Wavelet Transform Theory

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What is a Wavelet Transform?

- Decomposition of a signal into constituent parts
- Note that there are many ways to do this. Some are:
  - Fourier series: harmonic sinusoids; single integer index
  - Fourier transform (FT): nonharmonic sinusoids; single real index
  - Walsh decomposition: “harmonic” square waves; single integer index
  - Karhunen-Loeve decomp: eigenfunctions of covariance; single real index
  - Short-Time FT (STFT): windowed, nonharmonic sinusoids; double index
    - provides time-frequency viewpoint
  - Wavelet Transform: time-compacted waves; double index
- Wavelet transform also provides time-frequency view
  - Decomposes signal in terms of duration-limited, band-pass components
    - high-frequency components are short-duration, wide-band
    - low-frequency components are longer-duration, narrow-band
  - Can provide combo of good time-frequency localization and orthogonality
    - the STFT can’t do this
  - More precisely, wavelets give time-scale viewpoint
    - this is connected to the multi-resolution viewpoint of wavelets

General Characteristics of Wavelet Systems

- Signal decomposition: build signals from “building blocks”, where the building blocks (i.e. basis functions) are doubly indexed.
- The components of the decomposition (i.e. the basis functions) are localized in time-frequency
  - ON can be achieved w/o sacrificing t-f localization
- The coefficients of the decomposition can be computed efficiently (e.g., using $O(N)$ operations).

Specific Characteristics of Wavelet Systems

- Basis functions are generated from a single wavelet or scaling function by scaling and translation
- Exhibit multiresolution characteristics: dilating the scaling functions provides a higher resolution space that includes the original
- Lower resolution coefficients can be computed from higher resolution coefficients through a filter bank structure

Fourier Development vs. Wavelet Development

- Fourier and others:
  - expansion functions are chosen, then properties of transform are found
- Wavelets
  - desired properties are mathematical imposed
  - the needed expansion functions are then derived
- Why are there so many different wavelets
  - the basic desired property constraints don’t use all the degrees of freedom
  - remaining degrees of freedom are used to achieve secondary properties
    - these secondary properties are usually application-specific
    - the primary properties are generally application-nonspecific
- What kinds of signals are wavelets and Fourier good for?
  - Wavelets are good for transients
    - localization property allows wavelets to give efficient rep. of transients
  - Fourier is good for periodic or stationary signals

Why are Wavelets Effective?

- Provide unconditional basis for large signal class
  - wavelet coefficients drop-off rapidly
  - thus, good for compression, denoising, detection/recognition
  - goal of any expansion is
    - have the coefficients provide more info about signal than time-domain
    - have most of the coefficients be very small (***sparse*** representation)
  - FT is not sparse for transients
- Accurate local description and separation of signal characteristics
  - Fourier puts localization info in the phase in a complicated way
  - STFT can't give localization ***and*** orthogonality
- Wavelets can be adjusted or adapted to application
  - remaining degrees of freedom are used to achieve goals
- Computation of wavelet coefficient is well-suited to computer
  - no derivatives of integrals needed
  - turns out to be a digital filter bank

Multiresolution Viewpoint

Multiresolution Approach

- Stems from image processing field
  - consider finer and finer approximations to an image
- Define a nested set of signal spaces

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2$$

- We build these spaces as follows:
- Let V_0 be the space spanned by the integer translations of a fundamental signal $\phi(t)$, called the scaling function:

that is, ***if*** f(t) is in V_0 ***then*** it can be represented by:

$$f(t) = \sum_k a_k \phi(t-k)$$

- So far we can use just about any function $\phi(t)$, but we'll see that to get the nesting only certain scaling functions can be used.

Multiresolution Analysis (MRA) Equation

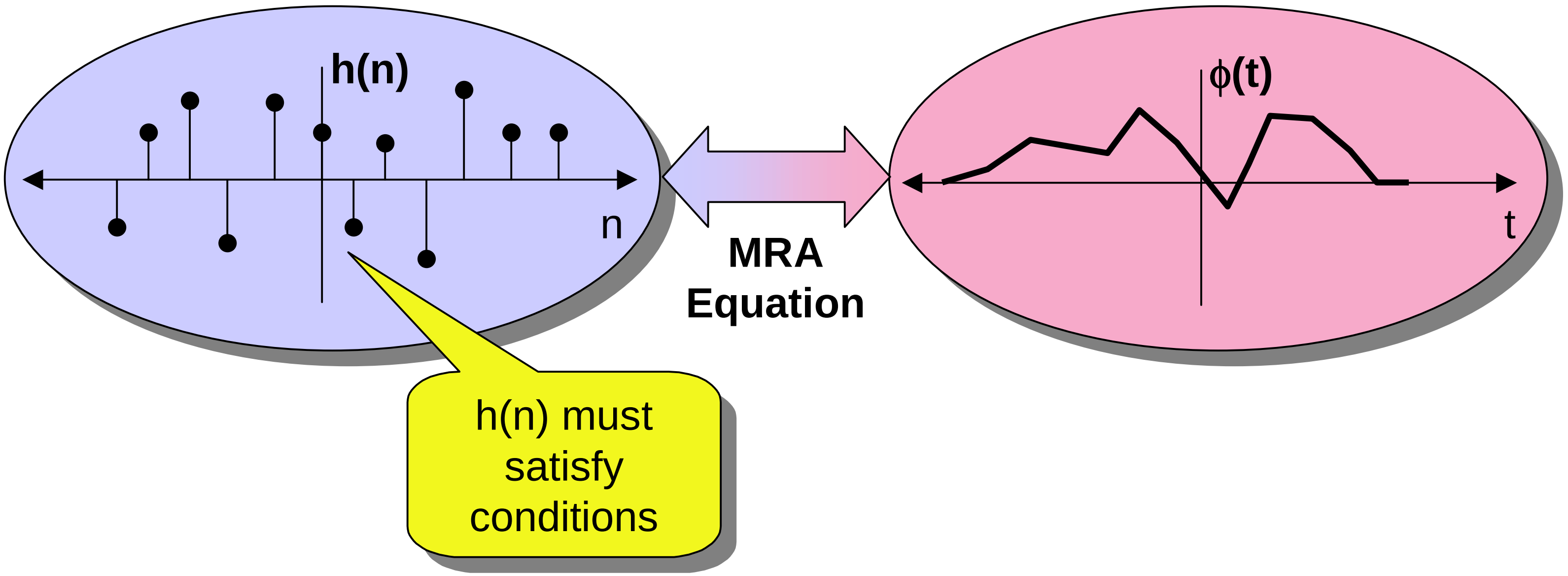
- Now that we have V_0 how do we make the others and ensure that they are nested?
- If we let V_1 be the space spanned by integer translates of $\phi(2t)$ we get the desired property that V_1 is indeed a space of functions having higher resolution.
- Now how do we get the nesting?
- We need that any function in V_0 also be in V_1; in particular we need that the scaling function (which is in V_0) be in V_1, which the requires that

$$\phi(t) = \sum_n h(n)\sqrt{2}\phi(2t - n)$$

  where the expansion coefficient is $h(n)\sqrt{2}$
- This is the requirement on the scaling function to ensure nesting: it must satisfy this equation
  - called the multiresolution analysis (MRA) equation
  - this is like a differential equation that the scaling function is the solution to

