Wavelet Transform Theory

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What is a Wavelet Transform?

• Decomposition of a signal into constituent parts

• Note that there are many ways to do this. Some are:
  – Fourier series: harmonic sinusoids; single integer index
  – Fourier transform (FT): nonharmonic sinusoids; single real index
  – Walsh decomposition: “harmonic” square waves; single integer index
  – Karhunen-Loeve decomp: eigenfunctions of covariance; single real index
  – Short-Time FT (STFT): windowed, nonharmonic sinusoids; double index
    • provides time-frequency viewpoint
  – Wavelet Transform: time-compacted waves; double index

• Wavelet transform also provides time-frequency view
  – Decomposes signal in terms of duration-limited, band-pass components
    • high-frequency components are short-duration, wide-band
    • low-frequency components are longer-duration, narrow-band
  – Can provide combo of good time-frequency localization and orthogonality
    • the STFT can’t do this
  – More precisely, wavelets give time-scale viewpoint
    • this is connected to the multi-resolution viewpoint of wavelets
General Characteristics of Wavelet Systems

• Signal decomposition: build signals from “building blocks”, where the building blocks (i.e. basis functions) are doubly indexed.
• The components of the decomposition (i.e. the basis functions) are localized in time-frequency
  – ON can be achieved w/o sacrificing t-f localization
• The coefficients of the decomposition can be computed efficiently (e.g., using $O(N)$ operations).

Specific Characteristics of Wavelet Systems

• Basis functions are generated from a single wavelet or scaling function by scaling and translation
• Exhibit multiresolution characteristics: dilating the scaling functions provides a higher resolution space that includes the original
• Lower resolution coefficients can be computed from higher resolution coefficients through a filter bank structure
Fourier Development vs. Wavelet Development

• Fourier and others:
  – expansion functions are chosen, then properties of transform are found

• Wavelets
  – desired properties are mathematically imposed
  – the needed expansion functions are then derived

• Why are there so many different wavelets
  – the basic desired property constraints don’t use all the degrees of freedom
  – remaining degrees of freedom are used to achieve secondary properties
    • these secondary properties are usually application-specific
    • the primary properties are generally application-nonspecific

• What kinds of signals are wavelets and Fourier good for?
  – Wavelets are good for transients
    • localization property allows wavelets to give efficient rep. of transients
  – Fourier is good for periodic or stationary signals
Why are Wavelets Effective?

• Provide unconditional basis for large signal class
  – wavelet coefficients drop-off rapidly
  – thus, good for compression, denoising, detection/recognition
  – goal of any expansion is
    • have the coefficients provide more info about signal than time-domain
    • have most of the coefficients be very small (*sparse* representation)
  – FT is not sparse for transients

• Accurate local description and separation of signal characteristics
  – Fourier puts localization info in the phase in a complicated way
  – STFT can’t give localization and orthogonality

• Wavelets can be adjusted or adapted to application
  – remaining degrees of freedom are used to achieve goals

• Computation of wavelet coefficient is well-suited to computer
  – no derivatives or integrals needed
  – turns out to be a digital filter bank
Multiresolution Viewpoint
Multiresolution Approach

• Stems from image processing field
  – consider finer and finer approximations to an image
• Define a nested set of signal spaces

\[ \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2 \]

• We build these spaces as follows:
• Let \( V_0 \) be the space spanned by the integer translations of a fundamental signal \( \phi(t) \), called the scaling function:

  that is, if \( x(t) \) is in \( V_0 \) then it can be represented by:

  \[ x(t) = \sum_k a_k \phi(t - k) \]

• So far we can use just about any function \( \phi(t) \), but we’ll see that to get the nesting only certain scaling functions can be used.
Multiresolution Analysis (MRA) Equation

• Now that we have $V_0$ how do we make the others and ensure that they are nested?
• If we let $V_1$ be the space spanned by integer translates of $\phi(2t)$ we get the desired property that $V_1$ is indeed a space of functions having higher resolution.
• Now how do we get the nesting?
• We need that any function in $V_0$ also be in $V_1$; in particular we need that the scaling function (which is in $V_0$) be in $V_1$, which then requires that

$$\phi(t) = \sum_n h(n)\sqrt{2}\phi(2t - n)$$

where the expansion coefficient is $h(n)2^{\frac{1}{2}}$
• This is the requirement on the scaling function to ensure nesting: it must satisfy this equation
  – called the multiresolution analysis (MRA) equation
  – this is like a differential equation for which the scaling function is the solution
The $h(n)$ Specify the Scaling Function

- Thus, the coefficients $h(n)$ determine the scaling function
  - for a given set of $h(n)$, $\phi(t)$
    - may or may not exist
    - may or may not be unique
- Want to find conditions on $h(n)$ for $\phi(t)$ to exist and be unique, and also:
  - to be orthogonal (because that leads to an ON wavelet expansion)
  - to give wavelets that have desirable properties

$h(n)$ must satisfy conditions
Whence the Wavelets?

