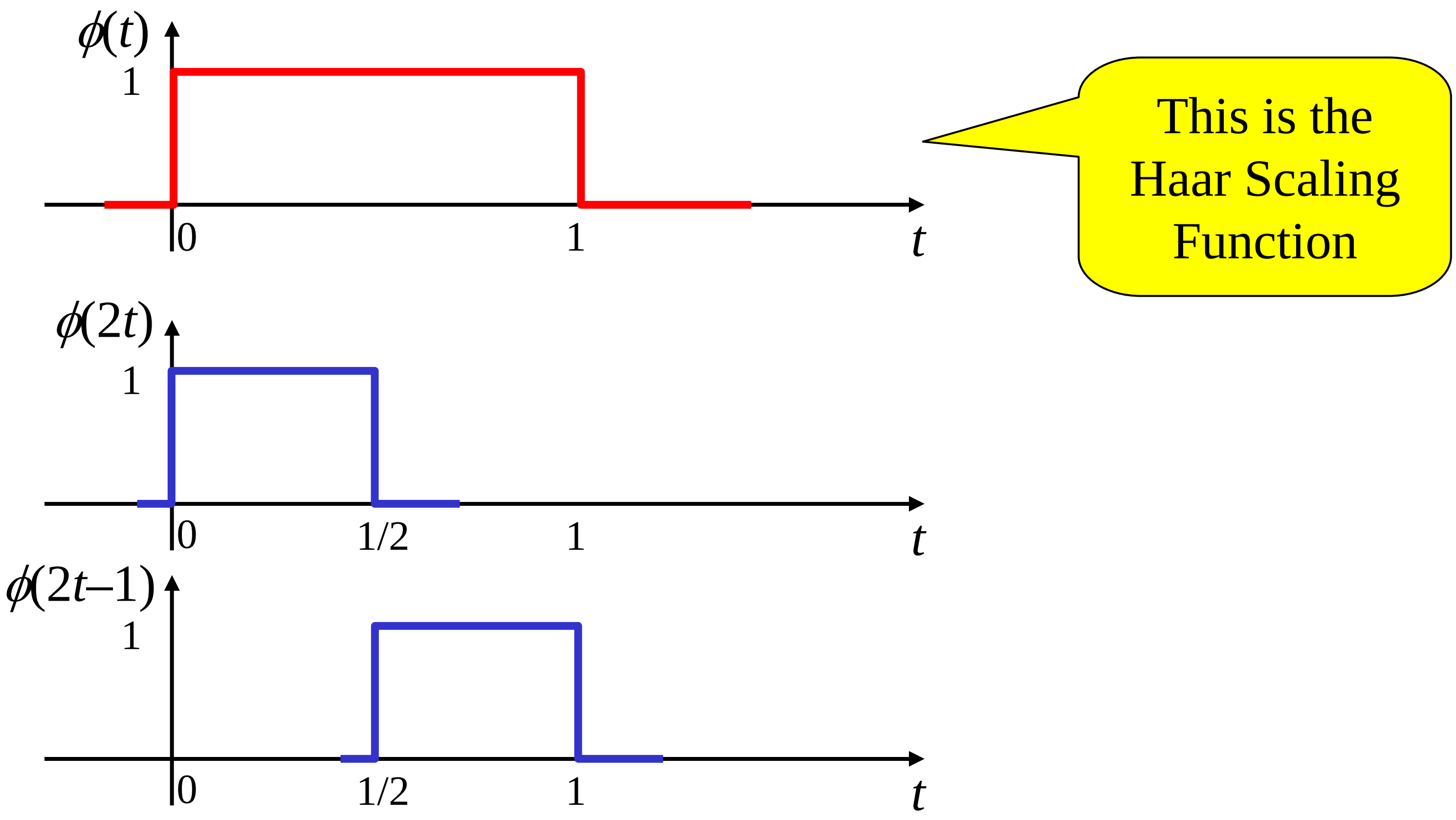
Wavelet Example: Haar Wavelet

Suppose we specify the MRE coefficients to be $h[n] = \left\{ \frac{1}{\sqrt{2}} \;,\; \frac{1}{\sqrt{2}} \right\}$

Then the MRE becomes $\phi(t) = \sum_n h(n)\sqrt{2}\phi(2t-n)$ ➡ $\varphi(t) = \varphi(2t) + \varphi(2t-1)$