The h(n) Specify the Scaling Function

- Thus, the coefficients h(n) determine the scaling function
  - for a given set of h(n), $\phi(t)$
    - may or may not exist
    - may or may not be unique
- Want to find conditions on h(n) for $\phi(t)$ to exist and be unique, and also:
  - to be **orthogonal** (because that leads to an ON wavelet expansion)
  - to give wavelets that have **desirable properties**



Whence the Wavelets?

- The spaces V_j represent increasingly higher resolution spaces
- To go from V_j to higher resolution V_{j+1} requires the addition of "details"
  - These details are the part of V_{j+1} not able to be represented in V_j
  - This can be captured through the "orthogonal complement of V_j w.r.t V_{j+1}
- Call this orthogonal complement space W_j
  - all functions in W_j are orthogonal to all functions in V_j
  - That is:

$$<\phi_{j,k}(t), \psi_{j,l}(t)> = \int \phi_{j,k}(t)\psi_{j,l}(t)dt = 0 \quad \forall j,k,l \in \mathbf{Z}$$

- Consider that V_0 is the lowest resolution of interest
- How do we characterize the space W_0 ?
  - we need to find an ON basis for W_0, say $\{\psi_{0,k}(t)\}$ where the basis functions arise from translating a single function (we'll worry about the scaling part later):

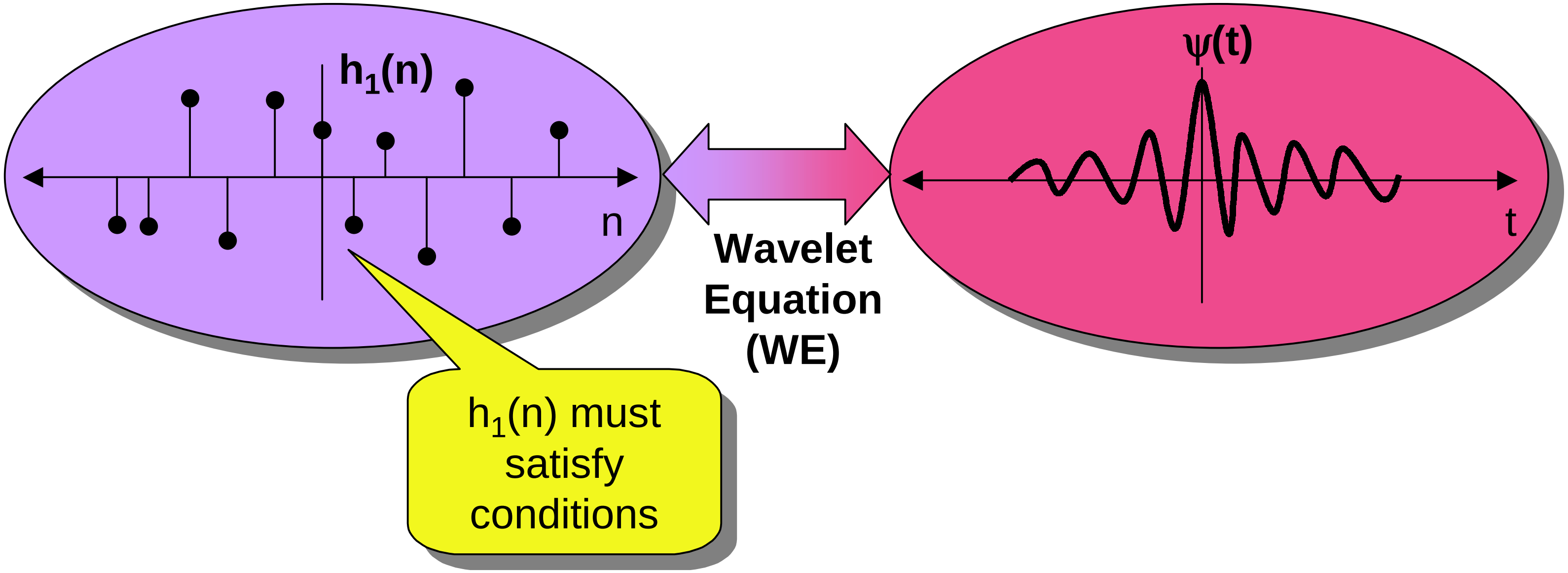
$$\psi_{0,k}(t) = \psi(t-k)$$

Finding the Wavelets

- The wavelets are the basis functions for the W_j spaces
  - thus, they lie in V_{j+1}
- In particular, the function $\psi(t)$ lies in the space V_1 so it can be expanded as

