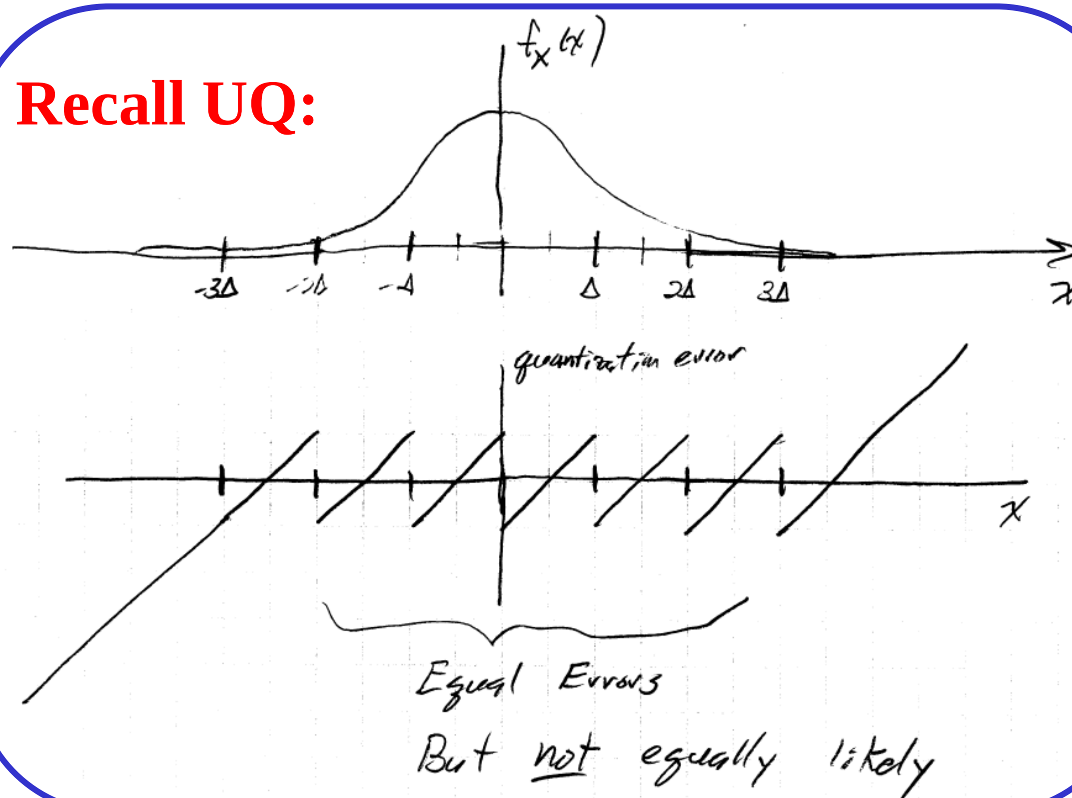
Ch. 9 Scalar Quantization

Non-Uniform Quantizers

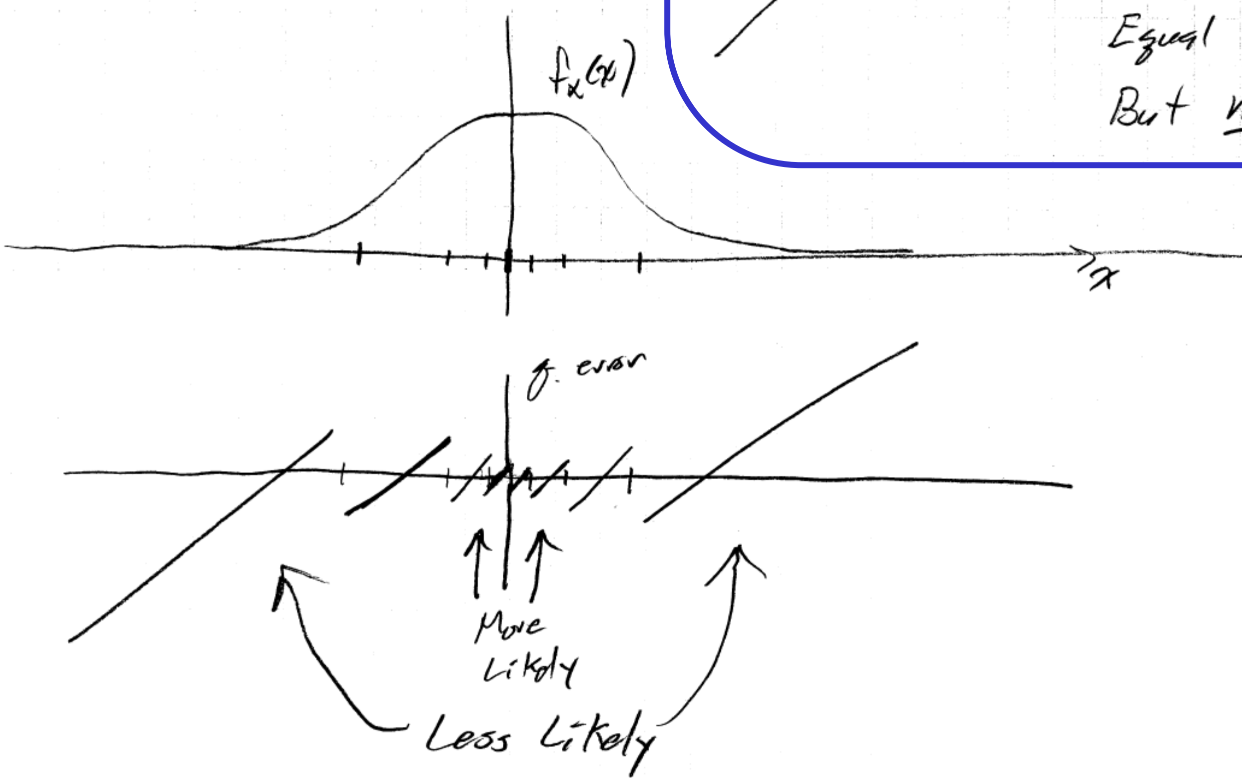
Motivation

Recall UQ:

