Writing on Wet Paper

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ABSTRACT

In this paper, we show that the communication channel known as writing in memory with defective cells [1][2] is a relevant information-theoretical model for a specific case of passive warden steganography when the sender embeds a secret message into a subset *C* of the cover object *X* without sharing the selection channel *C* with the recipient. The set *C* could be arbitrary, determined by the sender from the cover image using a deterministic, pseudo-random, or a truly random process. We call this steganography "writing on wet paper" and realize it using a simple variable-rate random linear code that gives the sender a convenient flexibility and control over the embedding process and is thus suitable for practical implementation. The importance of the wet paper scenario for covert communication is discussed within the context of adaptive steganography and perturbed quantization steganography [3]. Heuristic arguments supported by tests using blind steganalysis [4] indicate that the wet paper steganography provides improved steganographic security and is less vulnerable to attacks compared to existing methods with shared selection channels.

1. MOTIVATION

The importance of the informed coder channel of Gelfand-Pinsker [5] for steganography and watermarking has been widely recognized by many researchers. A special case of the informed sender channel that is highly relevant for robust watermarking and active warden steganography is Costa's writing on dirty paper [6] and its extensions [7][8]. In this paper, we point out the importance of another special case of the informed sender channel that is known as memory with defective cells [1][2]. When used in passive warden steganography, we coin a new term for this channel and call it "writing on wet paper", which intentionally evokes analogy with Costa's work. Indeed, similar to the result of Costa that the channel capacity is independent of the Gaussian noise known to the encoder but not to the decoder, in the wet paper scenario, quite similarly the sender can send the same number of bits to the recipient who is unaware of the selection channel (as described in the abstract).

To explain the metaphor "writing on wet paper", imagine that *X* is an image exposed to rain and the sender can only slightly modify the dry spots of the cover object *X* (the set *C*) but not the wet spots. During transmission, the stego image *Y* dries out and thus the recipient does not know which pixels were used by the sender. We note that the rain can be random, pseudo-random, completely determined by the sender, or an arbitrary mixture of all. This communication setup gives the sender complete freedom in choosing the dry pixels that will be used for embedding because the recipient does not need to determine the dry pixels from the stego image in order to read the message. In particular, the sender may formulate selection rules based on side information that is *in principle unavailable* to the recipient and thus to any attacker. This is likely to provide better security [9]–[13] than steganographic schemes with public selection rules [14]–[17].

The "wet paper" channel is highly relevant to steganography and arises in several different situations. One of them is adaptive steganography, where the sender selects the location of pixels that will carry message bits based on pixels' neighborhood in the cover image. A fundamental problem with adaptive schemes is that the requirement that the recipient be able to recover the same message-carrying samples from the stego object undermines the security of the algorithm because it

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gives an attacker a starting point for mounting an attack [17]. Another potential problem here is that the recipient may not be able to recover the same set of message carrying pixels from the stego image, which is modified by the embedding act itself. This problem is usually solved either by increasing the message redundancy using error correction to recover from random bit losses and inserts or by employing some artificial measures, such as special embedding operations matched to the selection rules [14][15]. These measures, however, usually limit the embedding capacity [38], complicate the embedding algorithm, and do not give the sender the ability to fully utilize his side-information – the cover image. Moreover, the pixel selection rule is often ad hoc and it is not always possible to justify it from the point of view of steganographic security. In fact, ideally, the sender should be able to use his side information in an unrestricted manner and perform embedding while focusing on important steganographic design principles and necessary security considerations rather than the receiver's ability to read the message.

A different case of the wet paper scenario occurs when the cover image is processed before embedding using an information-reducing operation, such as A/D conversion during image acquisition, lossy compression, dithering, recoloring, downsizing, etc. Most today's steganographic algorithms[1] disregard this side information (the raw cover image before processing) and work with the processed image only. *However, there is no reason why the steganographer could not utilize, for example, the knowledge of the raw, uncompressed cover image when embedding into its JPEG compressed form.* In fact, the sender will have access to the unquantized values of the processed image (e.g., non-rounded DCT coefficients during compression or non-rounded pixel values after downsampling) and may utilize this information for coefficient/pixel selection (Section 4). This side information is largely ignored by current steganographic algorithms because of the seemingly insurmountable obstacle that the receiver would not be able to read the message due to the fact that the coefficient/pixel selection is based on information that is practically completely removed during quantization. On the contrary, we view the fact that one can create an embedding scheme where the information about dry pixels is unavailable to the recipient (and any attacker) as a good property that could substantially improve the steganographic security and remove the above mentioned problem of adaptive steganography.

In Section 2, we describe a variable-rate random linear code for writing on wet paper and show that its average capacity reaches the channel capacity (Appendix A). In Section 3, we give a detailed description of the encoder and decoder while paying close attention to implementation issues. In the same section, we also discuss further improvements, such as minimizing the impact of embedding changes or using random linear codes with sparse matrices. In Section 4, we describe several practical steganographic schemes that use wet paper codes, most notably the Perturbed Quantization [3], and briefly evaluate their steganographic security using heuristic arguments and blind steganalysis. The paper is concluded in Section 5 where we outline future research directions and list several other applications of wet paper codes.

2. WET PAPER CODES FOR STEGANOGRAPHY

2.1 Basic concepts

Let us assume that the sender has a cover object X consisting of n samples $\{x_i\}_{i=1}^n$, $x_i \in J$, where J is the range of discrete values for x_i. For example, for an 8-bit grayscale image represented in the spatial domain, $J = \{0, 1, \ldots, 255\}$ and n is the number of pixels in the cover image X. The sender uses a Selection Rule (SR) to select k changeable samples $x_j, j \in C \subset \{0, 1, \ldots, n-1\}$. The changeable samples may be used and modified by the sender to communicate a secret message to the recipient. The remaining samples are not modified during embedding.

We further assume that the sender and the recipient agree on a public parity function P, which is a mapping $P: J \to \{0,1\}$. Although we do not consider it in this paper, this mapping could in principle depend on the sample position i and a secret stego key K shared by the sender and the recipient.

During embedding, the sender either leaves the changeable samples x_j, $j \in C$, unmodified or replaces x_j with y_j such that $P(x_j) = 1 - P(y_j)$. The stego object Y will consist, again, of n samples $\{y_i\}_{i=1}^n$. The recipient will decode message bits from the bit-stream of parities of samples from the stego object $\{P(y_i)\}_{i=1}^n$.

[1] A few notable exceptions include the fragile authentication by Marvel et al. [18] and the embedding-while dithering method [15].

Obviously, if the recipient could determine the same set of changeable samples from the stego object, the sender would be able to communicate up to $k = |C|$ bits, one parity bit per each changeable sample. However, as discussed in the introduction, there are two problems with this scenario. First, the requirement that the recipient be able to determine the same set of changeable samples imposes a limitation on the SR and the embedding modification. Second, the fact that the message-carrying samples can be determined from the stego object may help an attacker to mount an attack. Thus, we propose a different approach that solves both problems at once – the recipient can read the correct message but does not need to know the set of changeable samples (or even the SR or the embedding operation $x_j \rightarrow y_j$) because the message bits are not communicated directly as sample parities. All the recipient needs to share with the sender is a secret key and the public parity function P.

2.2 Writing on wet paper as memory with defective cells

Let $b_i = P(x_i)$ be the sequence of parities of all n samples from the cover object X. All the bits b_i are known to the sender. The sender can modify all k bits b_j, $j \in C$, but cannot modify the remaining $n - k$ bits. The recipient does not know the set C. This is an example of a channel known as an n-bit memory containing $n - k$ defective cells, which are stuck either at 0 or 1, introduced in 1974 by Tsybakov and Kusnetsov [1]. A simple random binning argument (see, e.g., [19]) can show that asymptotically the capacity of this channel is k. It is also known that the capacity of this channel is k [2][20][21]. For non-binary alphabet, this capacity can be achieved, for example, using an algebraic coding scheme that uses the cosets of an erasure correction code as bins [19]. A noisy generalization of this channel is given in [19], where it is shown that the early work on partitioned codes [22][23] are cases of nested linear codes capable of achieving the theoretical maximum capacity.

In steganographic applications, both the number of defective cells (wet samples) and the dry samples may be quite large. For example, in the double compression embedding (Perturbed Quantization) described in Section 4.2, for a typical JPEG image, $n \sim 10^6$ and $k \sim 10^4$. Due to the large variability of k from image to image, one cannot assume a reasonable upper bound on the number of defective cells without sacrificing the capacity. Thus, considering these specifics of our application, in the next section we describe a simple variable-rate random linear code that also enables the sender to communicate on average k bits. The simplicity of this code enables efficient implementation of the wet paper scenario for steganography. Another advantage of this code is its flexibility and control it gives to the sender to choose which samples should be modified, which further improves steganographic security and minimizes the impact of embedding changes (Section 3.1).

2.3 Wet paper codes

2.3.1 Encoder

In Sections 2 and 3, we use capital non-bold letters to denote the cover and stego object and their subsets, bold small letters for vectors, and bold capital letters for matrices. The proposed code can be viewed as a generalization of the selection channel proposed by Anderson and Petitcolas [10] where one message bit is embedded as the parity of a group of individual samples. In the selection channel, at most one sample value must be changed in order to match the parity of a group of samples to the message bit. The parity of the group is a sum modulo 2 of the individual sample parities. Now, if there are q changeable samples in the group, one can attempt to embed q message bits by forming q linearly independent linear combinations of sample parities instead of just one sum. This suggests the following approach to the wet paper code.

