Methods to Evaluate the Performance of Multiple Uncooperative Users for Green Cognitive Radio Networks

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***Abstract*—In this paper, we propose a method to calculate the throughput of multiple uncooperative cognitive radio users. We use a bank of Markov models to derive the fraction of time that each user spends in successful channel access. The mutual interference among the users is modeled into the transitional probabilities of the Markov models, which makes it tractable to calculate multiple user throughput in large cognitive radio networks (CRN). In addition, to evaluate the level of greenness of the multiple user CRN, we develop a method to optimize the ideal capacity of the fully cooperative CRN. This method relies on solving the sum-of-ratios linear fractional programming for optimization. Simulations are conducted to show the big gap between the uncooperative CRN performance and the ideal performance.**

I. Introduction

With the rapid increase of wireless communication services, wireless spectrum has become a scarce resource. The requirement of high data rate and long lifetime makes power efficiency of the communication system a critical factor. The efficient use of spectrum and energy resources is thus one of the major objectives for green wireless communications. Communication systems with high (or optimal) bandwidth/power efficiency contribute to a green wireless environment with less interference and less "CO_2" emission.

Green communications have attracted great interests during recent years [1]. One of the potential technologies to support green communications is the "cognitive radio network" (CRN) [2][3]. CRN can adapt its use of spectrum and power according to the environment. It can exploit the spectrum white spaces, i.e., spectrum not used by the primary users (PU), without creating detrimental interference to the environment. It can adapt its transmission power according the interference level of the environment. A green CRN is able to realize its communication objectives with less spectrum and power resource consumption.

As a unique feature, CRN conducts spectrum sensing and accesses the spectrum that is not occupied by the PU. It must vacate the spectrum if finding that the PU becomes active. There have been extensive research published in CRN, including areas such as spectrum sensing, transmission/modulation design, theoretical performance/capacity analysis, MAC/Network layer protocols, hardware/testbed development, security, etc. However, there have been very limited study on the greenness analysis, especially for CRN with a large number of uncooperative users. A typical research on CRN may involve developing some spectrum sensing algorithms and spectrum access strategies. But the performance may be evaluated by simulations only due to the complexity of the performance analysis of large distributed CRN [4]-[6].

In this paper, we develop a method to analyze the throughput performance of CRN with multiple uncooperative secondary users. We use an innovative tool, i.e., Markov Model Bank (MMB), to model the operation of multiple CRN users and their mutual interference. Each user has its own sub-Markov model, and the mutual interference among the secondary users is abstracted into the state transition probabilities. This makes it tractable to treat the complex coupling among all the users. Throughput of each user can thus be derived, under various spectrum sensing and spectrum access strategies. To simplify the presentation, we consider a generic spectrum sensing and spectrum access strategy only in this paper. Consideration of multiple strategies in heterogenous CRN will be reported elsewhere, such as [7].

In order to evaluate the level of greenness of the CRN, we need to study how far away the CRN performance (under certain spectrum sensing and spectrum access strategies) is from the ideal (or optimal) performance. More specifically, *the "greenness" in this paper is measured by what throughput the CRN can achieve given the spectrum and energy resource, compared with the ideal case*. For this purpose, we further develop a method to optimize the ideal capacity of the multiuser CRN, under the fully cooperative assumption. We will show that this can be realized by solving the sum-of-ratios linear fractional programming problems. Note that the optimal capacity of CRN has been a challenge, and has been studied for small networks involving a few users only. For example, [8] derived the optimal capacity under some special cases (such as heavy mutual interference).

The organization of this paper is as follows. In Section II, we give the system model. In Section III, we develop the method to derive the throughput of the multiuser CRN. Then in Section IV, we develop the method to calculate the ideal capacity. Simulations are conducted in Section V to compare the throughput of multiuser CRN to its ideal capacity. Conclusions are given in Section VI.