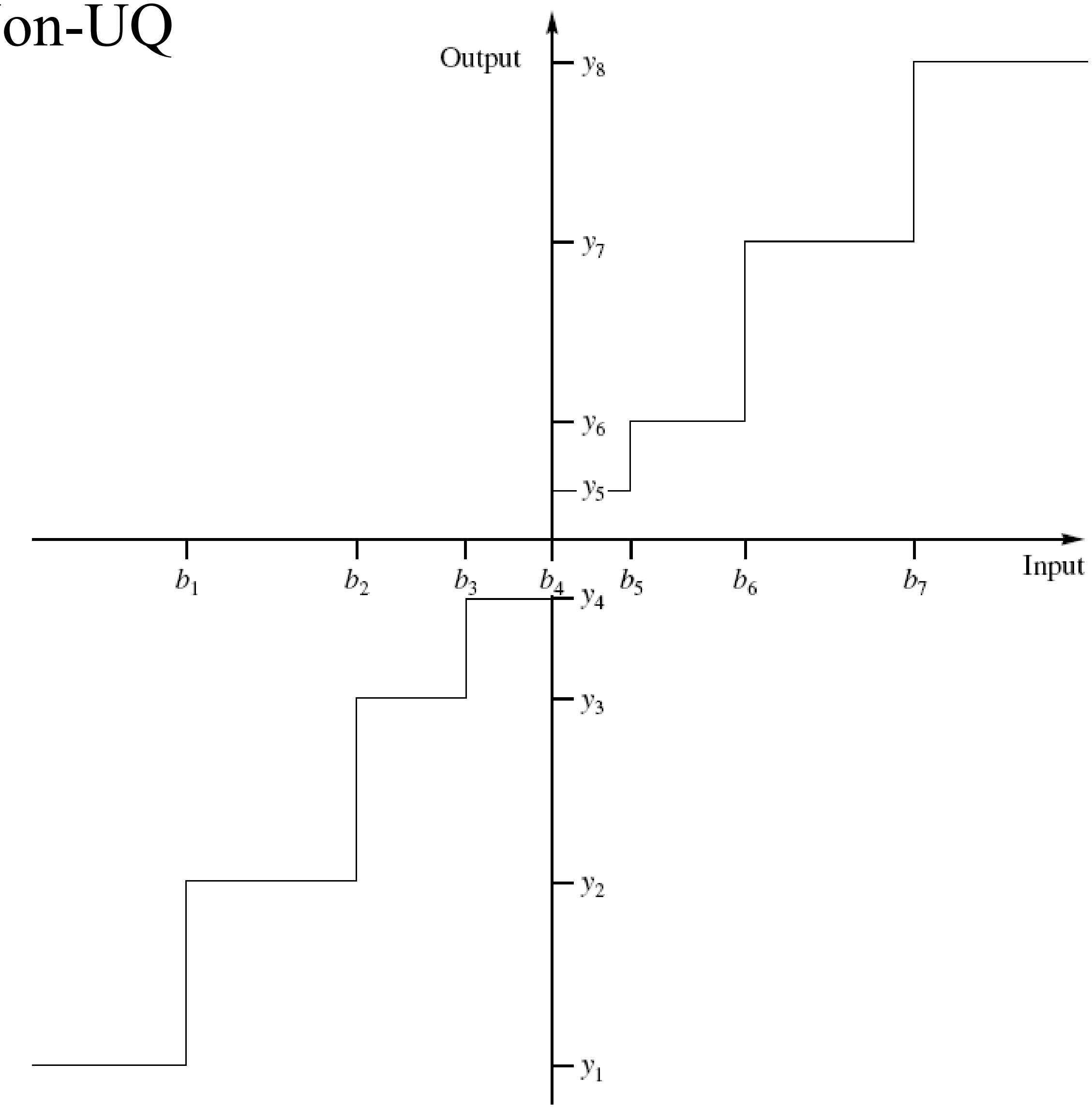


Instead of doing this…

We'd like to make the errors small in the regions where the signal is most likely



Mid-Riser Non-UQ



Lloyd-Max Quantizer

Optimize only w.r.t. <u>one</u> variable!

Recall for UQ optimization: $\min_{\Delta} \sigma_q^2(\Delta)$

Now more complex! Need to optimize w.r.t.
- DBs: $b_1, b_2, b_3, \ldots b_{M-1}$
- RLs: $y_1, y_2, y_3, \ldots y_M$

$$\text{minimize} \ \sigma_q^2(b_1, b_2, \ldots, b_{M-1}, y_1, y_2, \ldots y_M) = \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} (x - y_i)^2 f_X(x)dx$$

Note: only one term in this sum remains after the derivative…

First… set $\dfrac{d\sigma_q^2}{dy_j} = 0$

Solve for y_j to get:

$$y_j = \frac{\int_{b_{j-1}}^{b_j} x f_X(x)dx}{\int_{b_{j-1}}^{b_j} f_X(x)dx}$$

<u>Centroid</u> of PDF in the interval

Next… set $\dfrac{d\sigma_q^2}{db_j} = 0$ ➡ $b_j = \dfrac{y_{j+1} + y_j}{2}$