We repeat that the sender has a binary column vector $\boldsymbol{b} = \{b_i\}_{i=1}^{n}$ and a set of indices $C \subset \{0, 1, \ldots, n-1\}$, $|C| = k$, of those bits that can be modified to embed a message. The sender wants to communicate q bits $\boldsymbol{m} = \{m_1, \ldots, m_q\}^{\mathrm{T}}$. For a moment, let us assume that the recipient knows q. In the next section, we show how to relax this assumption. The sender and recipient use a shared stego key to generate a pseudo-random random binary matrix $\boldsymbol{D}$ of dimensions $q \times n$. The sender will modify b_j, $j \in C$, so that the modified binary column vector $\boldsymbol{b}' = \{b'_i\}_{i=1}^{n}$ satisfies

$$\boldsymbol{D}\boldsymbol{b}' = \boldsymbol{m} . \tag{1}$$

Thus, the sender needs to solve a system of linear equations in GF(2). The question of solvability of (1) is discussed in detail in Section 2.3.3. Note that the selection channel [10] is a special case of (1) when $\boldsymbol{D} = [1, \ldots, 1]$.

2.3.2 Decoder

The modified stego object $Y' = \{y'_i\}_{i=1}^n$ is sent to the recipient. The decoding is very simple because the recipient first forms the vector $b'_i = \text{Parity}(y_i')$ and then obtains the message $\boldsymbol{m} = \boldsymbol{D}\boldsymbol{b}'$ using the shared matrix $\boldsymbol{D}$. The biggest computational load is on the sender's side who needs to solve (1). Note that the recipient does not need to know the set of changeable elements C to read the message.

We now explain how to relax the assumption that the recipient knows q. The sender and recipient can generate the matrix $\boldsymbol{D}$ in a row-by-row manner rather than generating it as a two-dimensional array of $q \times n$ bits. In this way, the sender can reserve the first $\lceil \log_2 n \rceil$ bits of the message $\boldsymbol{m}$ for a header to inform the recipient of the number of rows in $\boldsymbol{D}$ – the message length q. The symbol $\lceil x \rceil$ is the smallest integer larger than or equal to x. The recipient first generates the first $\lceil \log_2 n \rceil$ rows of $\boldsymbol{D}$, multiplies them by the received vector $\boldsymbol{b}'$, and reads the header (the message length q). Then, he generates the rest of $\boldsymbol{D}$ and reads the message $\boldsymbol{m} = \boldsymbol{D}\boldsymbol{b}'$.

The decoding mechanism is similar to that of matrix embedding [24][25], where the recipient also extracts the message bits by multiplying the parity vector by an appropriate code matrix. The difference is that in matrix embedding the sender's goal is to maximize the embedding rate utilizing the *positions of the changes* to convey information. While in matrix embedding any element can be modified, in writing on wet paper the set of elements that can be modified is pre-determined by the sender (or the cover object, or some randomness) beforehand and is different for different objects.

2.3.3 Average capacity

We now investigate the issue of solvability of (1) and determine the average number of bits that the sender can communicate. Obviously, for small q, (1) will have a solution with a very high probability and this probability decreases with increasing q. We rewrite (1) to

$$\boldsymbol{D}\boldsymbol{v} = \boldsymbol{m} - \boldsymbol{D}\boldsymbol{b} \tag{2}$$

using the variable $\boldsymbol{v} = \boldsymbol{b}' - \boldsymbol{b}$ with non-zero elements corresponding to the bits the encoder must change to satisfy (1). In the system (2), there are k unknowns v_j, $j \in C$, while the remaining $n - k$ values v_i, $i \notin C$, are zeros. Thus, on the left hand side, we can remove from $\boldsymbol{D}$ all $n - k$ columns i, $i \notin C$, and also remove from $\boldsymbol{v}$ all $n - k$ elements v_i with $i \notin C$. Keeping the same symbol for $\boldsymbol{v}$, (2) now becomes

$$\boldsymbol{H}\boldsymbol{v} = \boldsymbol{m} - \boldsymbol{D}\boldsymbol{b}, \tag{3}$$

where $\boldsymbol{H}$ is a binary $q \times k$ matrix consisting of those columns of $\boldsymbol{D}$ corresponding to indices C, and $\boldsymbol{v}$ is an unknown $k \times 1$ binary vector. This system has a solution for an arbitrary message $\boldsymbol{m}$ as long as $rank(\boldsymbol{H}) = q$. The probability $P_{q,k}(s)$ that the rank of a random $q \times k$ binary matrix is s, $s \le \min(q, k)$, is [26]

$$P_{q,k}(s) = 2^{s(q+k-s)-qk} \prod_{i=0}^{s-1} \frac{(1-2^{i-q})(1-2^{i-k})}{(1-2^{i-s})}. \tag{4}$$

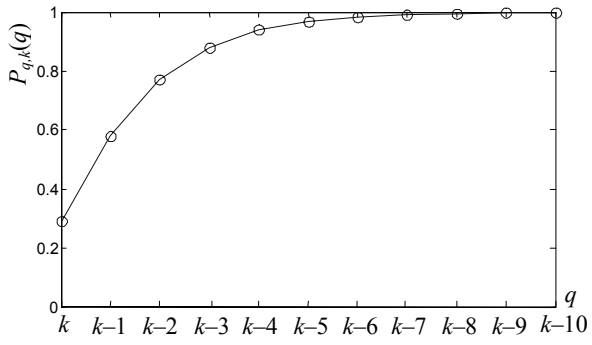

Figure 1. Probability that a random $q \times k$ binary matrix has rank q (for k=100).

From Lemma 1 and (A2) in Appendix A, it can be shown that for a large fixed k, $P_{q,k}(q) = 1 - O(2^{q-k})$, which very quickly approaches 1 with decreasing $q < k$ (see Figure 1). This suggests that the sender can on average communicate close to k bits to the recipient. We now prove that the expected maximal number of bits that can be communicated is exactly equal to k.

Trying to embed the longest possible message, the sender keeps on adding rows to $\boldsymbol{D}$ while (3) still has a solution. The probability that the sender can communicate at least $k - r$ ($r \geq 0$) bits will be denoted $p_{\geq k-r}$. This will happen when the first $k-r$ rows in $\boldsymbol{H}$ form a submatrix whose rank is $k-r$ or its rank is $k-r-i$ and each of the i linearly dependent rows is compatible with the corresponding bit on the right hand side, which will happen with probability 2^{-i}. Thus,

$$p_{\geq k-r} = \sum_{i=0}^{k-r} \frac{1}{2^i} P_{k-r,k}(k-r-i)\,, \tag{5}$$

while the probability that one can communicate at least $k + r$ ($r \geq 0$) bits is, using a similar argument,

$$p_{\geq k+r} = \sum_{i=0}^{k} \frac{1}{2^{r+i}} P_{k+r,k}(k-i)\,. \tag{6}$$

From (5–6), we calculate the expected maximum number $q_{\max}$ of bits that can be communicated using k changeable bits as

$$q_{\max}(k) = \sum_{i=1}^{\infty} i p_{=i} = \sum_{i=1}^{\infty} i(p_{\geq i} - p_{\geq i+1})\,, \tag{7}$$

where $p_{=i} = p_{\geq i} - p_{\geq i+1}$ is the probability that one can communicate exactly i bits. In Figure 2, we show the probability distribution $p_{=i}$, which appears to be symmetrical about $i=k$ and quickly falls to zero to the sides. This indicates that $q_{\max}(k) \approx k$, which is indeed the case. A precise formulation of this statement and its derivation is given in Appendix A. This result means that on average, using the wet paper code described above, the sender will be able to communicate k bits to the recipient.

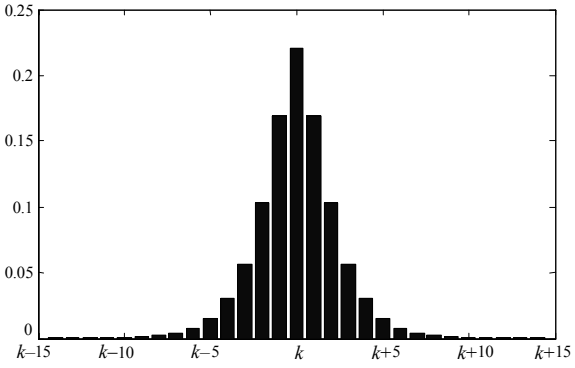

Figure 2. PDF $p_{=i}$ for the number of communicated bits.

3. PRACTICAL ENCODER/DECODER IMPLEMENTATION

The main complexity of this communication setup is on the side of the sender, who needs to solve q linear equations for k unknows in GF(2) (in binary arithmetic). Assuming that the maximal length message $q = k$ is sent, the complexity of Gaussian elimination for (3) is $O(k^3)$, which would lead to impractical performance for large payloads, such as $k > 10^5$. At this point, we would like to stress that the computational requirements are less of an obstacle in our application, which is steganography, when the coding and embedding is usually performed off-line, as opposed to coding for a communication channel where real time performance is essential. We set our goal to finding an implementation of the encoder/decoder with embedding of the order of a few seconds for typical steganographic scenarios, cover images, and payloads.

By far the best performance and most flexible method[2] for solving (3) was obtained using structured Gaussian elimination by dividing the bit-stream $\{b_i\}_{i=1}^{n}$ into β disjoint pseudo-random subsets B_i and using the Gaussian elimination on each subset separately. This can bring down the computational requirements substantially because the complexity of Gaussian elimination will decrease by the factor of β^3 while the number of solvings increases β-times. This gives performance improvement of β^2. The division into subsets, however, requires communication of the message length in each subset, which leads to a slight decrease in channel capacity (a few percent). Overall, the small decrease in capacity is well worth the significant improvement in speed. The performance of the structured Gaussian elimination is evaluated and compared to other solvers in Section 3.4.