Clearly the scaling function $\phi(t)$ as shown below satisfies this MRE



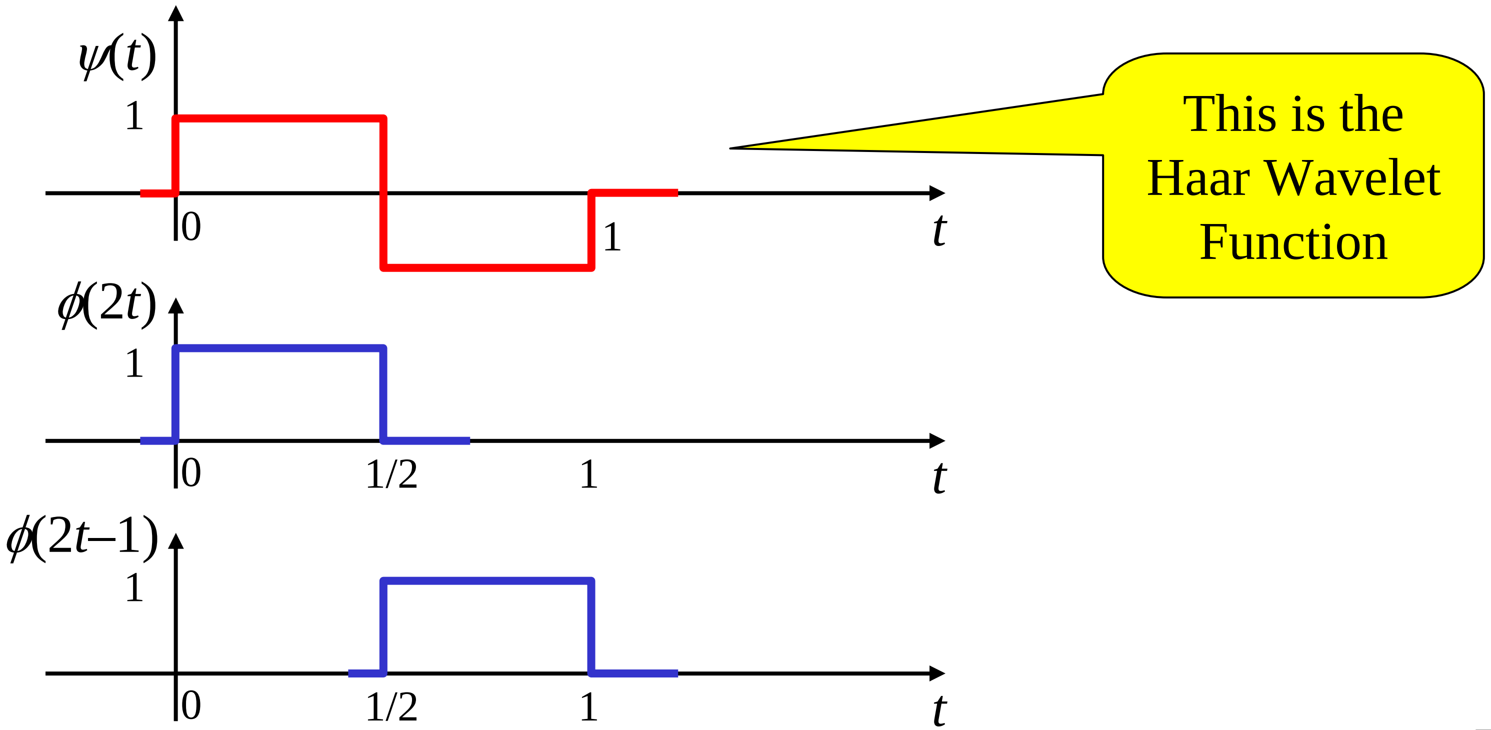
Next

- Special case: finite number N of nonzero $h(n)$ and ON wavelets & scaling functions
- Given the $h(n)$ for the scaling function, then the $h_1(n)$ that define the wavelet function are given by $h_1[n] = (-1)^n h(N-1-n)$ where N is the length of the filter

Thus the WE coefficients are $h_1[n] = \left\{ \frac{1}{\sqrt{2}} \; , \; -\frac{1}{\sqrt{2}} \right\}$

Then the WE becomes $\psi(t) = \sum_n h_1(n)\sqrt{2}\varphi(2t-n)$ ⇒ $\psi(t) = \varphi(2t) - \varphi(2t-1)$