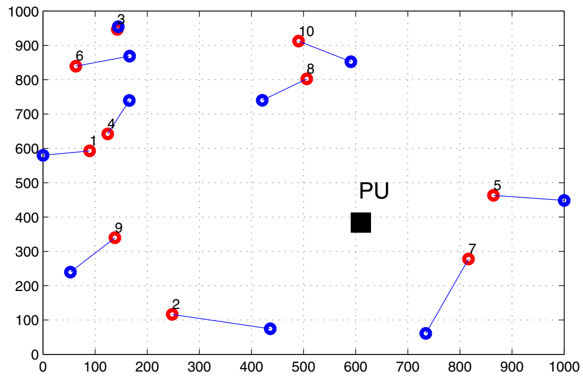

Fig. 1. An example of the CRN with $N = 10$ secondary users and one primary user (PU).

II. SYSTEM MODEL

The CRN we consider in this paper consists of a set of N secondary users and some primary users. Each secondary user is a transmitting-receiving pair of nodes, where the transmitting node transmits to the receiving node. The users are uniformly distributed. We assume that the users are either fixed or slowly moving, and conduct communications without the cooperation from other users. While transmitting to the receiving node, the transmitter creates interference to other users as well. An example of the network is shown in Fig. 1.

We assume there are K channels (spectrum resource) available for the N users to choose from. The channels may be occupied by some PU with probability $1 - \theta_k$, where $k = 1, \ldots, K$. Each secondary user conducts spectrum sensing to find all available channels [9], and then selects some available channels randomly to conduct transmission.

For each secondary user, we consider a simplified cognitive radio transmission model that includes three states: spectrum sensing, data transmission and channel switching, as shown in Fig. 2. The working sequence of a cognitive radio always begins with the spectrum sensing. If the spectrum sensing indicates the channel is available for secondary access, then the cognitive radio transmits a data packet, and the model shifts into the data transmission state. If the spectrum sensing indicates the channel is not available (due to either the PU activity or other secondary users activity), then the cognitive radio conducts channel switching, and the model shifts into the channel switching state. In order to simplify the analysis, we assume that the model always shifts back to the spectrum sensing state after the channel switching state or the data transmission state, whether the two operations are successful (without collisions) or not.

We can use a Markov Model Bank (MMB) to model the competitive spectrum access among the N CRN users. The MMB consists of a sub-Markov model for each user, as shown in Fig. 3, where π_{si},π_{di},π_{ci} are the probabilities of the user i staying in the channel sensing, data packet transmission and channel switching modes. The transitional probability q_{si} or q_{sj} denotes the probability that the channel is sensed as available. Although the two users' sub-Markov models look separated from each other, their transitional probabilities

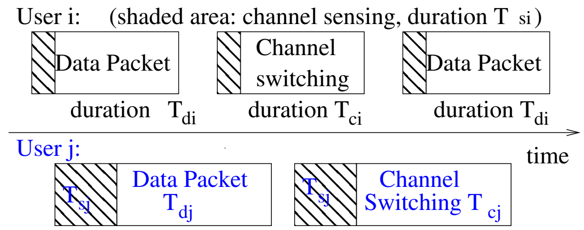

Fig. 2. Illustration of cognitive radio transmissions. Transmission structure with three basic operation modes: channel sensing, data packet transmission, and channel switching for each user i or j.

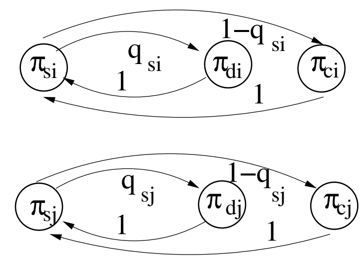

Fig. 3. Markov Model Bank (MMB) for multi-user CRN. MMB consists of a sub-Markov model for each user. Each sub-Markov model has three state probabilities (corresponding to the three basic operation modes), and has channel sensing result q_{si} as transitional probability.

and state probabilities are inter-related due to the mutual interference.

For each user i, let the durations of the spectrum sensing slot, data transmission slot, and channel switching slot be T_{si}, T_{di}, and T_{ci}, respectively. Usually the spectrum sensing duration T_{si} is much smaller than either T_{di} or T_{ci} for high throughput.

In the spectrum sensing slot, we assume that if the signal to noise ratio (SNR) is larger than the detection threshold Γ_s, then the cognitive radio will make a decision that the channel is occupied by primary users or other secondary users, and is thus not available [9]. As long as the interference emitted to the spectrum sensing slot makes the SNR larger than the sensing threshold Γ_s, the cognitive radios must vacate the channel and take the time-consuming channel switching procedure to negotiate a new one. We assume that the cognitive radios do not discriminate whether the interference comes from the primary users or other secondary users.