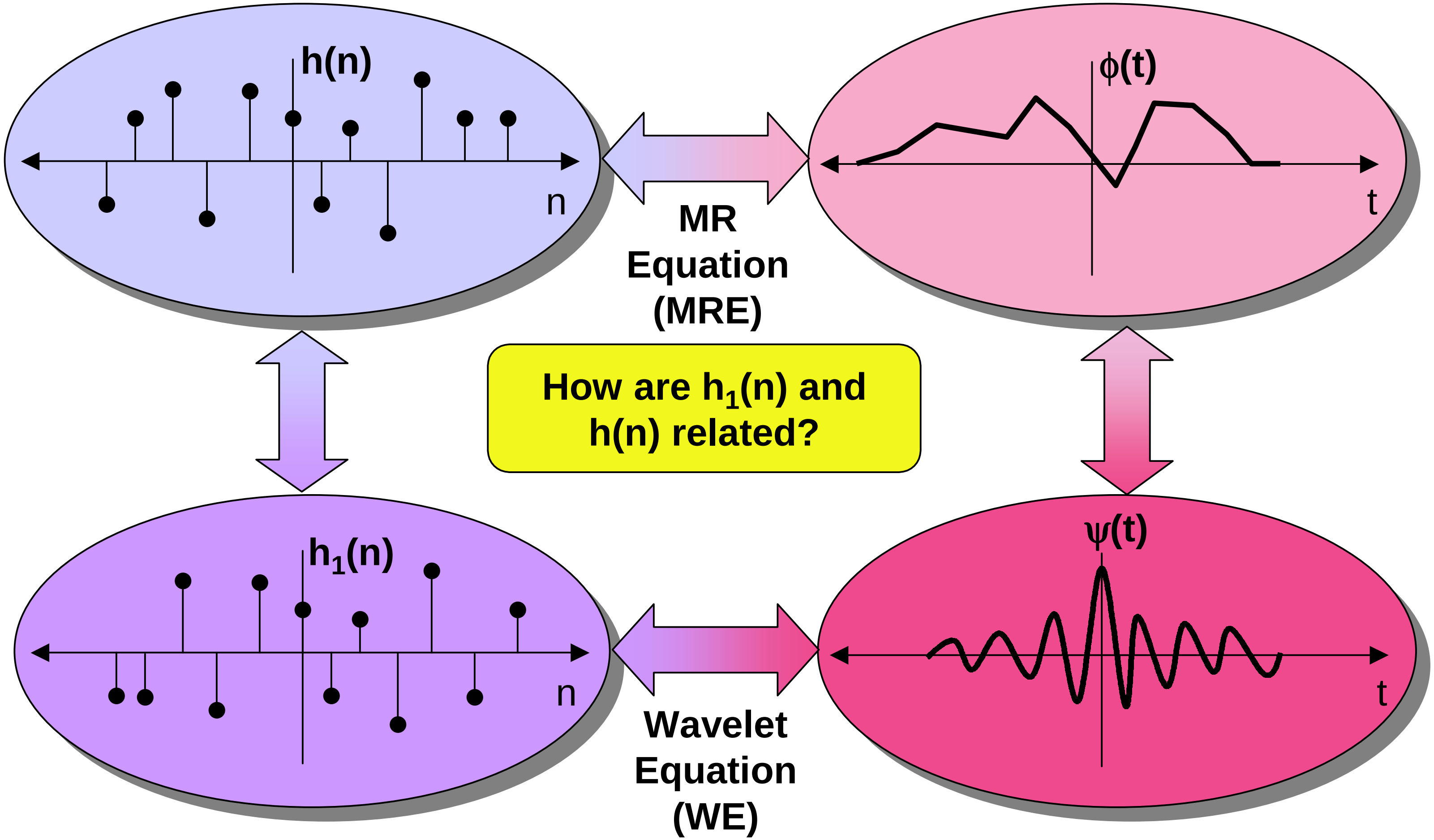
$$\psi(t) = \sum_n h_1(n)\sqrt{2}\phi(2t-n), \quad n \in \mathbf{Z}$$

- This is a fundamental result linking the scaling function and the wavelet
  - the $h_1(n)$ specify the wavelet, via the specified scaling function



Wavelet-Scaling Function Connection

- There is a fundamental connection between the scaling function and its coefficients h(n) , the wavelet function and its coefficients $h_1(n)$:



Relationship Between $h_1(n)$ and h(n)

- We state here the conditions for the important special case of
  - finite number N of nonzero h(n)
  - ON within V_0: $\int \phi(t)\phi(t-k)dt = \delta(k)$
  - ON between V_0 and W_0 : $\int \psi(t)\phi(t-k)dt = \delta(k)$
- Given the h(n) that define the desired scaling function, then the $h_1(n)$ that define the wavelet function are given by

$$h_1(n) = (-1)^n h(N-1-n)$$

- Much of wavelet theory addresses the origin, characteristics, and ramifications of this relationship between $h_1(n)$ and h(n)
  - requirements on h(n) and $h_1(n)$ to achieve ON expansions
  - how the MRE and WE lead to a filter bank structure
  - requirements on h(n) and $h_1(n)$ to achieve other desired properties
  - extensions beyond the ON case

The Resulting Expansions

- Let f(t) be in $L^2(R)$
- There are three ways of interest that we can expand f(t)

1 We can give an limited resolution approximation to f(t) via

$$f_j(t) = \sum_k a_k 2^{j/2} \phi(2^j t - k)$$

- increasing j gives a better (i.e., higher resolution) approximation

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2$$

- this is in general not the most useful expansion

The Resulting Expansions (cont.)

2 A low-resolution approximation plus its wavelet details

$$f(t) = \underbrace{\sum_k c_{j_0}(k) 2^{j_0/2} \phi(2^{j_0} t - k)}_{\text{Low-Resolution Approximation}} + \underbrace{\sum_k \sum_{j=j_0}^{\infty} d_j(k) 2^{j/2} \psi(2^j t - k)}_{\text{Wavelet Details}}$$

– Choosing j_0 sets the level of the coarse approximation

$$L^2 = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus W_{j_0+2} \oplus \cdots$$

– This is most useful in practice: j_0 is usually chosen according to application

- Also in practice, the upper value of j is chosen to be finite

The Resulting Expansions (cont.)

3 Only the wavelet details

$$f(t) = \sum_{k} \sum_{j=-\infty}^{\infty} d_j(k) 2^{j/2} \psi(2^j t - k)$$

– Choosing j_0=-∞ eliminates the coarse approximation leaving only details

$$L^2 = \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

– This is most similar to the “true” wavelet decomposition as it was originally developed

– This is not that useful in practice: j_0 is usually chosen to be finite according to application

The Expansion Coefficients $c_{j_0}(k)$ and $d_j(k)$

- We consider here only the simple, but important, case of ON expansion
  - i.e., the ϕ's are ON, the ψ's are ON, ***and*** the ϕ's are ON to the ψ's
- Then we can use standard ON expansion theory:

$$c_{j_0}(k) = \left\langle f(t), \varphi_{j_0,k}(t) \right\rangle = \int f(t)\varphi_{j_0,k}(t)\,dt$$

$$d_j(k) = \left\langle f(t), \psi_{j,k}(t) \right\rangle = \int f(t)\psi_{j,k}(t)\,dt$$

- We will see how to compute these without resorting to computing inner products
  - we will use the coefficients $h_1(n)$ and h(n) instead of the wavelet and scaling function, respectively
  - we look at a relationship between the expansion coefficients at one level and those at the next level of resolution

Summary of Multiresolution View

- Nested Resolution spaces:

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2$$

- Wavelet Spaces provide orthogonal complement between resolutions

$$L^2 = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus W_{j_0+2} \oplus \cdots$$