Clearly the scaling function $\phi(t)$ as shown below satisfies this MRE



Next

Define a nested set of signal spaces

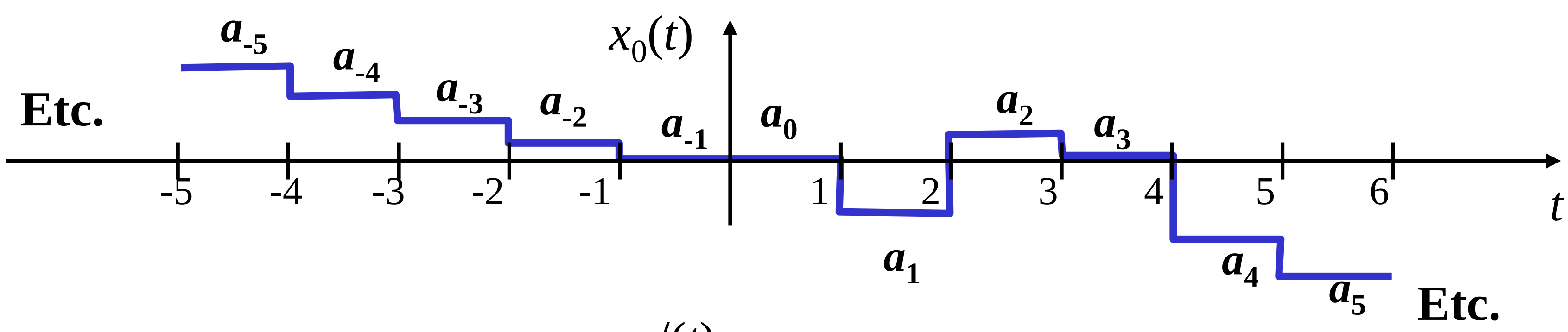
$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2$$

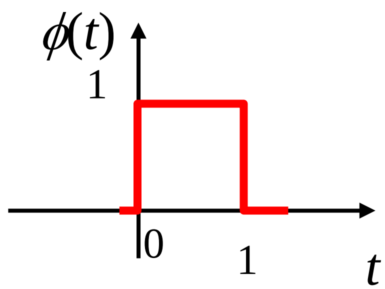
Let V_0 be the space spanned by the integer translations of scaling function $\phi(t)$ so that ***if*** $x_0(t)$ is in V_0 ***then*** it can be represented by:

$$x_0(t) = \sum_k a_k \varphi(t-k)$$

Q: For the Haar scaling function what kind of functions are in V_0??

A: Those that are "piece-wise" constant on the intervals $[k,k+1]$ for integer k…



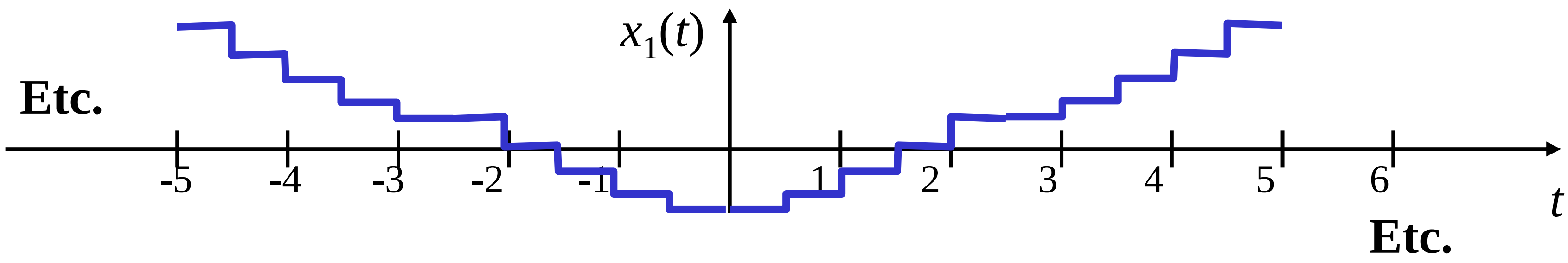


Next

If we let V_1 be the space spanned by integer translates of $\phi(2t)$ then V_1 is indeed a space of functions having higher resolution.