Let P_i^k be the transmission power of the user i spent in channel k, where $i = 1, \cdots, N$, and $k = 1, \cdots, K$. Assume the user i has maximum overall transmission power $\bar{P}_i$. Then we have

$$0 \leq \sum_{k=1}^{K} P_i^k \leq \bar{P}_i. \quad (1)$$

The signal received by the user i's receiver in the channel k is

$$y_i^k(n) = \sqrt{P_i^k} h_{ii} s_i(n) + \sum_{j=1, j \neq i}^{N} \sqrt{P_j^k} h_{ji} s_j(n) + v_i^k(n) \quad (2)$$

where the $s_i(n)$ is the discrete signal transmitted by the user i, h_{ji} is the complex Gaussian distributed flat fading channel from the user j's transmitter to the user i's receiver, and

$v_i^k(n)$ is the AWGN with zero-mean and power σ_i^{k2}. The instantaneous SNR of the user i in the channel k is thus

$$\gamma_i^k = \frac{P_i^k |h_{ii}|^2}{\sum_{j=1, j\neq i}^{N} P_j^k |h_{ji}|^2 + \sigma_i^{k2}} \tag{3}$$

As the performance metric, we consider the summation of the throughput R_i of all the N secondary users

$$R = \sum_{i=1}^{N} R_i. \tag{4}$$

Each R_i can be calculated from the probabilities of the three states in Fig. 3 and the corresponding slot lengths, as well as the transmission channel capacities based on the SNR, i.e., $\log(1+\gamma_i^k)$. Only the data transmission state is counted toward the throughput. Due to the ideal spectrum sensing assumption, we can omit the data packet collisions in this paper.

III. CRN Throughput analysis

Consider the CRN with a generic spectrum access strategy, i.e., picking randomly only one available channel to access at each time. Then we can skip the channel index k. Note that the spectrum sensing still covers all the K channels. Consider the user i's sub-Markov model in Fig. 3. There is only one unknown state transitional probability q_{si}. According to the steady state property of the Markov model, we can calculate the probabilities of the three states π_{si}, π_{di} and π_{ci} by solving the following equation

$$\begin{bmatrix} -1 & 1 & 1 \\ q_{si} & -1 & 0 \\ 1-q_{si} & 0 & -1 \end{bmatrix} \begin{bmatrix} \pi_{si} \\ \pi_{di} \\ \pi_{ci} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5}$$

This equation has infinitely many solutions. From (5), we can represent π_{di} and π_{ci} by π_{si}, i.e.,

$$\pi_{di} = q_{si}\pi_{si}, \quad \pi_{ci} = (1-q_{si})\pi_{si}. \tag{6}$$

To simplify expressions, without loss of generality, we assume that $T_{di} = T_{ci}$. This is reasonable considering the fact that most of the time the secondary user just needs to wait for one regular data transmission slot before doing channel sensing again. Then the fraction of time that the user i spends in data transmission session has a simple expression $\frac{q_{si}T_{di}}{T_{si}+T_{di}}$, which is also the probability of data packet transmission.

In order to find the state transmission probability q_{si}, we consider the spectrum sensing slot of the secondary user i. For the kth channel, the user i senses the SNR as γ_{si}. The transitional probability

$$q_{si} = P[\gamma_{si} < \Gamma_s]\theta \tag{7}$$

The SNR

$$\gamma_{si} = \frac{I_i}{\sigma_{si}^2} \tag{8}$$

where σ_{si}^2 is the noise power at the user i's sensing receiver (in channel k). The I_i denotes the overall interference received from all other secondary users [10]. Let the set S_i denote all the other secondary users that are using the kth channel simultaneously. Since each secondary user j's probability of being in the data transmission slot is $q_{sj}T_{dj}/(T_{sj}+T_{dj})$, we have

$$I_i = \sum_{j\in S_i} q_{sj}\frac{T_{dj}}{T_{sj}+T_{dj}} P_j |h_{ji}|^2. \tag{9}$$

With the state probabilities, we can define the average throughput of the user i as

$$R_i = \frac{\pi_{di}T_{di}}{\pi_{si}T_{si} + \pi_{di}T_{di} + \pi_{ci}T_{ci}} \log(1+\gamma_i), \tag{10}$$

where γ_i is the data transmission slot SNR, as per (3).