Let us assume that the communicating parties know the range of typical values of the rate $r = k/n$, $r_1 \le r \le r_2$. If the range is unknown or r_2/r_1 is too large, the sender can modify the pseudo-code below and communicate r to the recipient (see Section 3.3). The specific value of r will be influenced by the cover object content, the SR, and other specifics of the embedding algorithm. To keep the encoding time short, we desire approximately $k_{\text{avg}} \sim 250$ changeable bits in each subset. We also require all subsets to be of approximately the same size. Thus, we choose the number of sets $\beta = \lceil nr_2/k_{\text{avg}} \rceil$. The size n_i of each subset B_i will be $n_i \in \{\lfloor n/\beta \rfloor, \lceil n/\beta \rceil\}$ chosen so that $n_1+n_2+\ldots+n_\beta = n$. Both the encoder and decoder must follow the same pseudo-random process for dividing $\boldsymbol{b}$ into subsets. This process may use the stego key as a parameter.

The number of changeable bits k_i varies for each subset B_i and follows the hypergeometric distribution (see Section 3.2.1) with mean value k/β. The number of message bits embedded in each subset is denoted q_j and will be allocated *dynamically* during embedding by the sender (see the pseudo-code below). Without changing the notation, we will assume that the bits $\boldsymbol{b}$ are permuted using a pseudo-random permutation generated from a shared secret stego key. Then, the subsets B_i can simply be taken as segments of n_i consecutive bits and $\boldsymbol{b} = (\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}, \ldots, \boldsymbol{b}^{(\beta)})$, where $\boldsymbol{b}^{(i)}$ is a vector of n_i bits from B_i. We are now ready to describe the details of the encoding algorithm (follow Figure 3).

Encoder

E0. Using a PRNG, generate a random binary matrix $\boldsymbol{D}$ with $\lceil n/\beta \rceil$ columns and sufficiently many rows
E1. Determine the header size $h = \lceil \log_2(nr_2/\beta) \rceil + 1$, $q = |\boldsymbol{m}| + \beta h$
E2. $\boldsymbol{b}' \leftarrow \boldsymbol{b}$, $i \leftarrow 1$
E3. $q_i = \lceil k_i(q+10)/k \rceil$, $q_i = \min\{q_i, 2^h-1, |\boldsymbol{m}|\}$, $\boldsymbol{m}^{(i)} \leftarrow$ the next q_i bits in $\boldsymbol{m}$
E4. Select the first n_i columns and q_i rows from $\boldsymbol{D}$ and denote this submatrix $\boldsymbol{D}^{(i)}$. Solve q_i equations $\boldsymbol{H}^{(i)}\boldsymbol{v} = \boldsymbol{m}^{(i)} - \boldsymbol{D}^{(i)}\boldsymbol{b}^{(i)}$ for k_i unknowns $\boldsymbol{v}$, where $\boldsymbol{H}^{(i)}$ is a $q_i \times k_i$ submatrix of $\boldsymbol{D}^{(i)}$ consisting of those columns of $\boldsymbol{D}^{(i)}$ that correspond to changeable bits in B_i. If this system does not have a solution, the encoder decreases q_i till a solution is found

E5. According to the solution $\boldsymbol{v}$, obtain the i-th segment $\boldsymbol{b}'^{(i)}$ of the vector $\boldsymbol{b}'$ by modifying or leaving $\boldsymbol{b}^{(i)}$ unchanged
E6. Binary encode q_i using h bits and append them to $\boldsymbol{m}$
E7. Remove the first q_i bits from $\boldsymbol{m}$
E8. $q \leftarrow q - q_i$, $k \leftarrow k - k_i$, $i \leftarrow i + 1$
E9. IF $i < \beta$ GOTO 3
E10. IF $i = \beta$, $q_\beta \leftarrow q$
E11. Binary encode q_β using h bits and *prepend* to $\boldsymbol{m}$, $\boldsymbol{m}^{(q)} \leftarrow \boldsymbol{m}$
E12. Select the first n_β columns and q_β rows from $\boldsymbol{D}$ and denote this submatrix $\boldsymbol{D}^{(\beta)}$. Solve q_β equations $\boldsymbol{H}^{(\beta)}\boldsymbol{v} = \boldsymbol{m}^{(\beta)} - \boldsymbol{D}^{(\beta)}\boldsymbol{b}^{(\beta)}$ for k_β unknowns $\boldsymbol{v}$. If this system does not have a solution, exit and report failure to embed the message.
E13. According to the solution $\boldsymbol{v}$, obtain the β-th segment $\boldsymbol{b}'^{(\beta)}$ of the vector $\boldsymbol{b}'$ by modifying or leaving $\boldsymbol{b}^{(\beta)}$ unchanged

Decoder

D0. Using a PRNG, generate a random binary matrix $\boldsymbol{D}$ with $\lceil n/\beta \rceil$ columns and sufficiently many rows
D1. Determine the header length $h = \lceil \log_2(nr_2/\beta) \rceil + 1$
D2. $i \leftarrow \beta$
D3. Select the first with n_β columns and q_β rows from $\boldsymbol{D}$ and denote this submatrix $\boldsymbol{D}^{(\beta)}$.

[2] Alternative approaches to solving (3) are discussed in Section 3.4.

$\boldsymbol{D} \leftarrow$ the first h rows of $\boldsymbol{D}^{(\beta)}$, obtain h bits as $\boldsymbol{D}\boldsymbol{b}'^{(\beta)}$

D4. $\boldsymbol{D} \leftarrow$ the next $q_\beta - h$ rows of $\boldsymbol{D}^{(\beta)}$, $\boldsymbol{m} = \boldsymbol{D}\boldsymbol{b}'^{(\beta)}$

D5. $i \leftarrow i - 1$

D6. Decode q_i from the last h bits of $\boldsymbol{m}$ and remove the last h bits from $\boldsymbol{m}$

D7. Select the first n_i columns and q_i rows from $\boldsymbol{D}$ and denote this submatrix $\boldsymbol{D}^{(i)}$.
$\boldsymbol{D} \leftarrow$ the first q_i rows of $\boldsymbol{D}^{(i)}$, prepend $\boldsymbol{D}\boldsymbol{b}'^{(i)}$ to $\boldsymbol{m}$, $\boldsymbol{m} \leftarrow \boldsymbol{D}\boldsymbol{b}'^{(i)} \& \boldsymbol{m}$

D8. IF $i > 1$ GOTO 5

D9. ELSE $\boldsymbol{m}$ is the extracted message

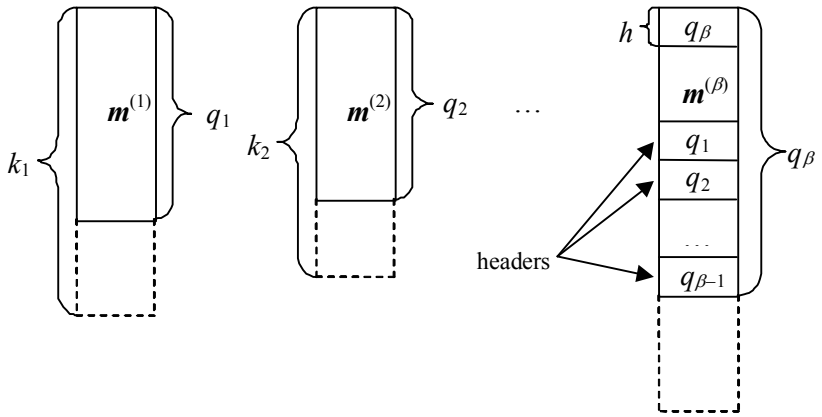

Figure 3. Placement of message bits and headers within blocks.

We now explain Steps E4 and E12 in more detail. In both steps, the sender forms an upper diagonal matrix from $\boldsymbol{H}^{(i)}$ using Gaussian elimination, exchanging columns as needed to obtain 1's everywhere on the main diagonal. Once q_i rows are successfully processed, the sender sets the remaining values $v_i = 0$ for $i = q_i+1, \ldots, k_i$ and calculates the unknowns v_i, $i = 1, \ldots, q_i$. This will ensure that the embedding rate will on average be 2 bits per change. Also, the encoder is allowed to decrease q_i whenever it cannot form an upper diagonal matrix (with ones on the diagonal) from $\boldsymbol{H}^{(i)}$ using Gaussian elimination and by exchanging columns. Note that the encoding process may fail in the last block because this is the only block in which the sender doesn't have the freedom to decrease q_β. To minimize the probability of this happening when q is close to k, we force the encoder to embed slightly more bits in all other blocks than in the last one. This is the reason why the sender starts dividing the message bits with $q+10$ rather than q.

Notice also that the sender reserves one more bit for headers to cover a possibly larger k_i in a block than the expected value k/β. Because the header in each block has h bits, the message length in one block must not exceed 2^h-1 (Step E3).

The maximum number of bits that can be communicated using this algorithm is approximately $k-\beta h = k-\beta\times\lceil \log_2(r_2 k/\beta)\rceil$. The performance of this algorithm for some "typical" values of n and k is discussed in Section 3.4.