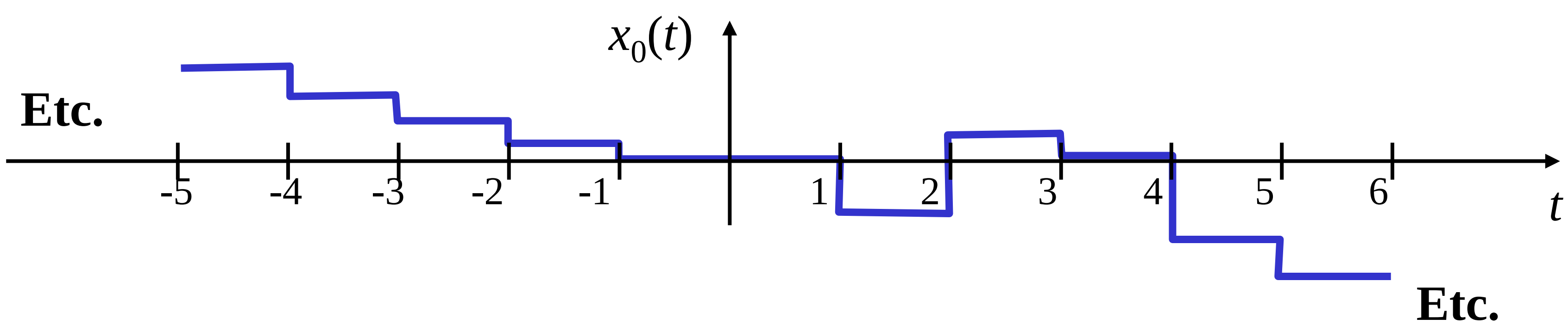
Q: For the Haar scaling function what kind of functions are in V_1??

A: Those that are "piece-wise" constant on the intervals $[k/2, k/2+ ½]$ for integer k



Note: $x_0(t)$ is also in V_1 because it is also "piece-wise" constant on $[k/2, k/2+ ½]$

In fact, $x_0(t)$ is also in every V_j for $j \geq 0$ … that is the nesting!!!



Next

If we keep going to higher j values we get finer and finer resolution and can ultimately express (in the limit of j) any finite energy signal