- Wavelet Series Expansion of a continuous-time signal f(t):

$$f(t) = \sum_k c_{j_0}(k) 2^{j_0/2} \phi(2^{j_0} t - k) + \sum_k \sum_{j=j_0}^{\infty} d_j(k) 2^{j/2} \psi(2^j t - k)$$

- MR equation (MRE) provides link between the scaling functions at successive levels of resolution:

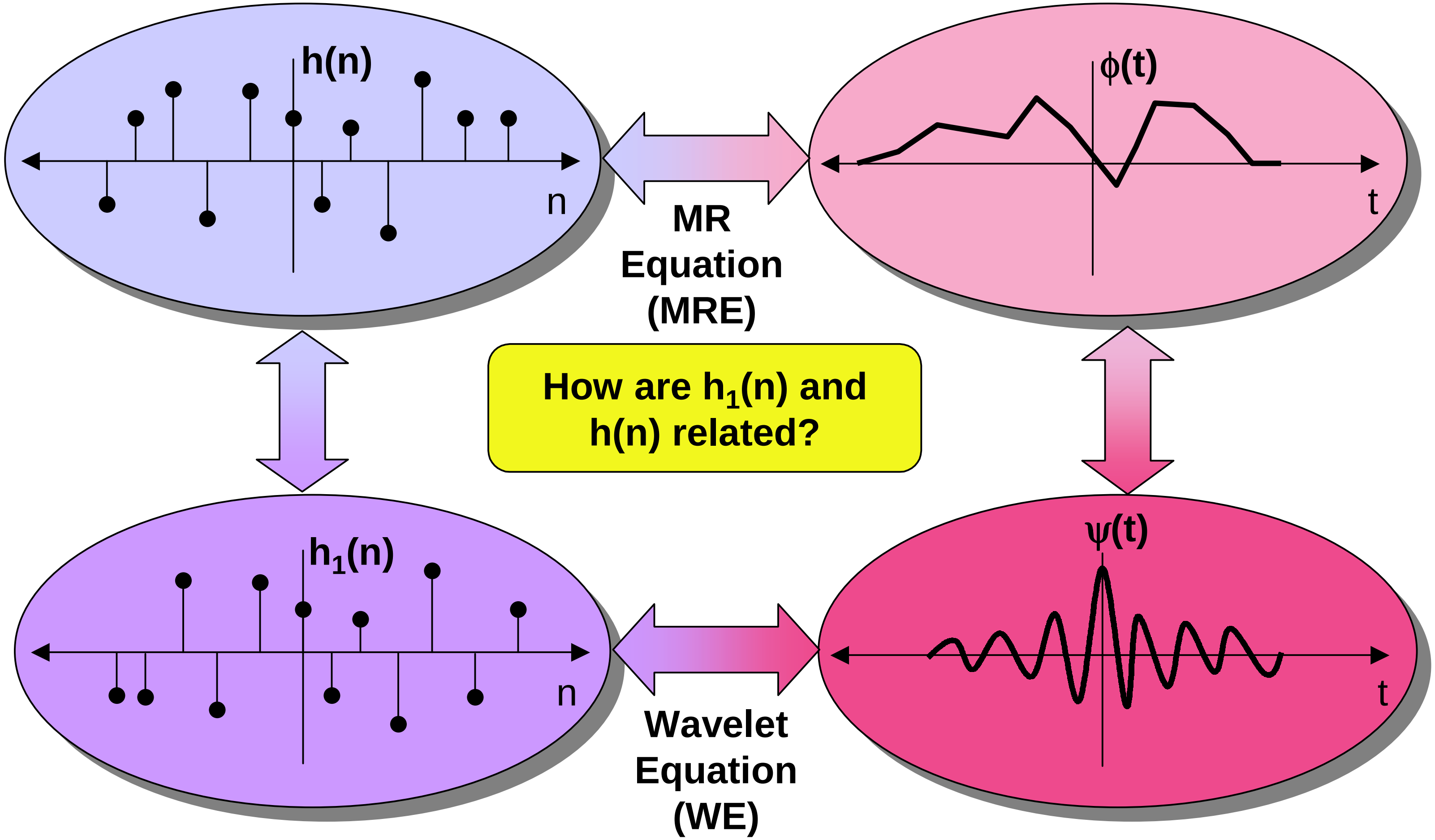
$$\phi(t) = \sum_n h(n) \sqrt{2} \phi(2t - n), \quad n \in \mathbf{Z}$$

- Wavelet equation (WE) provides link between a resolution level and its complement

$$\psi(t) = \sum_n h_1(n) \sqrt{2} \phi(2t - n), \quad n \in \mathbf{Z}$$

Summary of Multiresolution View (cont.)

- There is a fundamental connection between the scaling function and its coefficients h(n) , the wavelet function and its coefficients $h_1(n)$:



Filter Banks and DWT

Generalizing the MRE and WE

- Here again are the MRE and the WE:

$$\phi(t)=\sum_n h(n)\sqrt{2}\phi(2t-n) \qquad \psi(t)=\sum_n h_1(n)\sqrt{2}\phi(2t-n)$$

scale & translate: replace $t \to 2^j t - k$

- We get:

MRE

$$\phi(2^j t-k)=\sum_m h(m-2k)\sqrt{2}\phi(2^{j+1}t-m)$$

Connects V_j to V_{j+1}

WE

$$\psi(2^j t-k)=\sum_m h_1(m-2k)\sqrt{2}\phi(2^{j+1}t-m)$$

Connects W_j to V_{j+1}

Linking Expansion Coefficients Between Scales

- Start with the Generalized MRA and WE:

$$\phi(2^j t - k) = \sum_m h(m-2k)\sqrt{2}\phi(2^{j+1}t - m) \qquad \psi(2^j t - k) = \sum_m h_1(m-2k)\sqrt{2}\phi(2^{j+1}t - m)$$

$$c_j(k) = \left\langle f(t), \varphi_{j,k}(t) \right\rangle \qquad d_j(k) = \left\langle f(t), \psi_{j,k}(t) \right\rangle$$

$$c_j(k) = \sum_m h(m-2k)\left\langle f(t), 2^{(j+1)/2}\varphi(2^{j+1}t - m) \right\rangle \qquad d_j(k) = \sum_m h_1(m-2k)\left\langle f(t), 2^{(j+1)/2}\varphi(2^{j+1}t - m) \right\rangle$$

$$c_{j+1}(m)$$

$$c_j(k) = \sum_m h(m-2k)c_{j+1}(m) \qquad d_j(k) = \sum_m h_1(m-2k)c_{j+1}(m)$$

Convolution-Decimation Structure

New Notation For Convenience: h(n) $\rightarrow$ h_0(n)