3.1 Minimizing the impact of embedding

When embedding a shorter than maximal message, in Steps E4 and E12 the sender will have freedom in choosing which unknowns v_i should be set to 0 and which will be determined by the Gaussian elimination. This freedom can be used to further minimize the impact of embedding. The SR will usually be formulated in quantitative terms and thus it will be possible to associate with each changeable sample x_i a numerical value $f(x_i)$ that somehow expresses its "fitness" to be included in the set of changeable samples. In Steps E4 and E12, when solving q_i equations $\boldsymbol{H}^{(i)}\boldsymbol{v} = \boldsymbol{m}^{(i)} - \boldsymbol{D}^{(i)}\boldsymbol{b}^{(i)}$ for k_i unknowns $\boldsymbol{v}$, the sender can solve for those unknowns v_i that correspond to samples with the largest fitness and set the remaining v_i to zero. This way, the impact of embedding will be further minimized and the security further improved.

A different way to minimize the impact of embedding is to minimize the number of embedding changes (maximize the embedding efficiency). With a fixed set of changeable pixels C, the problem of maximizing the rate is a binary vector quantization problem. To see this, we repeat that the sender needs to solve the system of q linear equations (3) $\boldsymbol{H}\boldsymbol{v} = \boldsymbol{m} - \boldsymbol{D}\boldsymbol{b}$

for k unknowns $v_1, \ldots, v_k$. Also, recall that the non-zero elements of the vector $\boldsymbol{v}$ are the places where the sender needs to apply the perturbed quantizer. If $q < k$, the set of all solutions to (3) is of the form

$$\boldsymbol{v}_0 + Ker(\boldsymbol{H})$$

where $\boldsymbol{v}_0$ is one solution to (3) and $Ker(\boldsymbol{H})$ is the kernel of $\boldsymbol{H}$ formed by vectors $\boldsymbol{x}$, such that $\boldsymbol{Hx} = 0$. Minimizing the embedding distortion is equivalent to finding a vector $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{x}$ with the minimal Hamming weight. Thus, the sender needs to perform binary vector quantization, which is, however, known to be an NP complete problem. Nevertheless, there is a potential for improvement here even using suboptimal binary vector quantizers. This research direction will be part of our future effort.

3.2 Imposing structure on D to speed up the coding

An obvious question to ask is whether it is possible to solve (3) faster by imposing some structure on the matrix $\boldsymbol{D}$ (and thus indirectly on $\boldsymbol{H}$). Recall that $\boldsymbol{H}$ is obtained from $\boldsymbol{D}$ by selecting those columns of $\boldsymbol{D}$ that correspond to changeable samples. The matrix $\boldsymbol{D}$ is generated from a secret stego key and thus does not depend on the cover object or the secret message. Because the positions of changeable samples will be different for different covers, the sender has no control over the process of selecting the submatrix $\boldsymbol{H}$ from $\boldsymbol{D}$. One could, however, impose some structure on the columns in $\boldsymbol{D}$, such as requesting a certain number of ones in each column. Note that, however, introducing any regularity into $\boldsymbol{D}$ is likely to lead to codes with suboptimal performance because, as shown in Section 2, random matrices $\boldsymbol{D}$ achieve in a certain sense optimal performance. On the other hand, it may be worth sacrificing the embedding capacity a little in return for a faster and simpler performance (part of a future effort).

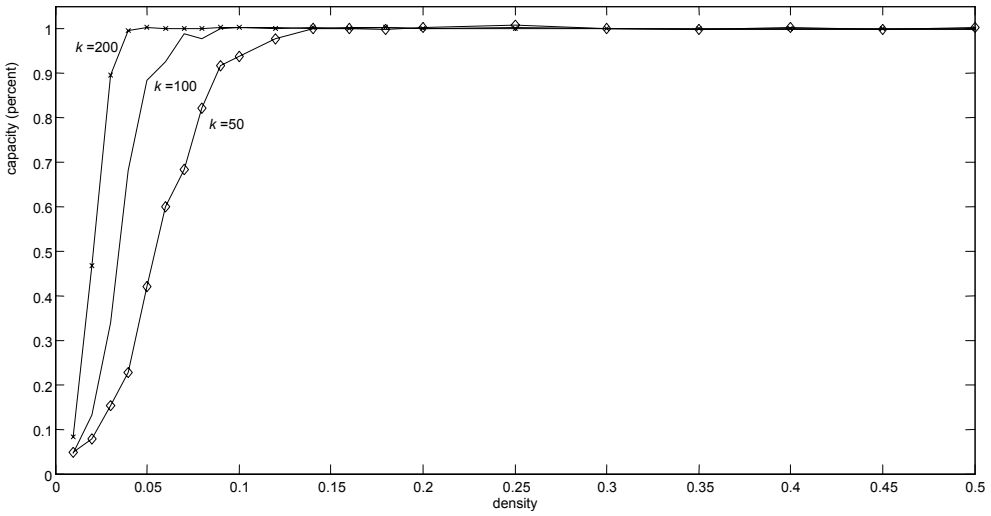

Figure 4. Capacity ratio $q_{\max}(\delta)/q_{\max}(1/2)$ as a function of δ for three values of k (averaged over 100 randomly generated matrices).

Another possibility to speed up the coding is to use sparse matrices $\boldsymbol{D}$ and $\boldsymbol{H}$. Let us assume that the elements of $\boldsymbol{H}$ are realizations of an i.i.d. random variable τ with range $\{0,1\}$ and $Prob(\tau=1) = \delta$ and $Prob(\tau=1) = 1-\delta$ with $\delta < 1/2$. The smaller the density δ, the faster the Gaussian elimination can be carried out in Steps E4 and E12. Also, allowing $\boldsymbol{H}$ to be sparse opens up new possibilities for solving (3) using solvers for sparse matrices (Section 3.4). What needs to be clarified, however, is how the sparseness influences the embedding capacity $q_{\max}(\delta)$, which now depends on δ. Figure 4 shows that the capacity stays very close to $q_{\max}(1/2)$ till a certain critical value of the density δ is reached. Then, it abruptly falls to zero.

According to the result proved by Cooper [28], the probability that a random $k \times k$ matrix $\boldsymbol{H}$ with density δ is nonsingular tends to $\lim_{k\to\infty} P_{k,k}(k) = 0.2889\ldots$, provided $\delta > (\log_2 k + d(k))/k$ for any $d(k) \to \infty$. Although this result does not tell us anything about the values $P_{k-r,k}(s)$ necessary to evaluate (7), it suggests that the critical density might be close to $(\log_2 k)/k$. Since this is an asymptotic result and in our applications k could be as small as 250, we need to verify the validity of

Cooper's result for "small" values of k. We have experimentally determined the value of the density $\delta_1(k)$ for which $1 - q_{\max}(\delta_1)/q_{\max}(1/2) < 0.01$. Figure 5 shows that $(\log_2 k)/k$ is, indeed, a good approximation of $\delta_1(k)$ even for small k.

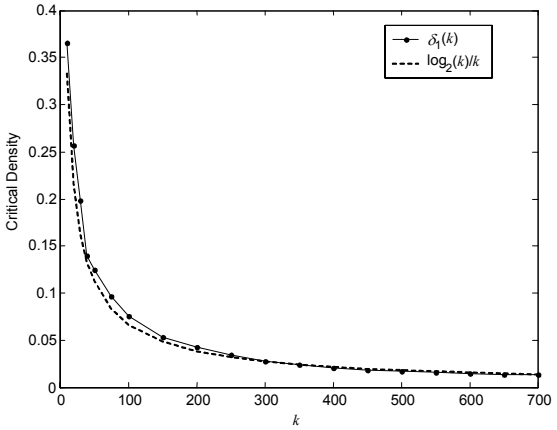

Figure 5. Comparison of critical density $\delta_1(k)$ and $(\log_2 k)/k$.

3.2.1 Using sparse matrices with structured Gaussian elimination

Because in the structured Gaussian elimination, the number of changeable samples k_i in each subset varies, when using sparse matrices we need to guarantee that in each subset B_i the density of 1s in $\boldsymbol{H}^{(i)}$ does not fall below the critical density $\delta(k_i) = (\log_2 k_i)/k_i$ for each i. Recall that the recipient does not know k (and thus k_i) and only knows that $nr_1 \le k \le nr_2$. Because $\delta(k)$ is decreasing, the critical density will be determined by the smallest k_i that one can encounter during embedding. Thus, the sender and recipient will set the density $\delta = (\log_2 k_{\min})/k_{\min}$, where k_{min} is the largest integer for which the probability $\text{Prob}(k_i < k_{\min}) \le p_1$, where p_1 is a small number (e.g., $p_1 = 0.01$) shared by both communicating parties.

The probability $P(r; n, k, n_i)$ that in a set of n_i randomly selected bits there will be t changeable bits (assuming we are selecting the bits from a set of n bits that contains k changeable bits) is

$$P(t; n, k, n_i) = \frac{\binom{n_i}{t}\binom{n-n_i}{k-t}}{\binom{n}{k}}. \tag{8}$$

Thus, the sender and the recipient determine the density

$$\delta = (\log_2 k_{\min})/k_{\min}, \tag{9}$$

where $k_{\min}$ is the largest integer satisfying the inequality

$$\sum_{t=0}^{k_{\min}} P(t; n, nr_1, \lceil n/\beta \rceil) \le p_1 . \tag{10}$$

3.3 Communicating the rate r

In applications where the range for $r = k/n$ is very large, the encoding algorithm may have suboptimal performance, such as longer computing times and smaller capacities. In this section, we briefly describe a slightly different implementation, in which the sender actually communicates the rate r, $0 < r < 1$, encoded using u bits, to the receiver. The communication of r makes the whole scheme more flexible especially when the covers are very diverse, such as images in raw, JPEG, palette

formats, and various audio formats, because the communicating parties do not need to research (and agree on) the typical range of r for all these different cover types.