- The spaces $V_j$ represent increasingly higher resolution spaces
- To go from $V_j$ to higher resolution $V_{j+1}$ requires the addition of “details”
  - These details are the part of $V_{j+1}$ not able to be represented in $V_j$
  - This can be captured through the “orthogonal complement” of $V_j$ w.r.t $V_{j+1}$
- Call this orthogonal complement space $W_j$
  - all functions in $W_j$ are orthogonal to all functions in $V_j$
  - That is:
    \[
    \langle \phi_{j,k}(t), \psi_{j,l}(t) \rangle = \int \phi_{j,k}(t) \psi_{j,l}(t) dt = 0 \quad \forall j, k, l \in \mathbb{Z}
    \]
- Consider that $V_0$ is the lowest resolution of interest
- How do we characterize the space $W_0$?
  - we need to find an ON basis for $W_0$, say $\{\psi_{0,k}(t)\}$ where the basis functions arise from translating a single function (we’ll worry about the scaling part later):
    \[
    \psi_{0,k}(t) = \psi(t - k)
    \]
Finding the Wavelets

- The wavelets are the basis functions for the $W_j$ spaces
  - thus, they lie in $V_{j+1}$
- In particular, the function $\psi(t)$ lies in the space $V_1$ so it can be expanded as
  \[
  \psi(t) = \sum_{n} h_1(n)\sqrt{2}\phi(2t - n), \quad n \in \mathbb{Z}
  \]
- This is a fundamental result linking the scaling function and the wavelet
  - the $h_1(n)$ specify the wavelet, via the specified scaling function
Wavelet-Scaling Function Connection

- There is a fundamental connection between the scaling function and its coefficients $h(n)$, the wavelet function and its coefficients $h_1(n)$:

$$h_1(n)$$

$\phi(t)$

$\psi(t)$

MR Equation (MRE)

Wavelet Equation (WE)

How are $h_1(n)$ and $h(n)$ related?
Relationship Between $h_1(n)$ and $h(n)$

• We state here the conditions for the important special case of
  – finite number $N$ of nonzero $h(n)$
  – ON within $V_0$: \[ \int \phi(t)\phi(t-k)dt = \delta(k) \]
  – ON between $V_0$ and $W_0$: \[ \int \psi(t)\phi(t-k)dt = \delta(k) \]

• Given the $h(n)$ that define the desired scaling function, then the $h_1(n)$ that define the wavelet function are given by

\[
h_1(n) = (-1)^n h(N - 1 - n)
\]

  where $N$ is the length of the filter

• Much of wavelet theory addresses the origin, characteristics, and ramifications of this relationship between $h_1(n)$ and $h(n)$
  – requirements on $h(n)$ and $h_1(n)$ to achieve ON expansions
  – how the MRE and WE lead to a filter bank structure
  – requirements on $h(n)$ and $h_1(n)$ to achieve other desired properties
  – extensions beyond the ON case
The Resulting Expansions

• Let $x(t)$ be in $L^2(R)$
• There are three ways of interest that we can expand $x(t)$

1. We can give a limited resolution approximation to $x(t)$ via

$$x_j(t) = \sum_k a_k 2^{j/2} \varphi(2^j t - k)$$

- increasing $j$ gives a better (i.e., higher resolution) approximation

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2$$

- this is in general not the most useful expansion
The Resulting Expansions (cont.)

2. A low-resolution approximation plus its wavelet details

\[
x(t) = \sum_{k} c_{j_0}(k)2^{j_0/2} \varphi(2^{j_0}t - k) + \sum_{j=j_0}^{\infty} \sum_{k} d_j(k)2^{j/2} \psi(2^jt - k)
\]

- Choosing \( j_0 \) sets the level of the coarse approximation

\[
L^2 = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus W_{j_0+2} \oplus \cdots
\]

- This is most useful in practice: \( j_0 \) is usually chosen according to application
  - Also in practice, the upper value of \( j \) is chosen to be finite
The Resulting Expansions (cont.)