From the probability of the user i staying in the data transmission slot, the throughput (10) can be changed to

$$\begin{aligned} R_i &= q_{si}\frac{T_{di}}{T_{si}+T_{di}}\log(1+\gamma_i) \qquad (11) \\ &= P\left[\sum_{j\in S_i} \frac{q_{sj}T_{dj}}{T_{sj}+T_{dj}} \frac{P_j|h_{ji}|^2}{\sigma_{si}^2} < \Gamma_s\right] \frac{\theta T_{di}\log(1+\gamma_i)}{T_{si}+T_{di}} \end{aligned}$$

In the throughput expression (11), the random variables are the channel coefficients h_{ji}. In order to evaluate the R_i, we define new random variables

$$x_{ji} = q_{sj}\frac{T_{dj}}{T_{sj}+T_{dj}}\frac{P_j}{\sigma_{si}^2}|h_{ji}|^2 \tag{12}$$

Obviously, each variable x_{ji} has exponential distribution with mean

$$E[x_{ji}] = q_{sj}\frac{T_{dj}}{T_{sj}+T_{dj}}\frac{P_j}{\sigma_{si}^2} \tag{13}$$

The summation of the multiple exponential distributions gives a new random variable

$$x_i = \sum_{j\in S_i} x_{ji} \tag{14}$$

which has Hypo-exponential distribution.

According to the cumulative distribution function of the Hypo-exponential random variables [11], we have

$$P[x_i < \Gamma_s] = 1 - \mathbf{a}e^{x_i\mathbf{B}}\mathbf{1}, \tag{15}$$

where $\mathbf{1}$ is a column vector with all elements being 1, $\mathbf{a}$ and $\mathbf{B}$ are vector and matrix whose dimensions are the number of secondary users using the channel k. As a result, we can numerically evaluate the throughput R_i.

To evaluate (15) and (11) is actually to calculate all transitional probabilities q_{si}. From (7), Obviously q_{si} is a function of all other q_{sj}, $j = 1, \cdots, N$. Many non-linear numerical algorithms can be used to find some solutions to q_{si}. Note that it is not necessary to find all the solutions or the globally optimal solutions.

As an example, let us consider a simple case with two secondary users only, i.e., $N = 2$. In this case, the complex Hypo-exponential distribution will be reduced to the simple exponential distribution, which gives us more clear idea of the throughput calculation.

Consider the user i, where $i = 1$ or 2. We use $j = 3 - i$ to denote the other secondary user. The transitional probability is

$$\begin{aligned} q_{si} &= P[\gamma_{si} < \Gamma_s]\theta \\ &= P[q_{sj}\frac{T_{dj}}{T_{sj}+T_{dj}}\frac{P_j}{\sigma_{si}^2}|h_{ji}|^2 < \Gamma_s]\theta \\ &= P[|h_{ji}|^2 < \frac{\Gamma_s\sigma_{si}^2}{q_{sj}P_j}\frac{T_{sj}+T_{dj}}{T_{dj}}]\theta \\ &= [1 - e^{-\frac{\Gamma_s\sigma_{si}^2}{q_{sj}P_j}\frac{T_{sj}+T_{dj}}{T_{dj}}}]\theta \end{aligned} \tag{16}$$

Note that we have used the fact that $|h_{ji}|^2$ is an exponential random variable. For the two secondary users, we have the following equation array to solve their transitional probabilities

$$\begin{cases} q_{si} = (1 - e^{-a_j/q_{sj}})\theta \\ q_{sj} = (1 - e^{-a_i/q_{si}})\theta \end{cases} \tag{17}$$

where $a_i = \frac{\Gamma_s\sigma_{si}^2}{P_j}\frac{T_{sj}+T_{dj}}{T_{dj}}$ and similarly for a_j. Therefore, the values of the two state transitional probabilities can be evaluated numerically. As a matter of fact, our simulation shows that we can simply solve them iteratively based on the form of (17).