Increasing j

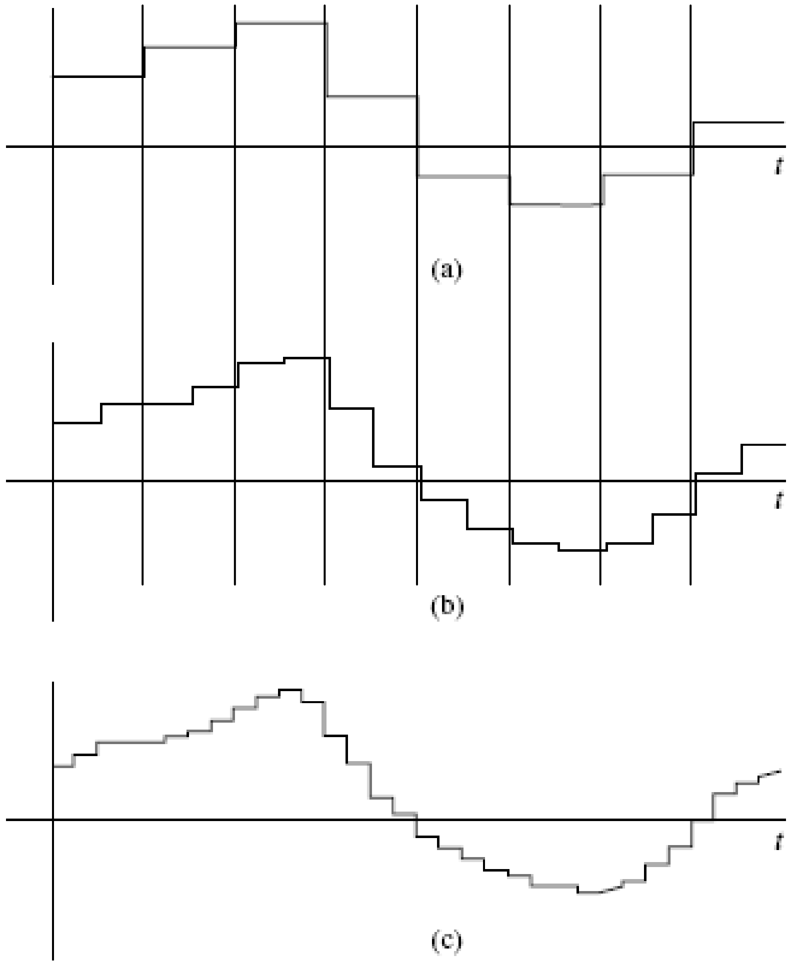


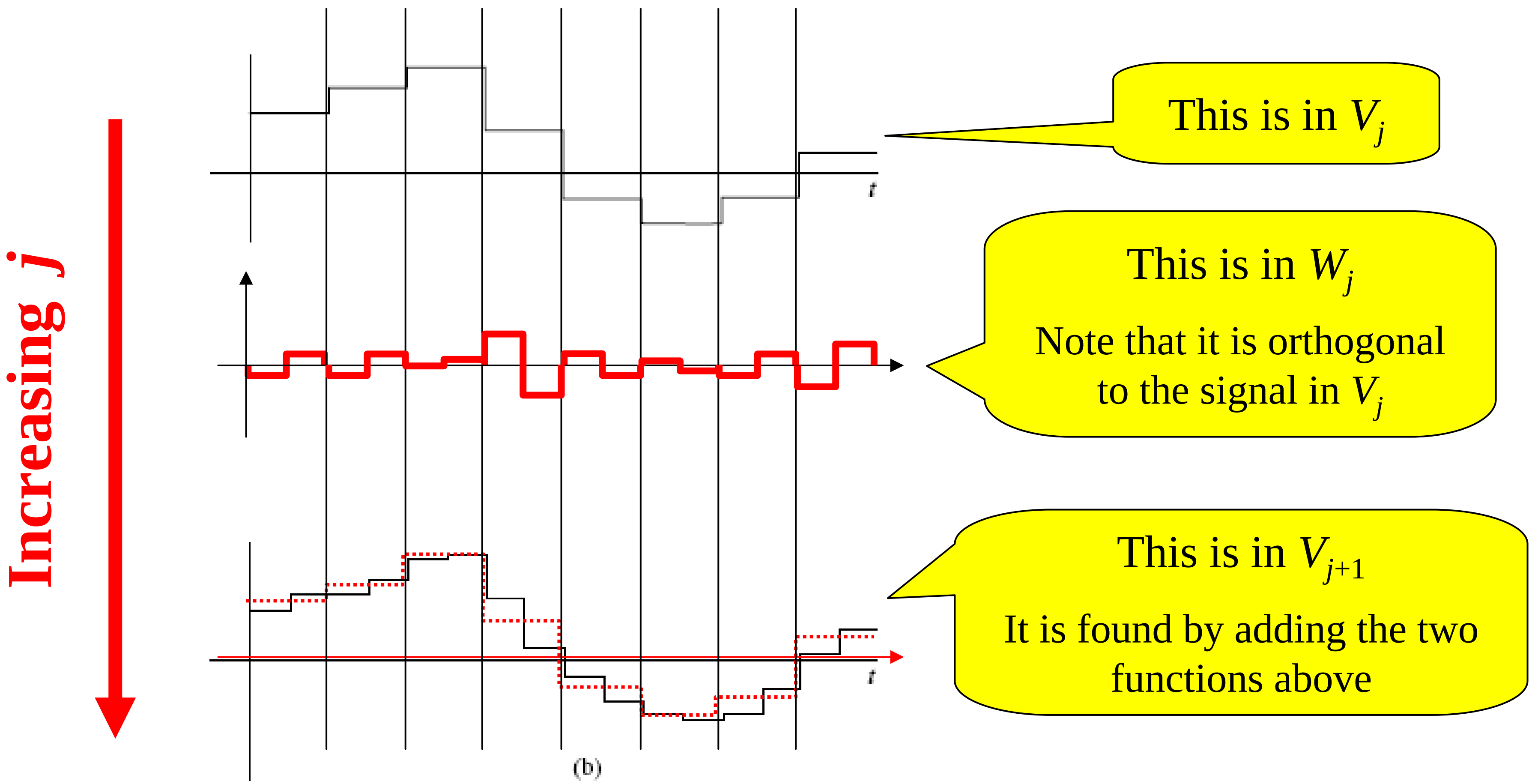
Figure 15.8 from Textbook

This MRA development started at V_0 and worked its way up to higher resolutions…

Next

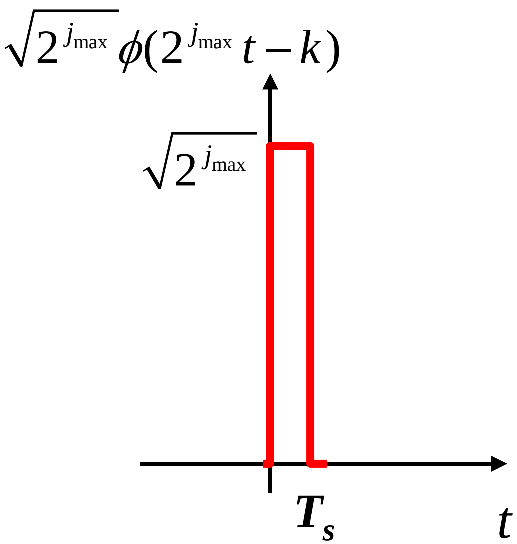
How do the wavelets enter into this?