3. Only the wavelet details

\[ x(t) = \sum_k \sum_{j=-\infty}^{\infty} d_j(k) 2^{j/2} \psi(2^j t - k) \]

- Choosing \( j_0 = -\infty \) eliminates the coarse approximation leaving only details

\[ L^2 = \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots \]

- This is most similar to the “true” wavelet decomposition as it was originally developed
- This is not that useful in practice: \( j_0 \) is usually chosen to be finite according to application
The Expansion Coefficients $c_{j_0}(k)$ and $d_j(k)$

- We consider here only the simple, but important, case of ON expansion
  - i.e., the $\phi$'s are ON, the $\psi$'s are ON, and the $\phi$'s are ON to the $\psi$'s
- Then we can use standard ON expansion theory:

$$c_{j_0}(k) = \langle x(t), \phi_{j_0,k}(t) \rangle = \int x(t)\phi_{j_0,k}(t) \, dt$$

$$d_j(k) = \langle x(t), \psi_{j,k}(t) \rangle = \int x(t)\psi_{j,k}(t) \, dt$$

- We will see how to compute these without resorting to computing inner products
  - we will use the coefficients $h_1(n)$ and $h(n)$ instead of the wavelet and scaling function, respectively
  - we look at a relationship between the expansion coefficients at one level and those at the next level of resolution
Summary of Multiresolution View

- Nested Resolution spaces:
  \[ \cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset L^2 \]

- Wavelet Spaces provide orthogonal complement between resolutions
  \[ L^2 = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus W_{j_0+2} \oplus \cdots \]

- Wavelet Series Expansion of a continuous-time signal \( x(t) \):
  \[
  x(t) = \sum_{k} c_{j_0}(k)2^{j_0/2} \phi(2^{j_0}t-k) + \sum_{k} \sum_{j=j_0}^{\infty} d_j(k)2^{j/2} \psi(2^j t-k)
  \]

- MR equation (MRE) provides link between the scaling functions at successive levels of resolution:
  \[ \phi(t) = \sum_{n} h(n)\sqrt{2} \phi(2t-n), \quad n \in \mathbb{Z} \]

- Wavelet equation (WE) provides link between a resolution level and its complement
  \[ \psi(t) = \sum_{n} h_1(n)\sqrt{2} \phi(2t-n), \quad n \in \mathbb{Z} \]
Summary of Multiresolution View (cont.)

- There is a fundamental connection between the scaling function and its coefficients $h(n)$, the wavelet function and its coefficients $h_1(n)$:

- Magnetic Resonance Equation (MRE)

- How are $h_1(n)$ and $h(n)$ related?

- Wavelet Equation (WE)
Filter Banks and DWT
Generalizing the MRE and WE

• Here again are the MRE and the WE:

\[ \phi(t) = \sum_{n} h(n) \sqrt{2} \phi(2t - n) \]

\[ \psi(t) = \sum_{n} h_1(n) \sqrt{2} \phi(2t - n) \]

scale & translate: replace \( t \to 2^j t - k \)

• We get:

\[ \phi(2^j t - k) = \sum_{m} h(m - 2k) \sqrt{2} \phi(2^{j+1} t - m) \]

Connects \( V_j \) to \( V_{j+1} \)

\[ \psi(2^j t - k) = \sum_{m} h_1(m - 2k) \sqrt{2} \phi(2^{j+1} t - m) \]

Connects \( W_j \) to \( V_{j+1} \)
Linking Expansion Coefficients Between Scales

- Start with the Generalized MRA and WE:

\[
\phi(2^j t - k) = \sum_{m} h(m - 2k)\sqrt{2}\phi(2^{j+1} t - m) \\
\psi(2^j t - k) = \sum_{m} h_1(m - 2k)\sqrt{2}\phi(2^{j+1} t - m)
\]

\[
c_j(k) = \left< f(t), \varphi_{j,k}(t) \right>
\]

\[
d_j(k) = \left< f(t), \psi_{j,k}(t) \right>
\]

\[
c_j(k) = \sum_{m} h(m - 2k)\left< x(t), 2^{(j+1)/2}\phi(2^{j+1} t - m) \right>
\]

\[
d_j(k) = \sum_{m} h_1(m - 2k)\left< x(t), 2^{(j+1)/2}\phi(2^{j+1} t - m) \right>
\]
Convolution-Decimation Structure

