other does not satisfy $0 \le y \le 1$. This can be explained if we form an equation in terms of q_{1k} rather than q_{2k} in (50). In this case we can show that there is only one valid solution to q_{1k}.

Obviously, R_{2k} increases with y, while R_{1k} decreases with y. The analysis of the competition between the two groups of SUs can be conducted similarly as Section IV-B. To save space, however, we do not repeat it since the expressions are substantially longer. We study some of their properties in simulations instead.

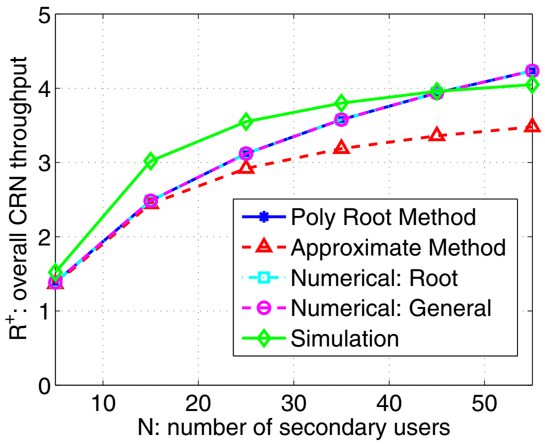

Fig. 2. Overall throughput of all the secondary users in all channels for CRN with various number of homogeneous secondary users.

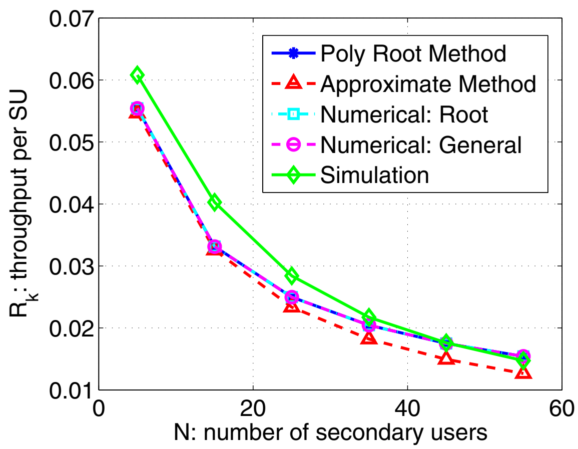

Fig. 3. Throughput per secondary user per channel in CRN with various number of homogeneous secondary users.

V. Simulations

In this section, we report our simulation of the numerical evaluation of the CRN throughput based on the analysis results derived in Sections III and IV. We verify them by the Monte-Carlo simulation of a real random CRN with N secondary users and M channels (**"Simulation"**). The nodes' positions were randomly generated within a square of 1000×1000 meters. The SNRs were calculated as $10^8 d_{ij}^{-2.6}$ where d_{ij} is the propagation distance. We evaluated the overall CRN throughput R^+ in addition to the throughput per SU.

First, we simulated the CRN with homogeneous SUs only for $M = 5, b = 10/11, a_k = 0.5, c = 2.1$. We numerically calculated the throughputs by using the equations (9)-(12) (**"Numerical:General"**), the equation (26) (**"Numerical:Root"**), the Proposition 1 (**"Poly Root Method"**), and the equation (31) (**"Approximate Method"**). We used $x_k = 1/M = 1/5$ since it was optimal. Simulation results were shown in Fig. 2 for the overall throughput R^+ and in Fig. 3 for per SU per channel throughput R_k. It can be clearly seen that the analysis results fit well with the Monte-Carlo simulation results, which verified that our analysis was valid.

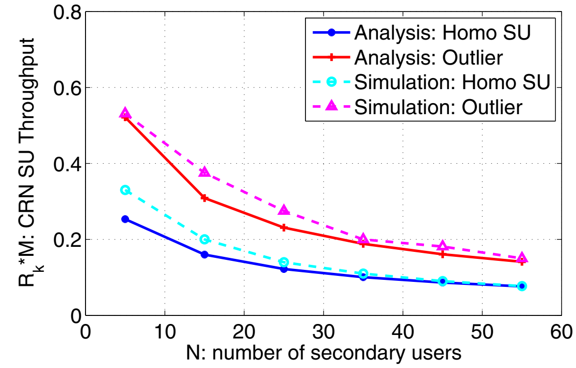

Fig. 4. Throughput of each secondary user achieved in all channels in the CRN with various number of homogeneous secondary users and one outlier.