For a given $r = k/n$, the sender communicates the integer l for which $r_1 = l2^{-u} \le r < (l+1)2^{-u} = r_2$, $0 \le l < 2^u$. In order to communicate l, Steps E1 and E12 in the pseudo-code for the encoder are modified as follows

E1. Set $\beta = \lceil nr_2/k_{\text{avg}} \rceil$. Determine the header size $h = \lceil \log_2(k_{\text{avg}}) \rceil + 1$, $q = |\boldsymbol{m}| + \beta h + u$

E12. Select the first n_β columns and $q_\beta - u$ rows from $\boldsymbol{D}$ and denote this submatrix $\boldsymbol{D}^{(\beta)}$. Using a PRNG seeded with the stego key, generate a random binary matrix $\overline{\boldsymbol{D}}$ with n columns and u rows with density $\bar{\delta} = 1/2$. Solve q_β equations

$$\begin{bmatrix} \overline{\boldsymbol{H}} \\ \boldsymbol{H}^{(\beta)} \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} bin(l) \\ \boldsymbol{m}^{(\beta)} \end{bmatrix} - \begin{bmatrix} \overline{\boldsymbol{D}}\overline{\boldsymbol{b}} \\ \boldsymbol{D}^{(\beta)}\boldsymbol{b}^{(\beta)} \end{bmatrix}, \tag{11}$$

for k_β unknowns $\boldsymbol{v}$, where

$$\overline{\boldsymbol{b}} = \begin{bmatrix} \boldsymbol{b}'^{(1)} \\ \boldsymbol{b}'^{(2)} \\ \dots \\ \boldsymbol{b}'^{(\beta-1)} \\ \boldsymbol{b}^{(\beta)} \end{bmatrix}$$

and $bin(l)$ is a column-wise binary encoding of l. If this system does not have a solution, exit and report failure to embed the message.

In (11), the equations $\boldsymbol{H}^{(\beta)}\boldsymbol{v} = \boldsymbol{m}^{(\beta)} - \boldsymbol{D}^{(\beta)}\boldsymbol{b}^{(\beta)}$ are solved together with u equations $\overline{\boldsymbol{H}}\boldsymbol{v} = bin(l) - \overline{\boldsymbol{D}}\overline{\boldsymbol{b}}$, where $\overline{\boldsymbol{D}}$ is a $u \times n$ random matrix with density $\bar{\delta} = 1/2$ and $\overline{\boldsymbol{H}}$ is a $u \times n$ submatrix of $\overline{\boldsymbol{D}}$ consisting of those columns of $\overline{\boldsymbol{D}}$ that correspond to changeable bits.

The decoder will first read u bits $bin(l) = \overline{\boldsymbol{D}}\boldsymbol{b}'$, calculate $r_1 = l2^{-u}$, $r_2 = (l+1)2^{-u}$, $\beta = \lceil nr_2/k_{\text{avg}} \rceil$, δ (from (9) and (10)), $h = \lceil \log_2(k_{\text{avg}}) \rceil + 1$, and then continues with Steps D0 to D9 for the decoder with a fixed range $[r_1, r_2]$. Note that since the decoder doesn't know β and δ before he decodes u, the matrix $\overline{\boldsymbol{D}}$ must have a fixed number of columns (say n) and a known density, which we set at 1/2.

3.4 Other methods for solving linear equations in GF(2)

In Section 3.2, we experimentally showed that $q_{\max}(k) \approx k$ holds for random sparse binary matrices $\boldsymbol{H}$ with the density of ones $\delta = (\log_2 k)/k$. This fact opens up new possibilities for solving (3) significantly faster using techniques designed for sparse matrices, such as the probabilistic algorithms of Lanczos [29] and Wiedemann [30]. Both methods have complexity proportional to $k(k+\omega k)(\log k)^c$, where ω is the average number of ones in each row of $\boldsymbol{H}$ and c is a small positive constant. We have implemented the Wiedemann and Lanczos methods and the Gaussian elimination and compared their timings on a set of regular square random $k \times k$ matrices with $\delta = (\log_2 k)/k$ for k = 250, … 30000. For each k, several matrices were generated and the running times averaged. For the two probabilistic algorithms, the total time needed to solve the system was recorded. All experiments were performed on a high end PC under Cygwin 1.5.9 running under Windows XP Professional. The CPU is Intel Pentium 4 with the HT technology running at 2.4GHz with the internal cache of the size of 8KB and the on-board L2 cache of the size of 512KB. The results are summarized in Table 1.

Wiedemann method. We implemented the basic algorithm described in the original paper [30]. The time complexity of our implementation is proportional to $k^2 \log k$, while its spatial complexity is proportional to $2k^2 + 8k$. A naïve implementation representing each element of the matrix with a byte (rather than a bit) requries about 760 MB of RAM for a problem of the size 20,000 while system of 30,000 equations already requires as much as 1.7 GB RAM. It is possible to save some space by storing the system as bits of indiviudal bytes (an approach taken for fast implementation of the Gaussian elimination described below). We implemented this algorithm also, but it actually led to a significant descrease in speed. We obtained the

best results for a combined solution in which some parts of the system store their results so that individual elements of the system matrix are arrays of bytes while the most time-critical part (the Berlekamp-Massey algoritm for finding the minimal polynomial) is implemented by storing elements of the system as bits.

Lanczos method. The implementation is based on the description of the algorithm in [29]. Because the exponentiation and logarithm tables for the finite field (which are jointly used to perform multiplication in the field) are accessed in a completely random way, the computation is much faster if these tables can reside in the cache permanently rather than being swapped back and forth between cache and main memory. The total size of these tables is 8×2^r integer variables when the field $GF(2^r)$ is used (i.e., algebraic extension of degree r of GF(2)), and so the memory requirement is $4\times 8\times 2^r$ bytes = 2^{r-5} kilobytes. This is consistent with our timings where we get best results for $r = 14$ and the cache size of 2^9 kilobytes. The timings shown in Table 1 were carried out with $r = 14$.

To simplify the coding, we used static arrays to store coordinates of non-zero positions of the matrix $\boldsymbol{H}$ and thus limited the number of ones in each row, which we set to 32 which is $2\times\log_2(50{,}000)$. This limit certainly has no influence on the results for $k \le 5{,}000$, shown in Table 1, and even for $k > 30{,}000$ its effect should be negligible (this limit can be set higher if needed).

Gaussian elimination. Since the system being solved is binary, we can improve the performance of the Gaussian elimination significantly by storing the elements of the system matrix as bits of the largest integer variable a given architecture/language offers. We implemented the code in the C++. Each row of the system is stored as an array of `int`s, where an `int` is 32 bit wide on our architecture. Given the fact that we work in GF(2), all the basic operations we need – addition and multiplication – can be very effictivelly carried out in "parallel" by using the C++ operators ^ (the bitwise eXclusive OR) and & (the bitwise AND) respectivelly. Because these operators are natively supported by the hardware, we obtain a constant time improvement of 32. Additional speed-up (up to a certain problem size) can be expected since the memory needed for the computations is reduced, again, by a factor of 32, which leads to an increased chance that more rows of the system will fit into the cache.

k_{avg}	Wiedemann	Lanczos	Gauss	Total solving time $k=10^5$
250	0.00971	0.00802	**0.00110**	0.440
500	0.0349	0.0322	**0.00499**	0.998
1000	0.134	0.147	**0.023**	2.30
2000	0.542	0.828	**0.122**	6.10
5000	3.74	7.73	**2.74**	54.8
10000	**15.5**	35.4	17.4	174
20000	**72.1**	172	106	530
30000	**U**	371	302	1208

Table 1 Average running times (the fastest times are highlighted in bold face).

The last column is the total time Steps E4 and E12 contribute to the total embedding time for the structured Gaussian method assuming the bit-stream is divided into β subsets, $\beta = k/k_{avg}$. It was calculated as $T\lceil k/k_{avg}\rceil$, where T is the solving time for the Gaussian elimination method (third column). For $k_{avg} = 30000$, the Wiedemann algorithm could not be run as the problem of this size did not fit into the main memory. Note that neither method leads to practical embedding times if applied to the whole bit-stream ($k = k_{avg}$) for $k \ge 10^4$. On the other hand, the structured Gaussian elimination gives the best performance and practical embedding times when applied to $\beta = k/k_{avg}$ subsets for $k_{avg} = 250$ or 500.

4. APPLICATIONS IN STEGANOGRAPHY

The proposed writing on wet paper enables constructions of new, more secure, steganographic schemes that were not possible before. We show examples of schemes that fit the wet paper communication setup and briefly discuss their steganographic security in comparison to existing schemes. Although we will be explaining the concepts on the example of

digital images, the considerations are clearly general enough to apply to other digital objects that allow insertion of steganographic content.

4.1 Adaptive steganography

A typical steganographic algorithm for digital media objects (images, audio, video) embeds one bit per object sample (pixel, DCT coefficient, index) by applying an embedding operation to the sample. This embedding operation is applied if the sender needs to adjust the "parity" of the sample to match the embedded message bit. The samples are selected either sequentially, randomly using a shared secret key, or adaptively based on the cover content. In adaptive steganography, the embedding distortion and selection of message-carrying samples in the cover object is in some way related to the cover content. This often undermines the security of the steganographic system because the set of adaptively selected samples may become available to the attacker [17]. Also, since the act of embedding itself modifies the image, care needs to be taken to make sure that the recipient correctly recovers the message. This is usually solved by employing some artificial ad hoc measures whose only purpose is to guarantee the message readability. These measures limit the sender in his choice of the embedding operation and sample selection and often severely limit the capacity. Next, we give a few examples of adaptive systems previously proposed in the literature.