New Notation For Convenience: $h(n) \rightarrow h_0(n)$

\[c_j(k) = \sum_m h_0(m - 2k)c_{j+1}(m)\]

\[d_j(k) = \sum_m h_1(m - 2k)c_{j+1}(m)\]

Convolution

\[y_0(n) = c_{j+1}(n) \ast h_0(-n)\]
\[= \sum_m h_0(m - n)c_{j+1}(m)\]

Decimation

\[n = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9\]

\[k = 0 \quad 1 \quad 2 \quad 3 \quad 4\]

\[n = 2k = 0 \quad 2 \quad 4 \quad 6 \quad 8\]
Summary of Progression to Convolution-Decimation Structure

MRE

\[ \phi(2^j t - k) = \sum_m h(m - 2k)\sqrt{2}\phi(2^{j+1} t - m) \]

\[ c_j(k) = \sum_m h(m - 2k)c_{j+1}(m) \]

WE

\[ \psi(2^j t - k) = \sum_m h_1(m - 2k)\sqrt{2}\phi(2^{j+1} t - m) \]

\[ d_j(k) = \sum_m h_1(m - 2k)c_{j+1}(m) \]

LPF

\[ h_0(-n) \]

\[ \downarrow 2 \]

\[ c_j(k) \]

HPF

\[ h_1(-n) \]

\[ \downarrow 2 \]

\[ d_j(k) \]
Computing The Expansion Coefficients

- The above structure can be cascaded:
  - given the scaling function coefficients at a specified level all the lower resolution c’s and d’s can be computed using the filter structure
Filter Bank Generation of the Spaces

\[ V_{j-1} \rightarrow \overset{V_j}{\rightarrow} \rightarrow V_{j+1} \]

\[ V_{j-2} \quad W_{j-2} \quad W_{j-1} \quad W_j \]

\[ \pi/8 \quad \pi/4 \quad \pi/2 \pi \]

\[ \text{LPF} \quad h_0(-n) \quad \downarrow 2 \]
\[ c_{j+1}(k) \quad V_{j+1} \]

\[ \text{HPF} \quad h_1(-n) \quad \downarrow 2 \]
\[ d_j(k) \quad W_j \]

\[ c_j(k) \quad V_j \]

\[ \text{LPF} \quad h_0(-n) \quad \downarrow 2 \]
\[ d_{j-1}(k) \quad W_{j-1} \]

\[ c_{j-1}(k) \quad V_{j-1} \]

\[ \text{HPF} \quad h_1(-n) \quad \downarrow 2 \]
\[ d_{j-2}(k) \quad W_{j-2} \]

\[ c_{j-2}(k) \quad V_{j-2} \]

\[ \text{LPF} \quad h_0(-n) \quad \downarrow 2 \]

\[ \text{LPF} \quad h_0(-n) \quad \downarrow 2 \]

\[ \text{LPF} \quad h_0(-n) \quad \downarrow 2 \]

\[ \text{LPF} \quad h_0(-n) \quad \downarrow 2 \]
Connection to Notation of Previous WT Notes

From Previous Note Set on WT

\[ X(s, \tau) = \int_{-\infty}^{\infty} x(t) \left[ \frac{1}{\sqrt{s}} h \left( \frac{t-\tau}{s} \right) \right] dt \]

\[ x(t) = \int_{0}^{\infty} \int_{-\infty}^{\infty} X(s, \tau) \left[ \frac{1}{\sqrt{s}} h \left( \frac{t-\tau}{s} \right) \right] \frac{ds d\tau}{s^2} \]

\[ X_{mn} = \int_{-\infty}^{\infty} x(t) \left[ 2^{-m/2} h \left( 2^{-m} t - n \right) \right] dt \]

\[ x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} X_{mn} \left[ 2^{-m/2} h \left( 2^{-m} t - n \right) \right] \]

From This Note Set on WT

Compute \( d_j(k) \)… & \( c_j(k) \)… using filter bank

\[ x(t) = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} d_j(k) \left[ 2^{j/2} \psi \left( 2^j t - k \right) \right] \]