In summary, the MMB model we developed in this paper simplifies the mutual interference among all the N users into the set of transitional probabilities. By increasing the complexity of the transitional probabilities, the complexity of the Markov model is much reduced, which makes it feasible to deal with large CRN with many users.

IV. IDEAL FULLY-COOPERATIVE CAPACITY ANALYSIS

The derivation in the Section III is based on the assumption that the secondary users do not cooperative with each other for channel scheduling, which may degrade the throughput severely. In order to find how far away it is from the best possible results, in this section we derive the optimized throughput of the secondary users in the ideal fully-cooperative CRN.

From the system model in Section II, recall the definition of P_i^k as the transmission power of the user i in the channel k, where $i = 1, \cdots, N$, and $k = 1, \cdots, K$. Recall also that θ_k is the probability that the channel k is not occupied by the PU. Considering the PU activity, not all channels are available simultaneously.

Since each user may use multiple available channels simultaneously, we need to consider a set of any L_m channels

$$C_m = \{k_1, \cdots, k_{L_m}\}. \tag{18}$$

The probability that only channels in C_m are available is

$$P[C_m] = \prod_{\ell=1}^{L_m}\theta_{k_\ell}\prod_{k=1, k\notin C_m}^{K}(1-\theta_k). \tag{19}$$

We have altogether 2^K such sets, i.e., $m = 1, \cdots, 2^K$. Therefore, for optimal throughput evaluation, we need to consider all possible sets, and find the average capacity.

For the ideal (optimized) throughput in each set C_m, we can choose appropriate transmission power P_i^k to maximize the sum of the SNRs of all the N users, i.e.,

$$\begin{aligned} f(C_m) &= \max_{\{P_i^k\}} \sum_{i=1}^{N}\sum_{\ell=1}^{L_m}\gamma_i^{k_\ell} \\ &\text{s.t.} \quad \sum_{\ell=1}^{L_m} P_i^{k_\ell} \le \bar{P}_i, \ \ P_i^{k_\ell} \ge 0, \ \ 1 \le i \le N \end{aligned} \tag{20}$$

Note that $P_i^k = 0$ for all $k \notin C_m$.

Define the transmission power vector

$$\mathbf{z}_m = \left[\frac{P_1^{k_1}}{\bar{P}_1}, \frac{P_1^{k_2}}{\bar{P}_1}, \cdots, \frac{P_N^{k_{L_m}}}{\bar{P}_N}\right]^T. \tag{21}$$

We can rewrite (20) into

$$\begin{aligned} f(C_m) = \ & \max_{\mathbf{z}_m} \ \sum_{i=1}^{N}\sum_{\ell=1}^{L_m}\frac{\mathbf{a}_{i\ell}^T\mathbf{z}_m}{\mathbf{b}_{i\ell}^T\mathbf{z}_m + 1} \\ & \text{s.t.} \quad \mathbf{A}\mathbf{z}_m \le \mathbf{1}, \ \ \mathbf{z}_m \ge \mathbf{0}. \end{aligned} \tag{22}$$

The vectors

$$\mathbf{a}_{i\ell} = [\cdots\cdots, |h_{ii}|^2, \cdots\cdots]^T\frac{\bar{P}_i}{\sigma_{si}^2}, \tag{23}$$

has the channel coefficients $|h_{ii}|^2$ in a position corresponding to the transmission power $P_i^{k_\ell}/\bar{P}_i$ in $\mathbf{z}_m$. The vector

$$\mathbf{b}_{i\ell} = [\cdots\cdots, \frac{|h_{ji}|^2\bar{P}_j}{\sigma_{si}^2}, \cdots\cdots] \tag{24}$$

consists of channel gains of the interference items. The $N \times NL_m$ matrix $\mathbf{A}$ has the structure

$$\mathbf{A} = \begin{bmatrix} \mathbf{1}^T & & \\ & \ddots & \\ & & \mathbf{1}^T \end{bmatrix}. \tag{25}$$

where $\mathbf{1}$ is the L_m dimensional vector with all entries as 1.