Example 1 (Adaptive Least Significant Bit Embedding). The sender chooses some local complexity measure σ that is calculated from the 7 Most Significant Bits (MSBs) of pixels located in a small neighborhood $N(x)$ of a given pixel x. The message bits are embedded in LSBs of those pixels x for which $\sigma(N(x)) > \sigma_0$, where σ_0 is an appropriately chosen threshold shared by the sender and the recipient. In other words, the sender is placing the message bits in the LSBs of pixels located in those parts of the cover image that have a certain minimal level of structure or noise. The recipient will be able to determine the same set of message-carrying pixels as the sender and thus read the message. This is because σ is a function of the 7 MSBs, which are *invariant* with respect to the embedding operation. Note that in this scenario, an attacker also has access to the message-carrying pixels.

Example 2 (Statistics-Preserving LSB Embedding). Franz [14] proposed to use the LSB embedding method only for pixels with colors c_1 and c_2 (differing only in their LSBs) that are statistically spatially independent. This independency is evaluated using the chi-square test for statistical independency of the values c_1 and c_2 occurring as spatially neighboring pixels in the cover image (this is done for several orientations of the neighboring pair). The LSBs of all pixels with spatially independent color pairs (c_1, c_2) are replaced with message bits pre-biased to match the relative counts of each color. This embedding mechanism is intended to prevent histogram-based steganalytic attacks [31] and it also guarantees that the recipient will determine the same message-carrying pixels because the chi-square statistics used for determining the color pairs is invariant with respect to embedding changes. Again, the public selection channel gives a starting point to the attacker [17].

Example 3 (Block Parity Embedding). This technique was proposed for color palette images in [15]. The image is divided into disjoint blocks B (for example 3×3 blocks) completely covering the image. In each block B, at most one bit will be embedded as the parity of the whole block (e.g., XOR of LSBs of all pixels in B) by changing one pixel in B. As in Example 1, a local block complexity measure σ is selected together with a threshold σ_0. If $\sigma(B) > \sigma_0$, the sender embeds the message bit, obtaining the modified block B', and immediately verifies that $\sigma(B') > \sigma_0$. If the embedding change leads to $\sigma(B') \le \sigma_0$, the sender makes the change anyway and re-embeds the same bit in the next block. The recipient will thus correctly read all message bits from blocks B satisfying $\sigma(B) > \sigma_0$. This method also suffers from the public selection channel. In addition, the necessity to use non-overlapping pixel blocks leads to a significant capacity decrease.

Note that in the examples above the encoder is forced to choose such combinations of the embedding operation and the selection rule that satisfy the requirement of message readability by the receiver, who does not know the cover object. Ideally, the sender should fully focus on the impact of embedding changes on detectability and choose the embedding operation and selection rule accordingly, rather than paying attention to the readability. This is exactly, however, what writing on wet paper enables the sender to do. The selection rule can be completely arbitrary (it can, in fact, contain an element of *true* randomness) and does not have to be shared with the recipient. The wet paper codes thus solve one of the fundamental problems of adaptive steganography and improve the steganographic security because less information is now available to the attacker.

4.2 Perturbed quantization

In this section, we give a short description and analysis of an embedding method called Perturbed Quantization that was previously proposed by the authors of this paper [3]. Let us assume that before embedding the sender processes a digital cover image using some information-reducing process F, such as A/D conversion, lossy compression, downsizing, color quantization, etc. The process F typically consists of a real-valued transformation T and an integer quantizer Q. The sender has access to all numerical values before quantization occurs. The largest quantization errors occur for those values that are close to the middle of the quantization intervals of Q. Due to the noise that is commonly present in digital images, the quantization of these values is dominated by the noise and thus closely resembles a random process[3]. The sender may designate such samples (pixels, DCT coefficients) as changeable and use them, together with the wet paper code, for steganography. The remaining samples will be quantized without any changes (those are the wet pixels or "defects" in the "memory"). We call this method Perturbed Quantization (PQ) because the sender slightly perturbs the quantization process in order to embed message bits.

Because the downgrading process is *information-reducing*, an attacker cannot easily recover those fine details of the original image that would enable him to find statistical evidence that some of the samples in the stego image were quantized "incorrectly" (imagine, for example, obtaining a good-enough approximation to the uncompressed cover image from its JPEG compressed form). This is difficult because the sender used side information (the unquantized values) that is essentially removed during quantization and is unavailable to the attacker. Also, the sender can accept additional coefficient selection rule(s) to further decrease the probability of introducing detectable artifacts and thus improve the security. For example, the sender may avoid changing coefficients in those areas of the cover image where the attacker could predict the original coefficient values with better accuracy.

4.2.1 Information-reducing operations

We proceed with providing a more formal description of the embedding process. Let us assume that the cover image X is represented with a vector $x \in I^m$, where I is the range of its pixel/coefficient/color/index values depending on the format of X. For example, for an 8-bit grayscale image, $I=\{0, \ldots, 255\}$. The downgrading process F will be modeled as a transformation

$$F = Q \circ T\colon I^m \to J^n, \tag{12}$$

where J is the integer dynamic range of the downgraded image $Y = F(X)$ represented with an n-dimensional integer vector $y \in J^n$, $m \geq n$. The transform $T\colon I^m \to \mathbf{R}^n$ is a real-valued transformation and $Q\colon \mathbf{R}^n \to J^n$ is a quantizer. The intermediate "image" $T(X)$ will be denoted as U and represented using an n-dimensional vector $u \in \mathbf{R}^n$. We give several examples of image downgrading operations F that could be used for steganography based on Perturbed Quantization (PQ).

Example 1 (Resizing). For grayscale images, the transformation T maps a square $m_1 \times m_2$ matrix of integers x_{ij}, $i=0, \ldots, m_1-1$, $j=0, \ldots, m_2-1$ into an $n_1 \times n_2$ matrix of real numbers u_{rs}, $n_1 < m_1$, $n_2 < m_2$ using a resampling algorithm. The quantizer Q is a uniform scalar quantizer (rounding to integers), applied to the vector u by coordinates.

$$Q(z) = round(z), \tag{13}$$

Example 2 (Decreasing the color depth by *d* bits). The transformation T maps a square $m_1 \times m_2$ matrix of integers x_{ij} in the range $I=\{0, \ldots, 2^b-1\}$, $i=0, \ldots, m_1-1$, $j=0, \ldots, m_2-1$ into a $m_1 \times m_2$ matrix of real numbers u_{ij}, $u_{ij}=x_{ij}/2^d$. The quantizer Q is the same uniform scalar quantizer as in Example 1.

Example 3 (JPEG compression). For grayscale images, the transformation T maps a square $m_1 \times m_2$ matrix of integers x_{ij}, into a $8\lceil m_1/8 \rceil \times 8\lceil m_2/8 \rceil$ matrix of real numbers u_{ij} in a block-by-block manner ($\lceil z \rceil$ denotes the smallest integer larger than or equal to z). In each 8×8 pixel block B_x, the corresponding block B_u in u_{ij} is $\mathrm{DCT}(B_x)./q$, where DCT is the 2D DCT transform, q is the quantization matrix, and the operation "./" is an element-wise division. The quantizer Q is given by (13).

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[3] The authors are currently working on a better justification of this heuristic statement using statistical modeling and prove for a certain image model that the embedding is ε-secure in the Cachin's sense [11].

4.2.2 Perturbed quantizer

As discussed above, one of the simplest SRs that the sender can formulate is to require the intermediate values u_i of changeable samples $y_i = Q(u_i)$ to be ε-close to the middle of the quantization intervals of Q:

$$C = \{i | i \in \{0, \ldots, n\}, u_i \in [L+0.5-\varepsilon, L+0.5+\varepsilon] \text{ for some integer } L\}. \quad (14)$$

The tolerance ε could in principle be adaptive and depend on the neighborhood of the pixel x_i. It can also be made key dependent, if desired. For this SR, the act of embedding a a random message in the cover image X is well modeled with the probabilistic process $X \to Q_\varepsilon \circ T(X) = Y'$, where Q_ε is the perturbed quantizer and L is an integer,

$$Q_\varepsilon(z) = \begin{cases} L & \text{for } L \le z < L + 0.5 - \varepsilon \\ L+1 & \text{for } L + 0.5 + \varepsilon \le z < L + 1 \\ L \text{ or } L+1 & \text{with equal probability for } L + 0.5 - \varepsilon \le z < L + 0.5 + \varepsilon, \end{cases} \quad (15)$$

and Y' is the stego image represented using an integer vector $y' \in J^m$. Note that $Q_\varepsilon = Q$ for $\varepsilon = 0$. The quantizers Q and Q_ε are identical with the exception of the interval $[L+0.5-\varepsilon, L+0.5+\varepsilon)$ where their output differs in 50% of cases. It can be easily shown that, assuming u is a random variable uniformly distributed on [0, 1], the average quantization error $u - Q(u)$ introduced by the scalar quantizer (13) is 1/4, while for the perturbed quantizer (15) it is $1/4+\varepsilon^2$. Thus, the difference between the average error of both quantizers is ε^2, which for $\varepsilon = 0.1$ is at least by one order of magnitude smaller than the average quantization error. Also, note that $-2\varepsilon \le |u - Q(u)| - |u - Q_\varepsilon(u)| \le 2\varepsilon$ for all u.

4.2.3 Embedding while double compressing

The SR can be defined differently based on other heuristics, the image format, and properties of image pixels/coefficients. For example, in our previous work [3] a different example of a SR is given when the information-reducing transformation is recompression of the cover JPEG image using a lower JPEG quality factor. Because this steganography is discussed in detail in [3], in this paper we discuss it very briefly just to illustrate the point that the wet paper codes enable construction of steganographic techniques that have substantially better steganographic security than previously proposed schemes.