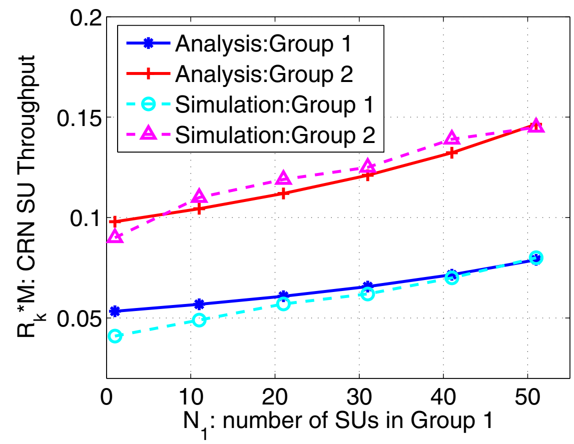

Fig. 5. Throughput of each secondary user achieved in all channels in the CRN with various number of homogeneous secondary users in two different groups.

Next, we simulated the CRN with $N-1$ homogeneous SUs plus one outlier. Parameters are similar to the first experiment, except that the outlier has $a_{2k} = 0.9$. Simulation results were shown in Fig. 4, which shows the average throughputs of each homogeneous SU and the outlier achieved in all the $M = 5$ channels. The analysis results fit well with the Monte-Carlo simulation results, and the outlier can gain much more throughput by increasing its offered load.

Finally, we simulated the CRN with N_1 homogeneous SUs in Group 1 and N_2 homogeneous SUs in Group 2. The total number of SUs were $N = 52$. Parameters are similar to the first experiment, except that the Group 2 SUs have $a_{2k} = 0.9$. Simulation results were shown in Fig. 5, which shows the average throughputs of each homogeneous SU in the two groups achieved in all the $M = 5$ channels. The analysis results fit well with the Monte-Carlo simulation results.

VI. Conclusion

We developed an efficient approach to evaluate the competition of heterogeneous CRN with a large number of different users. Based on the Markov Model Bank, we applied some network decorrelation techniques to reduce the complex problem into the problem of looking for the roots of some special polynomials. This polynomial rooting approach provides feasible ways to analyze the throughput and competition among the secondary users. We considered three different CRN cases with homogeneous CRN plus some outliers. We set up the polynomials, studied their properties, and developed approximate closed-form solutions that were convenient for various optimizations. Water-filling principles were obtained for optimal performance of the CRN and for the competition among the secondary users.

Appendix
A Class of Constrained Optimizations with Water-Filling Solutions

For the constrained optimization problem

$$\max J_1 = \sum_{k=1}^{M} \frac{a_k + b_k x_k}{c_k + d_k x_k}, \quad s.t., \quad \sum_{k=1}^{M} x_k = 1, \; x_k \geq 0 \quad (54)$$

we can first change J into the following form

$$J_1 = \sum_{k=1}^{M} \frac{b_k}{d_k} + \sum_{k=1}^{M} \frac{\frac{a_k}{c_k} - \frac{b_k}{d_k}}{1 + \frac{d_k}{c_k} x_k}. \quad (55)$$

Since the first summation in (55) is a constant, the optimization (55) is equivalent to the following standard optimization

$$\max J_2 = \sum_{k=1}^{M} \frac{\alpha_k}{1 + \beta_k x_k}, \quad s.t., \quad \sum_{k=1}^{M} x_k = 1, \; x_k \geq 0. \quad (56)$$

With the Lagrange multiplier method, define $J_3 = \sum_{k=1}^{M} (\frac{\alpha_k}{1+\beta_k x_k} + \lambda x_k) - \lambda$. From $\frac{\partial J_3}{\partial x_k} = 0$, we get $\lambda = \frac{\alpha_k \beta_k}{(1+\beta_k x_k)^2}$, from which we can get $x_k = \sqrt{\frac{\alpha_k}{\beta_k}} \frac{1}{\sqrt{\lambda}} - \frac{1}{\beta_k}$. Since x_k are probabilities, we have constraints

$$\frac{\alpha_k \beta_k}{(1 + \beta_k)^2} \leq \lambda^* \leq \alpha_k \beta_k. \quad (57)$$

From $\sum_{k=1}^{M} x_k = 1$, we have

$$\lambda^* = \left(\frac{\sum_{k=1}^{M} \sqrt{\frac{\alpha_k}{\beta_k}}}{1 + \sum_{k=1}^{M} \frac{1}{\beta_k}} \right)^2. \quad (58)$$

Note that λ^* should satisfy (57). In other words, we can look iteratively for λ^* by selecting the channels k with the strongest $\alpha_k \beta_k$ and by applying (58) until both (57) and (58) are satisfied. The optimal variables are then

$$x_k = \begin{cases} \sqrt{\frac{\alpha_k}{\beta_k}} \frac{1+\sum_{j=1}^{M} \frac{1}{\beta_j}}{\sum_{j=1}^{M} \sqrt{\frac{\alpha_j}{\beta_j}}} - \frac{1}{\beta_k}, & \text{if (57) is satisfied} \\ 0, & \text{otherwise} \end{cases} \quad (59)$$

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