After obtaining the optimal transmission powers and the optimal SNRs by solving (22), we can then calculate the ideal capacity of the CRN. Specifically, for each available channel set C_m, the user i gains capacity

$$R_i(C_m) = \sum_{\ell=1}^{L_m} R_i^{k_\ell}, \tag{26}$$

where

$$R_i^{k_\ell} = \log(1 + \gamma_i^{k_\ell}). \tag{27}$$

The overall capacity of the user i is thus

$$R_i = \sum_{m=1}^{2^K} R_i(C_m)P[C_m]. \tag{28}$$

By maximizing (22), we can maximize the capacity for all the secondary users. Note that (22) is in the form of sum-of-ratios linear fractional programming (LFP). Sum-of-ratios LFP has been found in many applications, which stimulated decades of research. Although the problem in the general setting (with arbitrarily large number of ratios, number of variables, as well as number of constraints) may still be challenging and deserve more investigation, there are many sophisticated algorithms that can be used to solve the special case in our setting, such as those in [12]-[16].

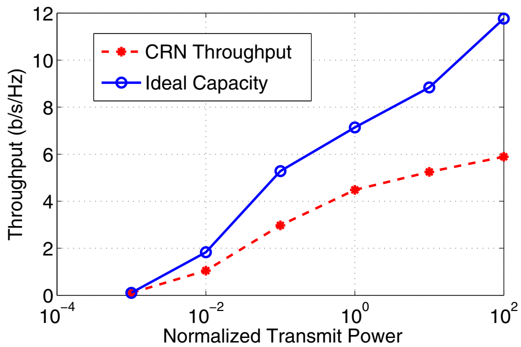

Fig. 4. CRN throughput and the ideal capacity under various (normalized) transmission powers.

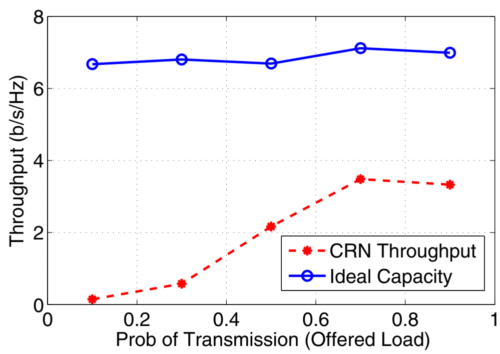

Fig. 5. CRN throughput and the ideal capacity as functions of the offered load (i.e., the probability that each user has data packet to transmit).

V. Simulations

In this section, we report our simulation of the numerical evaluation of the uncooperative CRN throughput as well as the ideal CRN capacity. We simulated a random CRN of $N = 2$ secondary users competing for accessing 1 channel. The nodes' positions were randomly generated within a square of 1000×1000 meters. The edge SNRs were calculated as $10^8 d_{ij}^{-2.6}$ where d_{ij} is the propagation distance.

First, we use the expressions in Section III to calculate the throughput of the uncooperative CRN users and use the expressions in Section IV to optimize the ideal capacity of the CRN users, under various (normalized) secondary user transmission powers $\bar{P}_i$. Simulation results were shown in Fig. 4. It can be clearly seen that the uncooperative CRN can achieve just $1/2$ of the optimized ideal capacity. Therefore, there are still much room to improve the greenness of the CRN.

Then, we calculate the CRN throughput and the ideal capacity under various offered load, where "offered load" refers to the probability that each secondary user has data packet to transmit at any time. Simulation results in Fig. 5 shows that the CRN throughput improves with offered load, until reaching a saturation point (which is because we have assumed almost perfect collision resolution). For green CRN, under light CRN user traffic load, much more users can be allowed to share each channel.

VI. Conclusion

In this paper, we study the greenness of the CRN with a special setting of N uncooperative users. We first develop a method to calculate the throughput of each CRN user. We use a Markov Model Bank (MMB) to model the N users' spectrum sensing, channel access, and channel switching procedure, as well as the mutual interference. Then we develop a method to optimize the ideal capacity under the ideal fully-cooperative assumption. We use the sum-of-ratios linear fractional programming for the optimization. Simulations are then conducted to show that the throughput of the uncooperative CRN is very far away from the ideal capacity.

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