During recompression of a JPEG file, certain values of the DCT coefficients in the cover JPEG file occur in the middle of quantization intervals during the second compression. These coefficients will be the changeable coefficients. Due to the decompression to the spatial domain, rounding and clipping errors are introduced and these errors make the second quantization of changeable coefficients resemble a random rather than deterministic process. Thus, the sender can round the changeable coefficients up or down and use them as the set of "dry" coefficients in a wet paper code.

For certain combinations of quality factors for both JPEG compressions, this embedding technique provides a very large capacity of approximately 0.5 bits per non-zero DCT coefficient of the stego file. At the same time, the blind JPEG steganalyzer of [4] was unable to distinguish between purely double compressed images and fully embedded double compressed images [3]. Table 2 shows the detection accuracy $\rho = 2A-1$, where A is the area under the ROC curve, for a simple linear classifier trained on 1400 cover and 1400 stego (fully embedded) images and tested on 400 never seen images. The table also shows the detection accuracy for other state-of-the-art steganographic techniques for JPEG images. It is very apparent that the new method offers significantly better resistance to steganalysis than the other tested techniques.

5. CONCLUSIONS

The main contributions of this paper are as follows. First of all, this paper reveals an important relationship between memories with defective cells [1] and steganography. The defective cells correspond to those cover object elements designated by the sender to be avoided for embedding and are *not* shared with the recipient. Because in steganography the number of defective cells could be quite large, we coin a new term for this steganographic channel – *writing on wet paper*. This is a metaphor for a steganographic channel in which the sender embeds message bits into a subset of elements of the cover object and communicates the message to the recipient, who does not have any information about the selection rule applied by the sender. If the selection rule is determined by side information available only to the sender but in principle unavailable to the recipient (and any attacker), this scenario provides improved steganographic security compared to schemes with a public selection rule [14]–[17].

Second, we propose a simple variable-rate random linear code for memories with a *large number* of defects and show how it can be applied for our steganographic channel. We prove that this code enables on average communication of k bits given k "dry" elements (n–k defective cells). The code lends itself to efficient practical implementations and offers flexibility and control to the sender over which cover object elements will be modified. This further minimizes the impact of embedding changes (Section 3.1).

bpc	F5	F5_111	OG	MB1	MB2	PQ
0.05	0.2410	0.6451	0.8789	0.2197	0.1631	~ 0
0.1	0.5386	0.9224	0.9929	0.4146	0.3097	0.0484
0.2	0.9557	0.9958	0.9991	0.7035	0.5703	0.0979
0.4	0.9998	0.9999	U	0.9375	0.8243	0.1744
0.6	1.0000	1.0000	U	0.9834	U	U
0.8	1.0000	1.0000	U	0.9916	U	U

Table 2 Detection reliability ρ for F5 (F5) [25], F5 with matrix embedding (1,1,1) (F5_111), OutGuess 0.2 (OG) [36], Model based Steganography [37] without and with deblocking (MB1 and MB2, respectively), and the proposed Perturbed Quantization [3] during double compression for different embedding rates expressed using bpc = bits per non-zero stego DCT coefficient (U = unachievable rate). All but the PQ algorithm, were tested with $Q = 80$. The PQ algorithm was tested with $Q_1 = 85$ and $Q_2 = 70$.

Third, we illustrate how wet paper codes can be used to solve some fundamental problems of adaptive steganography and we briefly discuss a new approach to steganography for digital media called Perturbed Quantization [3]. In Perturbed Quantization, the sender embeds a secret message while downgrading the cover object using some information-reducing operation, such as lossy compression, A/D conversion, downsampling, etc. The sender uses his knowledge of the *unprocessed* object and embeds data into those pixels/coefficients whose values are the most "uncertain" after the processing. We illustrate the methodology on the example of recompressing a JPEG image with a lower quality factor. Using heuristic arguments supported with blind steganalysis [4], it is shown that Perturbed Quantization is significantly less detectable than existing steganographic methods for JPEG images while providing a relatively large capacity.

We note that the writing on wet paper and the proposed wet paper code can be thought of as a generalization of the selection channel [10]. The wet paper is also a special case of the general problem of communication with informed sender [5]. While the Costa's dirty paper code [6] is relevant for watermarking [7][8], the wet paper is a suitable model for steganography. Both channels are different special cases of the general problem of communication with informed sender.

There are numerous applications of the wet paper code in steganography and general data embedding. For example, we name the removal of shrinkage in the F5 algorithm [25] and improving its embedding efficiency. Obviously, nullifying a DCT in F5 embedding coefficient will no longer be a problem for the decoder if the wet paper code is employed. Another application is constructing steganographic schemes that, besides the secret shared stego key, contain an element of true randomness and thus cannot be subjected to brute force stego key searches [27]. As the last application, we mention data hiding in binary images proposed by Wu [38]. In this application, the sender first identifies the set of "flippable" pixels that can be modified for embedding. Because this set of pixels is not shared with the recipient, block embedding combined with random shuffling is proposed in her work [38]. The block embedding however, leaves most of the flippable pixels unused and only a fraction of the embedding capacity is used. Because this problem exactly corresponds to writing on wet paper, the capacity of this data hiding method can be dramatically improved.

In the future, we plan to investigate in more detail the steganographic security of Perturbed Quantization. In particular, it seems plausible to prove its ε-security in the Cachin's sense [11] assuming an appropriate model of the cover object. Finally, we plan to further study methods for increasing the embedding efficiency (Section 3.1) and simplifying the coding process by imposing structure on matrix H (Section 3.2).

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APPENDIX A (Calculating the average coding rate)

Lemma 1. *The function* $\pi(n) = \prod_{i=1}^{n}\left(1-2^{-i}\right)$, $n\geq 0$, *is monotonically decreasing with* $\lim_{n\to\infty}\pi(n) = \pi(\infty) = 0.2889...$

Furthermore, $\pi(n) = \pi(\infty)(1+\rho_\pi(n))$, where $0 < \rho_\pi(n) < 2^{2-n}$ *for* $n\geq 0$. (A1)

Proof.

$$\pi(n) = \prod_{i=1}^{\infty}\left(1-2^{-i}\right)\prod_{i=n+1}^{\infty}\left(1-2^{-i}\right)^{-1} = \pi(\infty)\exp\left(-\sum_{i=n+1}^{\infty}\ln(1-2^{-i})\right) = \pi(\infty)\exp\left(\sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=n+1}^{\infty}\left(2^{-k}\right)^{i}\right) = \pi(\infty)\exp\left(\sum_{k=1}^{\infty}\frac{1}{k(2^k-1)2^{kn}}\right)$$

by expanding the natural logarithm using Taylor expansion and exchanging the sums. Thus, for $n > 1$

$\pi(\infty) < \pi(n) < \pi(\infty)\exp\left(\sum_{k=1}^{\infty}2^{kn}\right) = \pi(\infty)\exp\left(\frac{1}{2^n-1}\right) < \pi(\infty)\exp\left(2^{1-n}\right) < \pi(\infty)(1+2^{2-n})$ and $\pi(\infty) = 0.2889...$ by direct calculation.□

Using $\pi(n)$, we rewrite (6), (7), and (8)

$$P_{q,k}(s) = 2^{s(q+k-s)-qk}\frac{\pi(q)\pi(k)}{\pi(s)\pi(q-s)\pi(k-s)}. \tag{A2}$$

$$p_{\geq k-r} = \sum_{i=0}^{k-r}\frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)}\frac{\pi(k-r)\pi(k)}{\pi(k-r-i)} = \pi(\infty)\sum_{i=0}^{k-r}\frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)}\frac{(1+\rho_\pi(k-r))(1+\rho_\pi(k))}{1+\rho_\pi(k-r-i)} \tag{A3}$$

$$p_{\geq k+r} = \frac{1}{2^r}\sum_{i=0}^{k}\frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)}\frac{\pi(k+r)\pi(k)}{\pi(k-i)} = \frac{\pi(\infty)}{2^r}\sum_{i=0}^{k}\frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)}\frac{(1+\rho_\pi(k+r))(1+\rho_\pi(k))}{1+\rho_\pi(k-i)} \tag{A4}$$

Lemma 2.

For $0 \leq r \leq k/2$, $p_{\geq k-r} = \pi(\infty)\sum_{i=0}^{\infty}\frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)} + \rho_1(k)$, where $|\rho_1(k)| < 2^{3-k/4}$. (A5)

For $k/2 \leq r \leq k$, $1 - p_{\geq k-r} < 2^{3-k/2}$. (A6)

For $r \geq 0$, $p_{\geq k+r} = \frac{\pi(\infty)}{2^r}\sum_{i=0}^{\infty}\frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)} + \rho_2(k)$, where $|\rho_2(k)| < 2^{3-k/2}$. (A7)

For $r \geq 0$, $p_{\geq k+r} < 2^{3-r}$. (A8)

Proof: We first prove (A5). Let $Q(a,b) = \pi(\infty)\sum_{i=a}^{b}\frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)}\frac{(1+\rho_\pi(k-r))(1+\rho_\pi(k))}{1+\rho_\pi(k-r-i)}$ and

$Q_0(a,b) = \pi(\infty)\sum_{i=a}^{b}\frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)}$. Using Lemma 1, it is easy to show that for any $0 < l < l' < k-r$

$$Q_0(0,l)(1-2^{2-k+r+l}) < \frac{1}{1+\rho_\pi(k-r-l)}Q_0(0,l) < Q(0,l) < Q_0(0,l)(1+\rho_\pi(k/2))^2 < Q_0(0,l)(1+2^{4-k/2}) \tag{A9}$$

$$0 < Q_0(l,l') < Q_0(l,\infty) < 2^{1-l^2} \text{ and } 0 < Q(l,l') < Q(l,\infty) < 2^{1-l^2} \tag{A10}$$

Writing $p_{\geq k-r} = Q(0, k/4) + Q(k/4+1, k-r)$ and using (A9) and (A10)

$$(1-2^{2-k/4})\left(Q_0(0,\infty) - Q_0(k/4+1,\infty)\right) < Q(0,k/4) + Q(k/4+1,k-r) < \left(Q_0(0,\infty) - Q_0(k/4+1,\infty)\right)(1+2^{4-k/2}) + 2^{1-k^2/16}.$$

After applying (A10) to $Q_0(k/4+1,\infty)$ and replacing the bounds with less tight bounds, we finally obtain

$p_{\geq k-r} = Q(0,k-r) = Q_0(0,\infty) + \rho_1(k)$, where $|\rho_1(k)| < 2^{3-k/4}$.