$$c_j(k) = \sum_m h_0(m-2k)c_{j+1}(m)$$

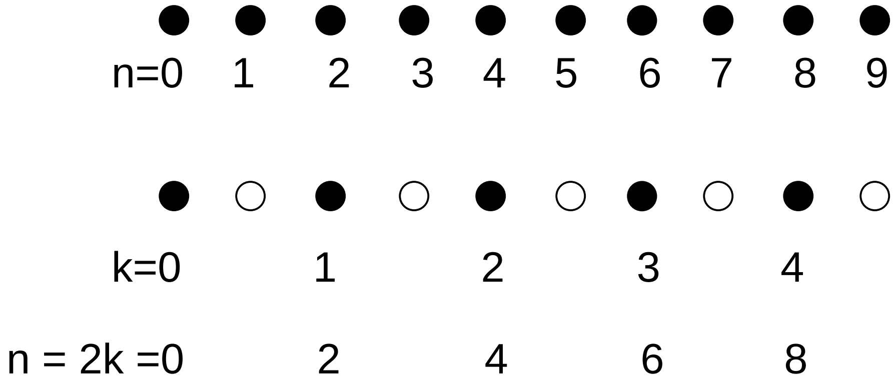
$$d_j(k) = \sum_m h_1(m-2k)c_{j+1}(m)$$

Convolution

$$y_0(n) = c_{j+1}(n) * h_0(-n) = \sum_m h_0(m-n)c_{j+1}(m)$$

$$y_1(n) = c_{j+1}(n) * h_1(-n) = \sum_m h_1(m-n)c_{j+1}(m)$$

Decimation



Summary of Progression to Convolution-Decimation Structure

MRE

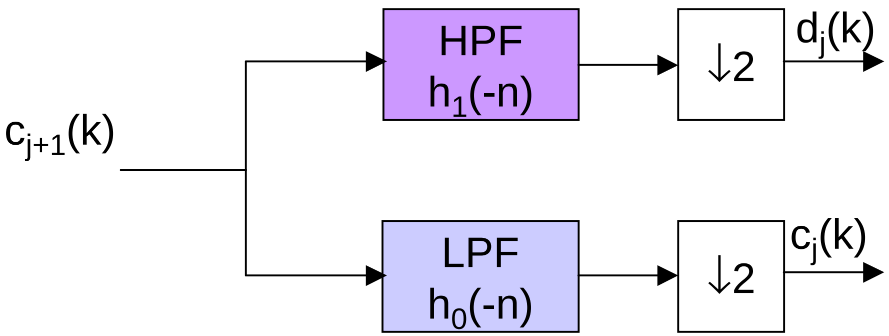
$$\phi(2^j t - k) = \sum_m h(m-2k)\sqrt{2}\phi(2^{j+1}t - m)$$

$$c_j(k) = \sum_m h(m-2k)c_{j+1}(m)$$

WE

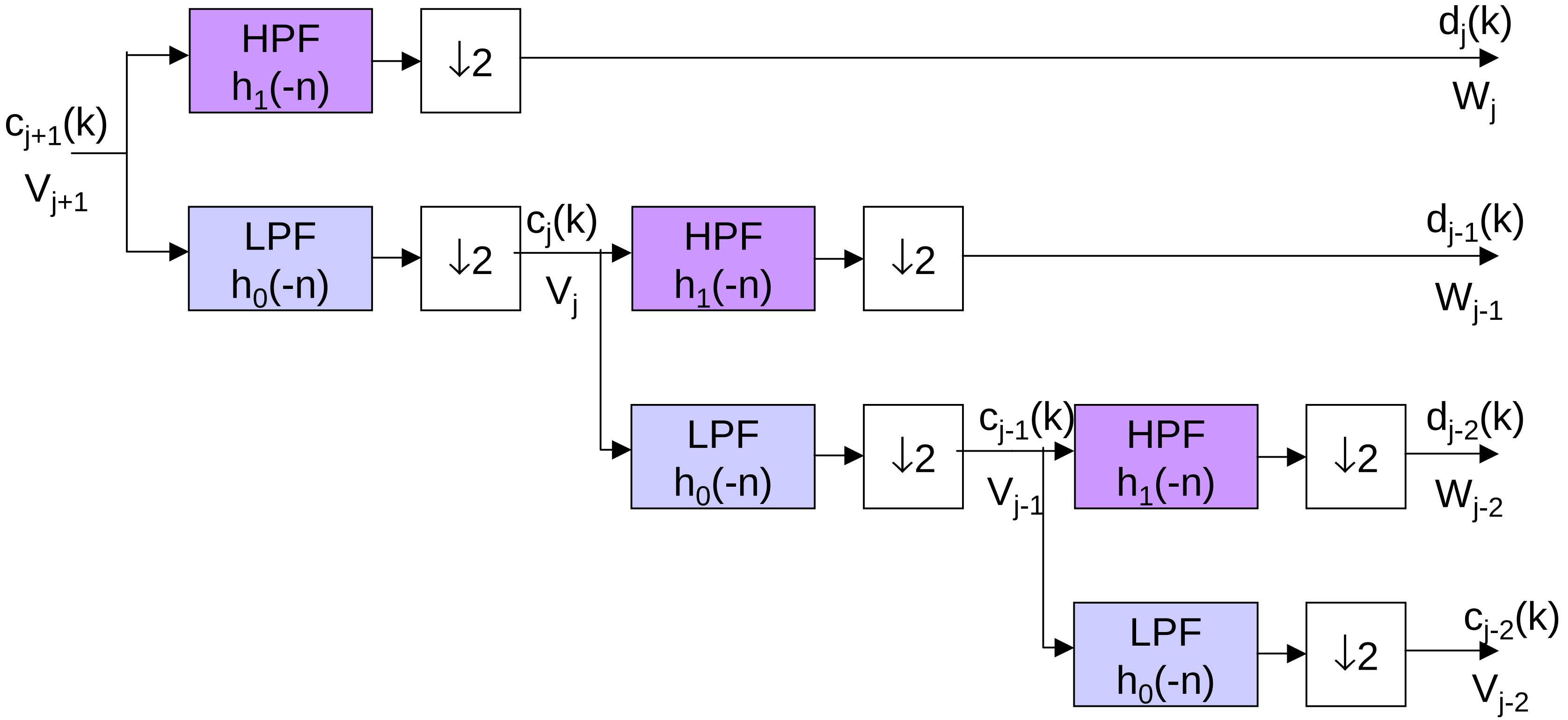
$$\psi(2^j t - k) = \sum_m h_1(m-2k)\sqrt{2}\phi(2^{j+1}t - m)$$