- To go from V_j to higher resolution V_{j+1} requires the addition of "details"
  - These details are the part of V_{j+1} not able to be represented in V_j
  - This is captured through W_j the "orthogonal complement" of V_j w.r.t V_{j+1}



The filterbank viewpoint that the MRA analysis lead to starts from some high-level resolution and works down… so let's see how that works…

We'll start at the resolution level where the scaled version of $\phi(t)$ has width of the sampling interval T_s



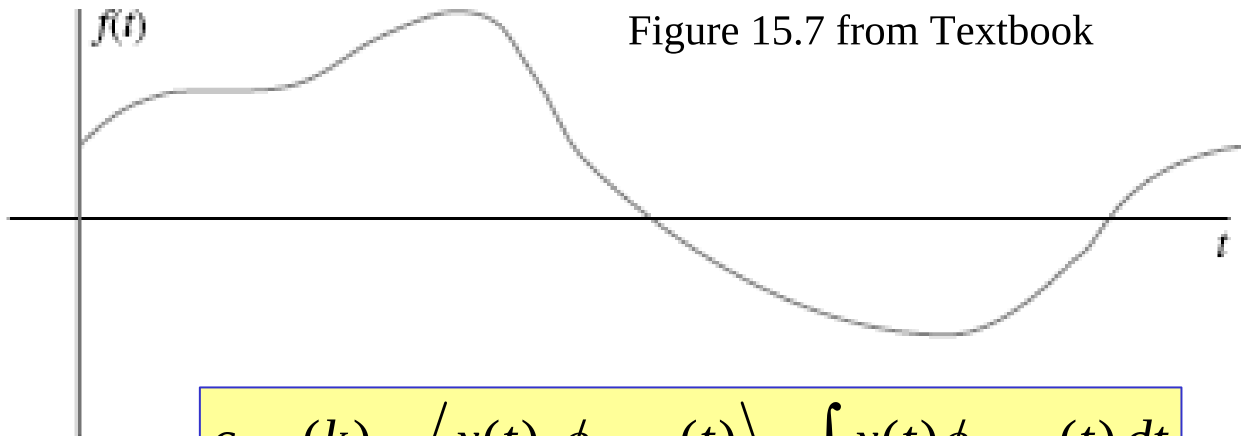


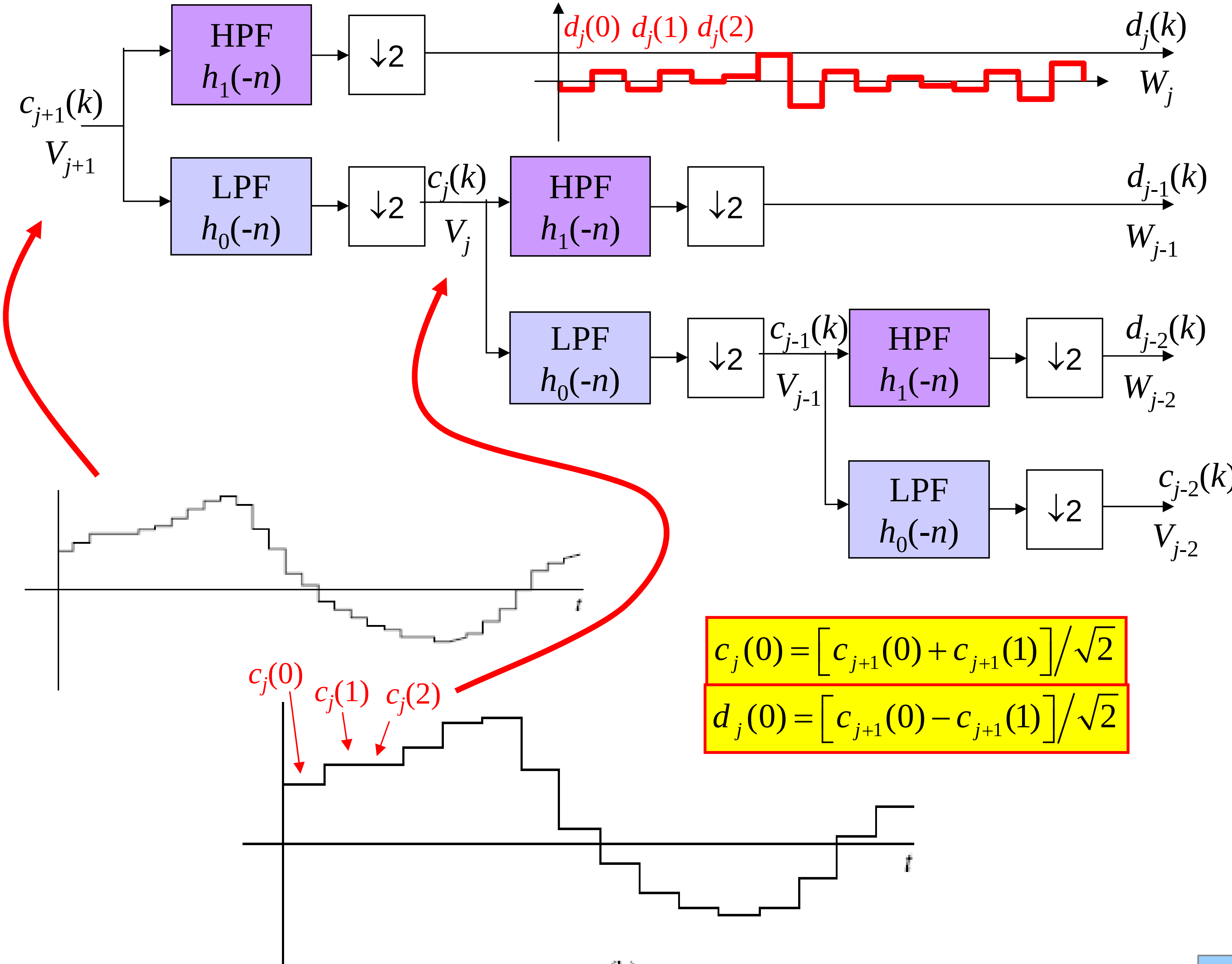
Figure 15.7 from Textbook

$$c_{j_{\max}}(k) = \left\langle x(t), \phi_{j_{\max},k}(t) \right\rangle = \int x(t)\phi_{j_{\max},k}(t)\,dt$$

$$\sim \int_{kT_s}^{(k+1)T_s} x(t)dt \approx x(kT_s)$$

➔ Samples are approximately proportional to the scale coefficients at $j_{\max}$

Next



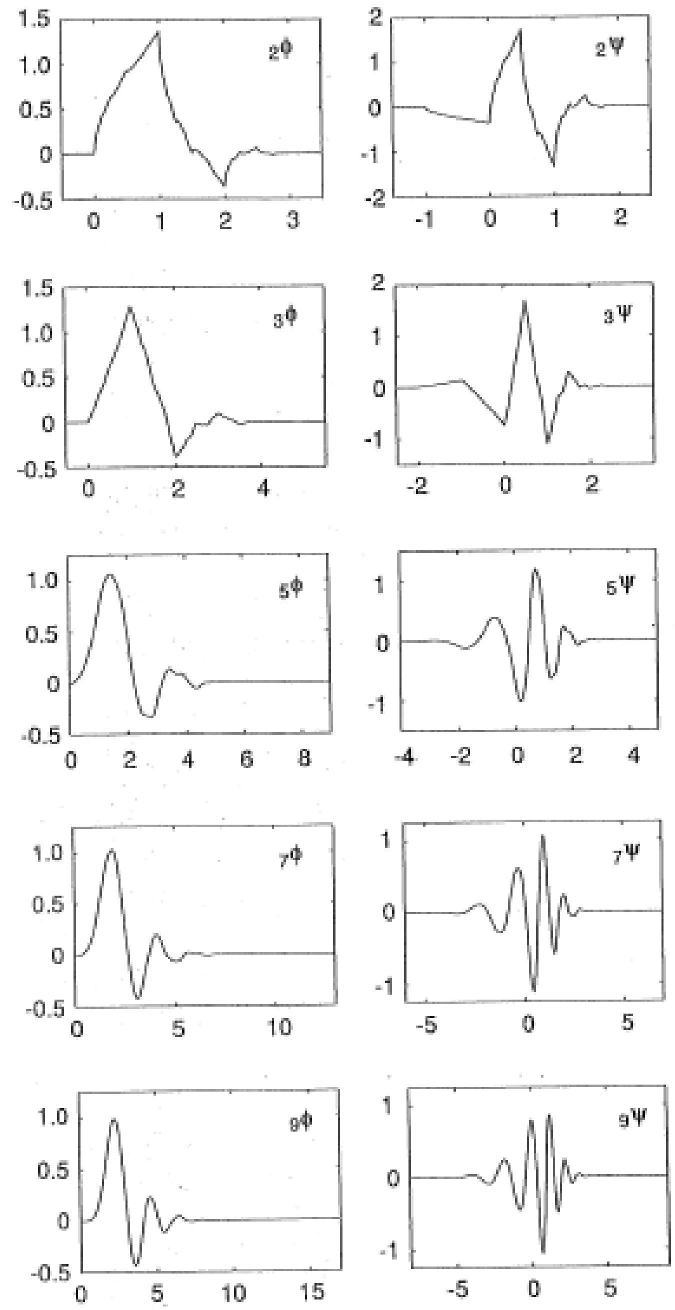
$$c_j(0) = \left[c_{j+1}(0) + c_{j+1}(1) \right] / \sqrt{2}$$