To prove (A6), we write

$$1 - 2^{2-k/2} < 1 - \rho_\pi(k/2) < p_{\geq k+r} = \frac{\pi(k)}{\pi(r)} + \sum_{i=1}^{\infty} \frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)} \frac{\pi(k-r)\pi(k)}{\pi(k-r-i)} < \frac{1+\rho_\pi(k)}{1+\rho_\pi(r)} + 2^{1-r} < 1 + 2^{3-k/2} .$$

To prove (A7), we define

$$R(a,b) = \pi(\infty)2^{-r} \sum_{i=a}^{b} \frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)} \frac{(1+\rho_\pi(k+r))(1+\rho_\pi(k))}{1+\rho_\pi(k-i)}$$

$$R_0(a,b) = \pi(\infty)2^{-r} \sum_{i=a}^{b} \frac{2^{-i^2-ri-i}}{\pi(i)\pi(i+r)} .$$

Using Lemma 1, we obtain

$$(1-2^{2-k/2})R_0(0,k/2) < \frac{R_0(0,k/2)}{1+\rho_\pi(k/2)} < R(0,k/2) < R_0(0,k/2)(1+\rho_\pi(k))^2 < R_0(0,k/2)(1+2^{3-k}) \qquad \text{(A11)}$$

$$0 < R_0(k/2+1,k) < 2^{-k^2/4} \text{ and } 0 < R(k/2+1,k) < 2^{-k^2/4} . \qquad \text{(A12)}$$

Writing $p_{\geq k+r} = R(0, k/4) + R(k/4+1, k-r)$ and using (A11) and (A12),

$$(1-2^{2-k/2})\big(R_0(0,\infty) - R_0(k/2+1,\infty)\big) < R(0,k) < \big(R_0(0,\infty) - R_0(k/2+1,\infty)\big)(1+2^{3-k}) + 2^{-k^2/4} .$$

Thus, $p_{\geq k+r} = R(0,k) = R_0(0,\infty) + \rho_2(k)$, where $|\rho_2(k)| < 2^{3-k/2}$.

The inequality (A8) is easily proved directly from (A4). □

Lemma 3.

$p_{=k-r} = p_{=k+r} + \rho_3(k)$, where $|\rho_3(k)| < 2^{5-k/4}$ for $0 \leq r \leq k/2$
$p_{=i} < 2^{4-k/2}$ for $0 \leq i < k/2$
$p_{=i} < 2^{4-i+k}$ for $i > k+k/2$.

The second and third inequalities are easily proved from (A6) and (A8). The approximate symmetry $p_{=k-r} \cong p_{=k+r}$ for $r \leq k/2$ is proved as follows

$$p_{\geq k-r+1} = \pi(\infty)\sum_{i=0}^{\infty} \frac{2^{-i^2-(r-1)i-i}}{\pi(i)\pi(i+r-1)} + \rho_1(k) = \pi(\infty)\sum_{i=0}^{\infty} \frac{2^{-i^2-ri-i}(2^i - 2^{-r})}{\pi(i)\pi(i+r)} + \rho_1(k) = -p_{\geq k+r} + \sum_{i=0}^{\infty} \frac{2^{-i(i+r)}}{\pi(i)\pi(i+r)} + \rho_1(k) + \rho_2(k) ,$$

or

$$p_{\geq k-r+1} + p_{\geq k+r} = \sum_{i=0}^{\infty} \frac{2^{-i(i+r)}}{\pi(i)\pi(i+r)} + \rho_1(k) + \rho_2(k) .$$

Thus, $p_{=k-r} - p_{=k+r} = p_{\geq k-r} + p_{\geq k+r+1} - (p_{\geq k-r+1} + p_{\geq k+r}) = \sum_{i=0}^{\infty} \frac{2^{-i(i+r)}\left(1 - \frac{2^{r+1}}{2^{r+i+1}-1}\right)}{\pi(i)\pi(i+r)} + \rho_3(k) = \rho_3(k)$, where $|\rho_3(k)| < 2^{5-k/4}$. We now prove that the sum is indeed equal to zero. After some tedious but straightforward algebra, we can rewrite the sum as

$$\sum_{i=0}^{\infty} \frac{2^i (2^{r+i+1} - 1 - 2^{r+1})}{2^{r+1} \prod_{k=1}^{r+1} \left(1 - \frac{1}{2^k}\right) \prod_{k=1}^{i} (2^{k+r+1} - 1)(2^k - 1)}. \tag{A13}$$

To prove that (A13) is zero, we first prove by induction with respect to n that for r, n positive and $t \neq 0$

$$s_{r,n} = \sum_{i=0}^{n} \frac{t^i (t^{r+i+1} - 1 - t^{r+1})}{t^{r+1} \prod_{k=1}^{r+1} \left(1 - \frac{1}{t^k}\right) \prod_{k=1}^{n} (t^{k+r+1} - 1)(t^k - 1)} = -\frac{1}{t^{r+1} \prod_{k=1}^{r+1} \left(1 - \frac{1}{t^k}\right) \prod_{k=1}^{n} (t^{k+r+1} - 1)(t^k - 1)}. \tag{A14}$$

For $n = 0$ we need to prove that

$$\frac{t^{r+1} - 1 - t^{r+1}}{t^{r+1} \prod_{k=1}^{r+1} \left(1 - \frac{1}{t^k}\right)} = -\frac{1}{t^{r+1} \prod_{k=1}^{r+1} \left(1 - \frac{1}{t^k}\right)}$$

which is true.

Assuming (A14) is true for n, we write

$$s_{r,n+1} = \frac{-1}{t^{r+1} \prod_{k=1}^{r+1} \left(1 - \frac{1}{t^k}\right) \prod_{k=1}^{n} (t^{k+r+1} - 1)(t^k - 1)} + \frac{t^{n+1} (t^{r+n+2} - 1 - t^{r+1})}{t^{r+1} \prod_{k=1}^{r+1} \left(1 - \frac{1}{t^k}\right) \prod_{k=1}^{n+1} (t^{k+r+1} - 1)(t^k - 1)} =$$

$$= \frac{-(t^{n+r+2} - 1)(t^{n+1} - 1) + t^{n+1} (t^{r+n+2} - 1 - t^{r+1})}{t^{r+1} \prod_{k=1}^{r+1} \left(1 - \frac{1}{t^k}\right) \prod_{k=1}^{n+1} (t^{k+r+1} - 1)(t^k - 1)} = -\frac{1}{t^{r+1} \prod_{k=1}^{r+1} \left(1 - \frac{1}{t^k}\right) \prod_{k=1}^{n+1} (t^{k+r+1} - 1)(t^k - 1)},$$

which completes the induction step.

The fact that (A13) is zero is now easily proved by observing that for a fixed r and $t = 2$, $\lim_{n \to \infty} s_{r,n} = 0$. □

Lemma 4. $q_{\max}(k) = \sum_{i=1}^{\infty} i p_{=i} = \sum_{i=1}^{\infty} i (p_{\geq i} - p_{\geq i+1}) = k + \rho_4(k)$, where $|\rho_4(k)| < k^2 2^{8-k/4}$.

Proof.

$$\sum_{i=1}^{\infty} i p_{=i} = \sum_{i=0}^{k/2} i p_{=i} + \sum_{i=k/2+1}^{k+k/2} i p_{=i} + \sum_{i=k+k/2+1}^{\infty} i p_{=i}$$

Using Lemma 3, $\sum_{i=0}^{k/2} i p_{=i} < \frac{k(k/2+1)}{2} 2^{4-k/2} < k^2 2^{2-k/2}$,

$\sum_{i=k+k/2+1}^{\infty} i p_{=i} < \sum_{i=k+k/2+1}^{\infty} i 2^{4-i+k} < 2^{4+k} \frac{2^{-k-k/2-1}}{1-1/2} (k + k/2 + 1 + 1) < k 2^{5-k/2}$ because $\sum_{i=a}^{b} i q^i \leq \frac{q^a}{1-q} \left(a + \frac{q}{1-q}\right)$ for any $0 < a < b$, $0 \leq q < 1$. Also,

$\sum_{i=k/2+1}^{k+k/2} i p_{=i} = k + \rho(k)$, where $\rho(k) < k^2 2^{6-k/4}$.

Adding all three sums together, we obtain $q_{\max}(k) = k + \rho_4(k)$, where $\rho_4(k) < k^2 2^{8-k/4}$. □