$$d_j(k) = \sum_m h_1(m-2k)c_{j+1}(m)$$

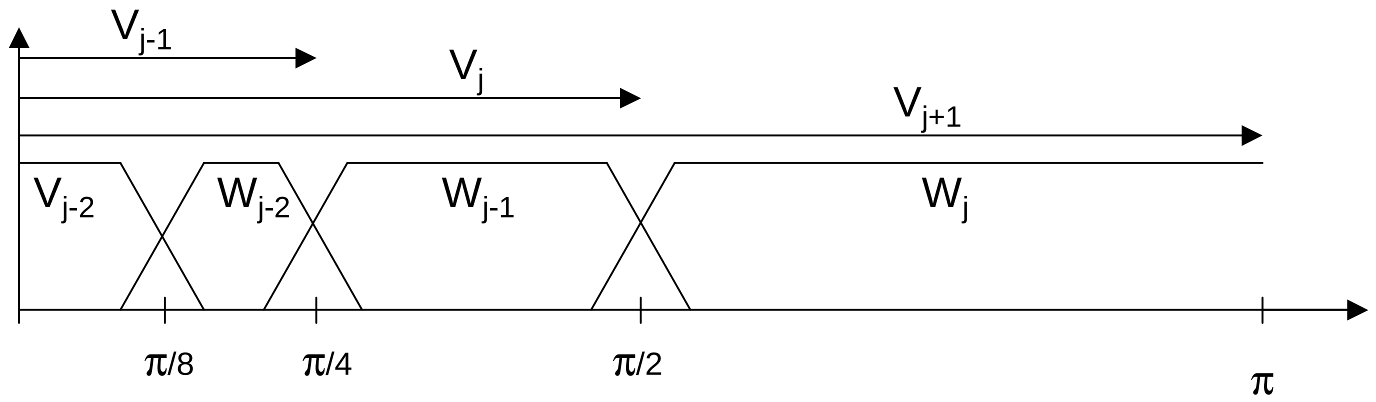


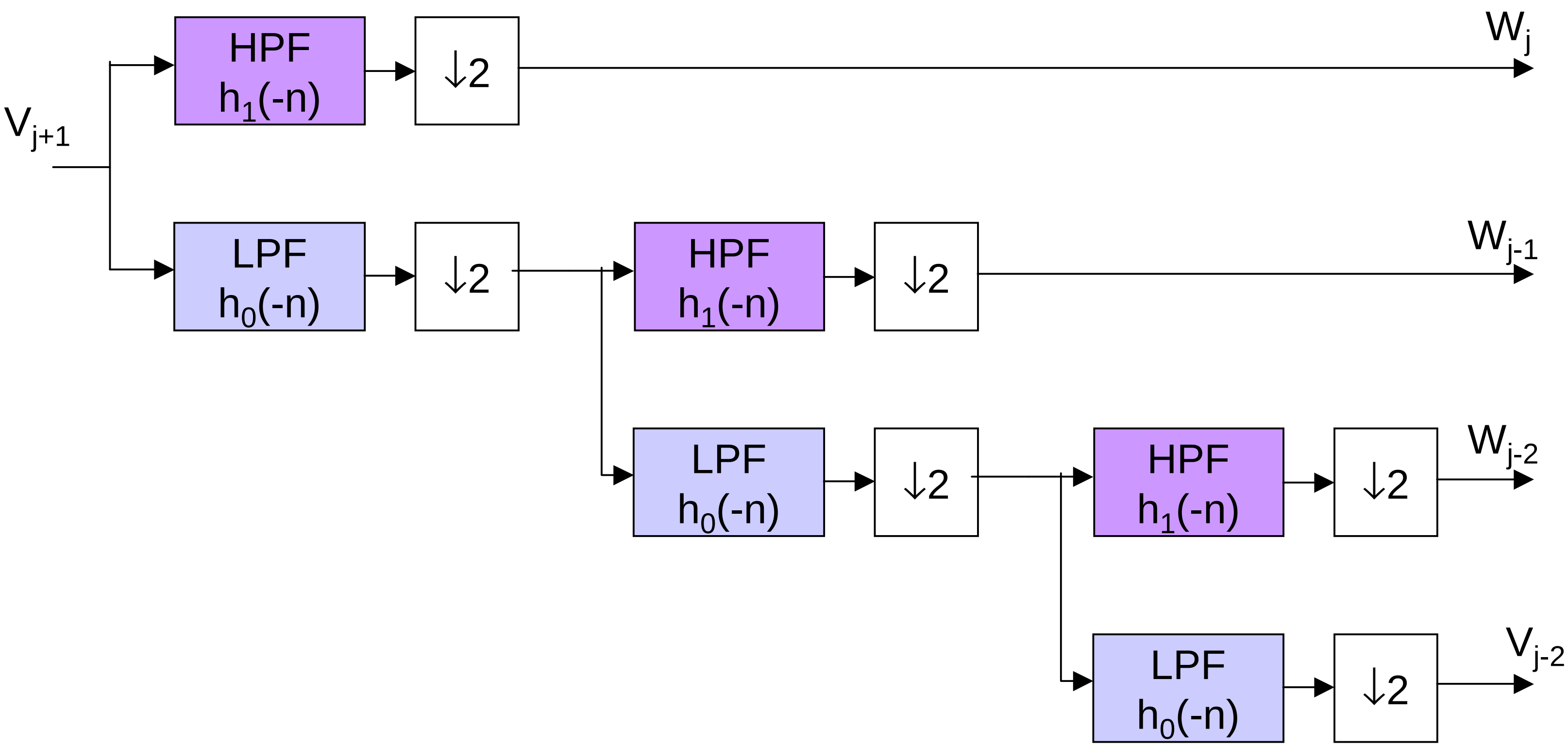
Computing The Expansion Coefficients

- The above structure can be cascaded:
  - given the scaling function coefficients at a specified level all the lower resolution c's and d's can be computed using the filter structure