$$d_j(0) = \left[c_{j+1}(0) - c_{j+1}(1) \right] / \sqrt{2}$$

Next

Daubechies' Compactly-Supported Wavelets

	n	$_N h_n$
$N=2$	0	.4829629131445341
	1	.8365163037378077
	2	.2241438680420134
	3	−.1294095225512603
$N=3$	0	.3326705529500825
	1	.8068915093110924
	2	.4598775021184914
	3	−.1350110200102546
	4	−.0854412738820267
	5	.0352262918857095
$N=4$	0	.2303778133088964
	1	.7148465705529154
	2	.6308807679298587
	3	−.0279837694168599
	4	−.1870348117190931
	5	.0308413818355607
	6	.0328830116668852
	7	−.0105974017850690
$N=5$	0	.1601023979741929
	1	.6038292697971895
	2	.7243085284377726
	3	.1384281459013203
	4	−.2422948870663823
	5	−.0322448695846381
	6	.0775714938400459
	7	−.0062414902127983
	8	−.0125807519990820
	9	.0033357252854738
$N=6$	0	.1115407433501095
	1	.4946238903984533
	2	.7511339080210959
	3	.3152503517091982
	4	−.2262646939654400
	5	−.1297668675672625
	6	.0975016055873225
	7	.0275228655303053
	8	−.0315820393174862
	9	.0005538422011614
	10	.0047772575109455
	11	−.0010773010853085
$N=7$	0	.0778520540850037
	1	.3965393194818912
	2	.7291320908461957
	3	.4697822874051889
	4	−.1439060039285212
	5	−.2240361849938412
	6	.0713092192668272
	7	.0806126091510774
	8	−.0380299369350104
	9	−.0165745416306655
	10	.0125509985560986
	11	.0004295779729214
	12	−.0018016407040473
	13	.0003537137999745
$N=8$	0	.0544158422431072
	1	.3128715909143166
	2	.6756307362973195
	3	.5853546836542159
	4	−.0158291052563823
	5	−.2840155429615824
	6	.0004724845739124
	7	.1287474266204893
	8	−.0173693010018090
	9	−.0440882539307971
	10	.0139810279174001
	11	.0087460940474065
	12	−.0048703529934520
	13	−.0003917403733770
	14	.0006754494064506
	15	−.0001174767841248
$N=9$	0	.0380779473638778
	1	.2438346746125858
	2	.6048231236900955
	3	.6572880780512736
	4	.1331973858249883
	5	−.2932737832791663
	6	−.0968407832229492
	7	.1485407493381256
	8	.0307256814793385
	9	−.0676328290613279
	10	.0002509471148340
	11	.0223616621236798
	12	−.0047232047577518
	13	−.0042815036824635
	14	.0018476468830563
	15	.0002303857635232
	16	−.0002519631889427
	17	.0000393473203163
$N=10$	0	.0266700579005473
	1	.1881768000776347
	2	.5272011889315757
	3	.6884590394534363
	4	.2811723436605715
	5	−.2498464243271598
	6	−.1959462743772862
	7	.1273693403357541
	8	.0930573646035547
	9	−.0713941471663501
	10	−.0294575368218399
	11	.0332126740593612
	12	.0036065535669870
	13	−.0107331754833007
	14	.0013953517470688
	15	.0019924052951925
	16	−.0006858566949564
	17	−.0001164668551285
	18	.0000935886703202
	19	−.0000132642028945



From Ch. 6 of I. Daubechies, *Ten Lectures on Wavelets*, SIAM 1992

End