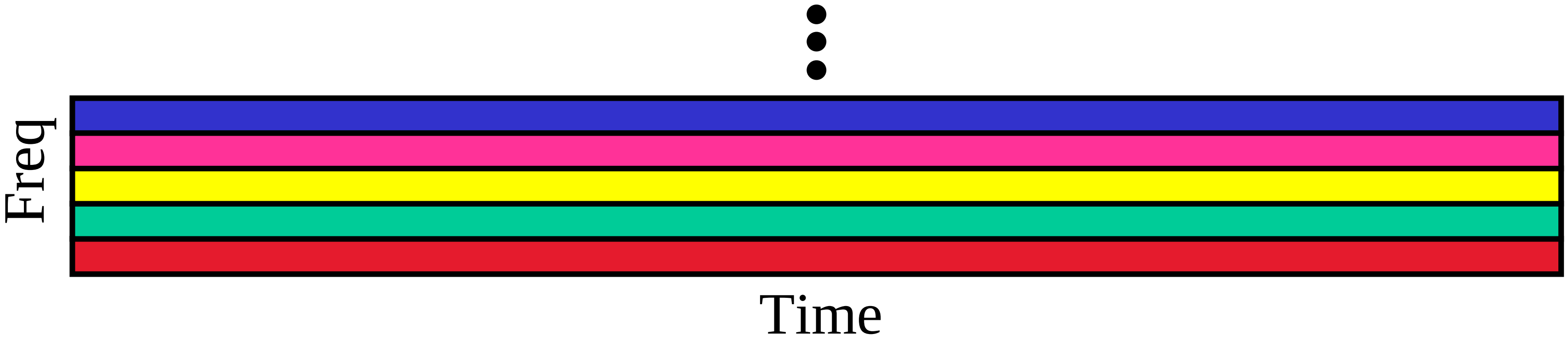


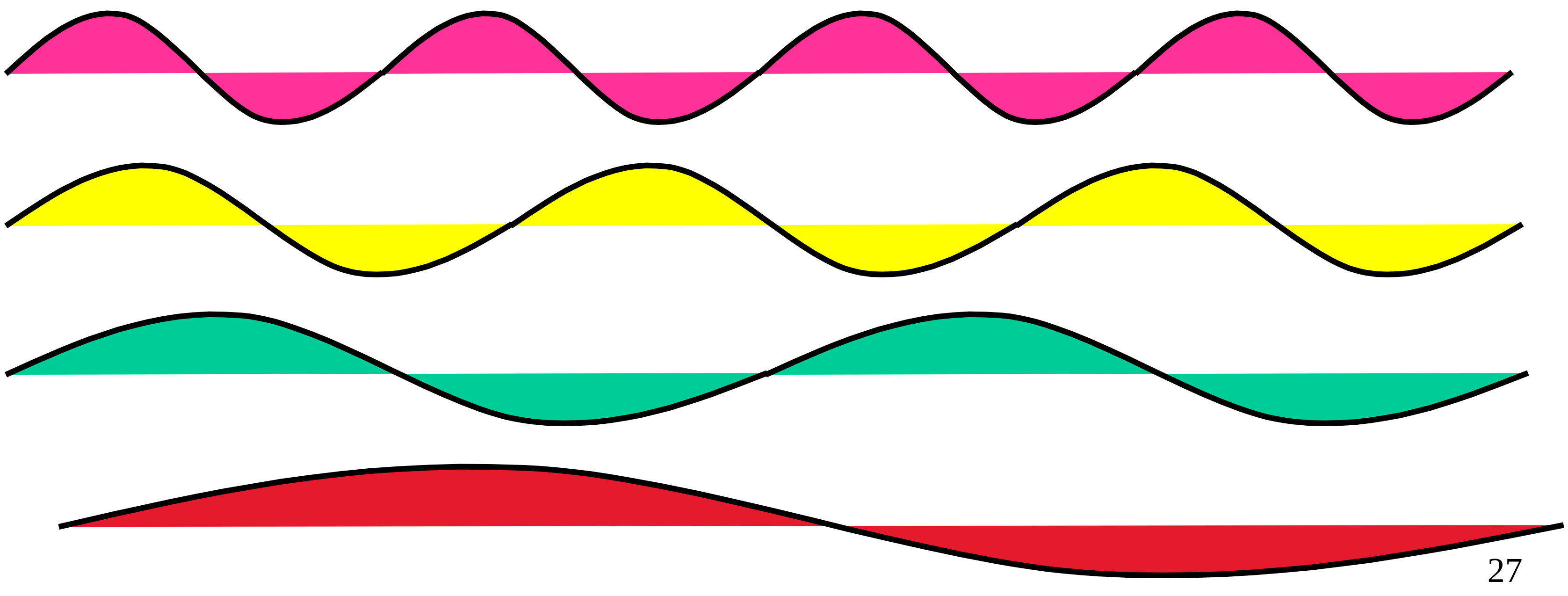
Filter Bank Generation of the Spaces



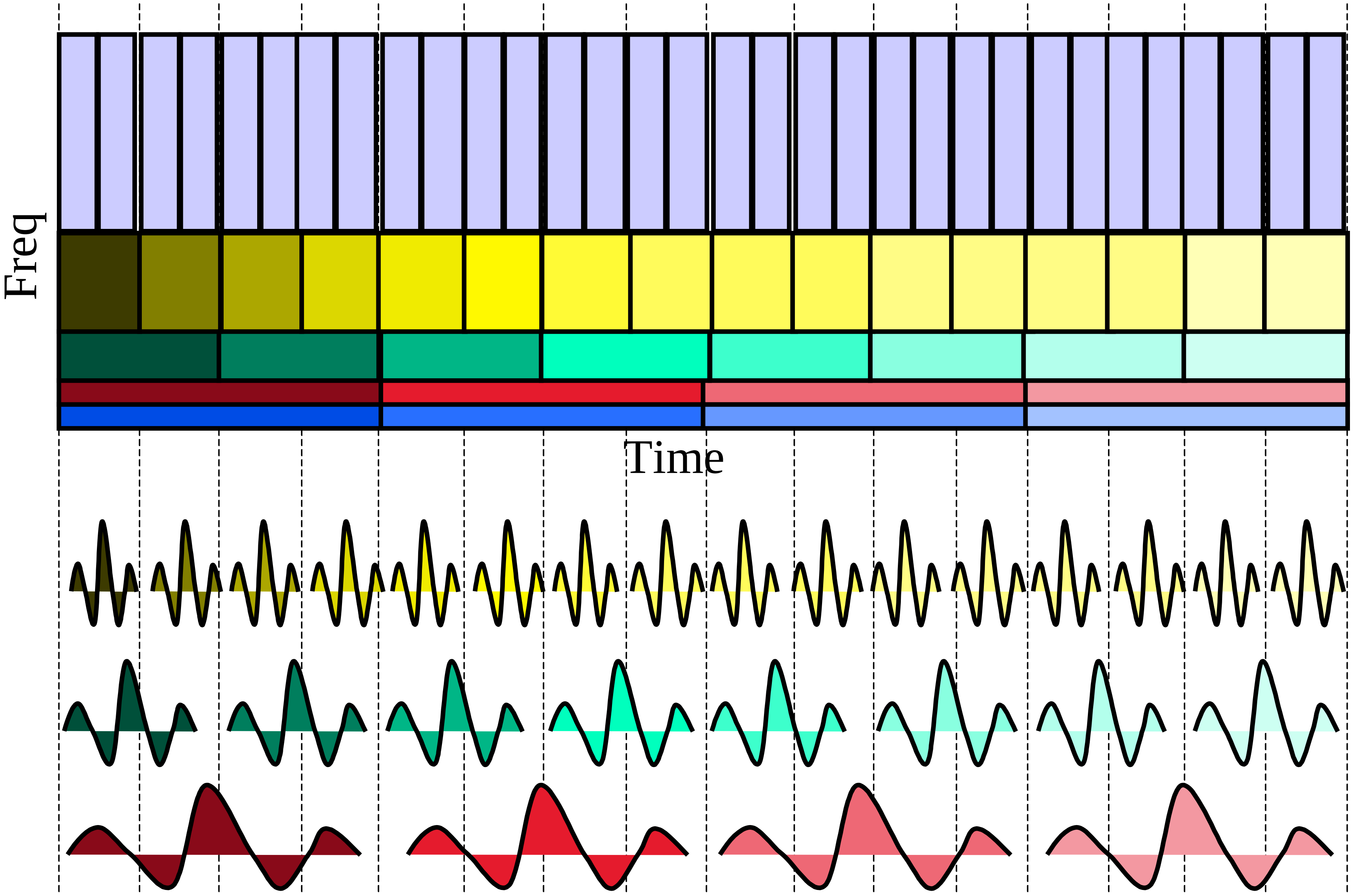


Discrete Fourier Transform

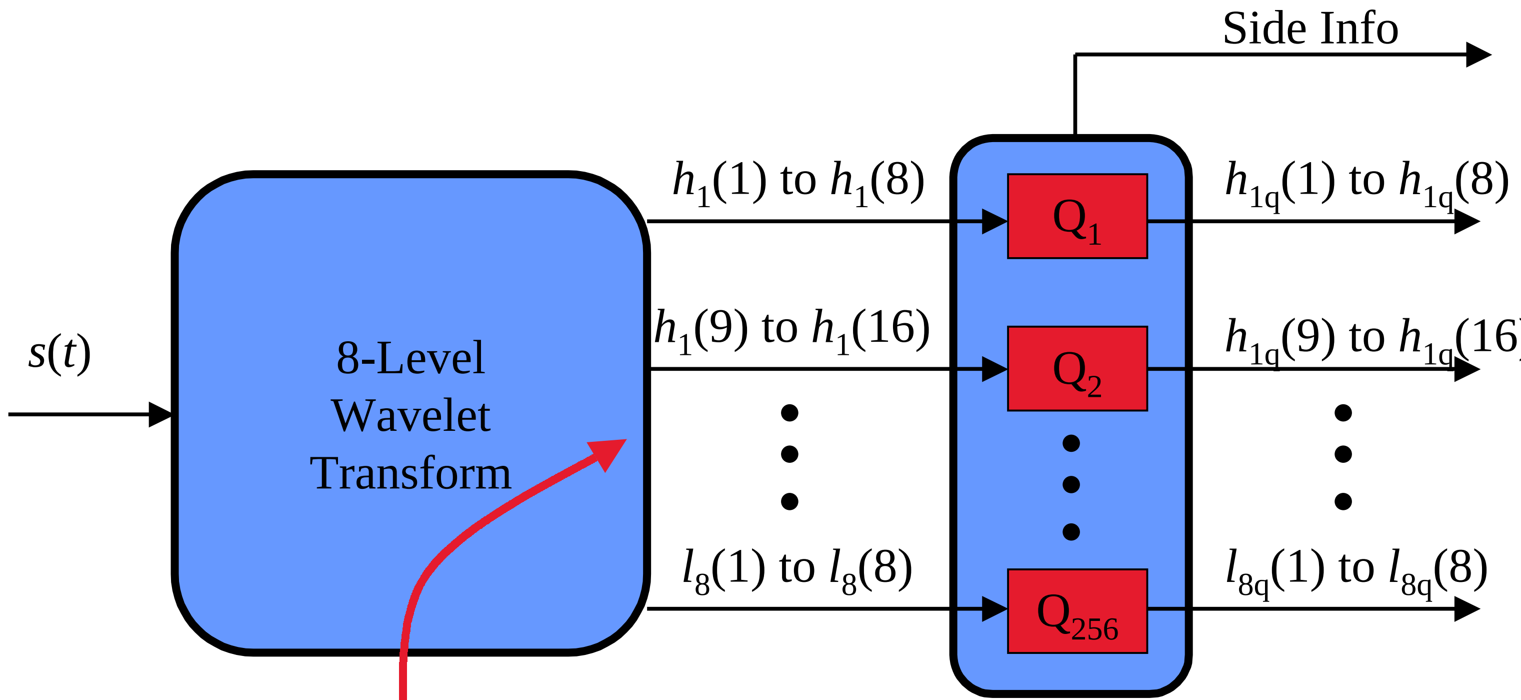


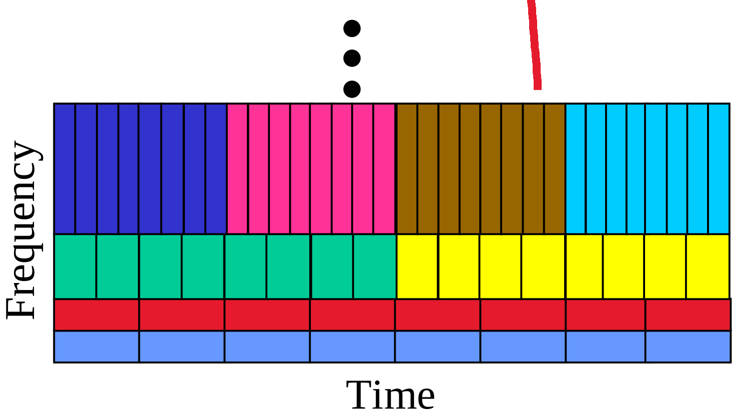


WAVELET TRANSFORM



WT-Based Compression Example





- Bits allocated to quantizers to minimize MSE
- Then allocations less than B_{min} are set to zero
  - » Eliminates negligible cells
- Side info sent to describe allocations