<u>Midpoint</u> between RLs

<Recall for UQ it was the other way around… RLs were midpoints of DLs>

Leads to two coupled equations… solved <u>iteratively</u> and <u>numerically</u> to give the "Lloyd-Max Quantizer"   *<See book for details on the algorithm>*

$$y_j = \frac{\int_{b_{j-1}}^{b_j} x f_X(x)dx}{\int_{b_{j-1}}^{b_j} f_X(x)dx}$$

$$b_j = \frac{y_{j+1} + y_j}{2}$$

TABLE 9.6 Quantizer boundary and reconstruction levels for nonuniform Gaussian and Laplacian quantizers.

	Gaussian			Laplacian		
Levels	b_i	y_i	SNR	b_i	y_i	SNR
4	0.0	0.4528		0.0	0.4196	
	0.9816	1.510	9.3 dB	1.1269	1.8340	7.54 dB
6	0.0	0.3177		0.0	0.2998	
	0.6589	1.0		0.7195	1.1393	
	1.447	1.894	12.41 dB	1.8464	2.5535	10.51 dB
8	0.0	0.2451		0.0	0.2334	
	0.7560	0.6812		0.5332	0.8330	
	1.050	1.3440		1.2527	1.6725	
	1.748	2.1520	14.62 dB	2.3796	3.0867	12.64 dB

8-Level PDF-Opt. UQ: 14.27 dB … 11.39 dB

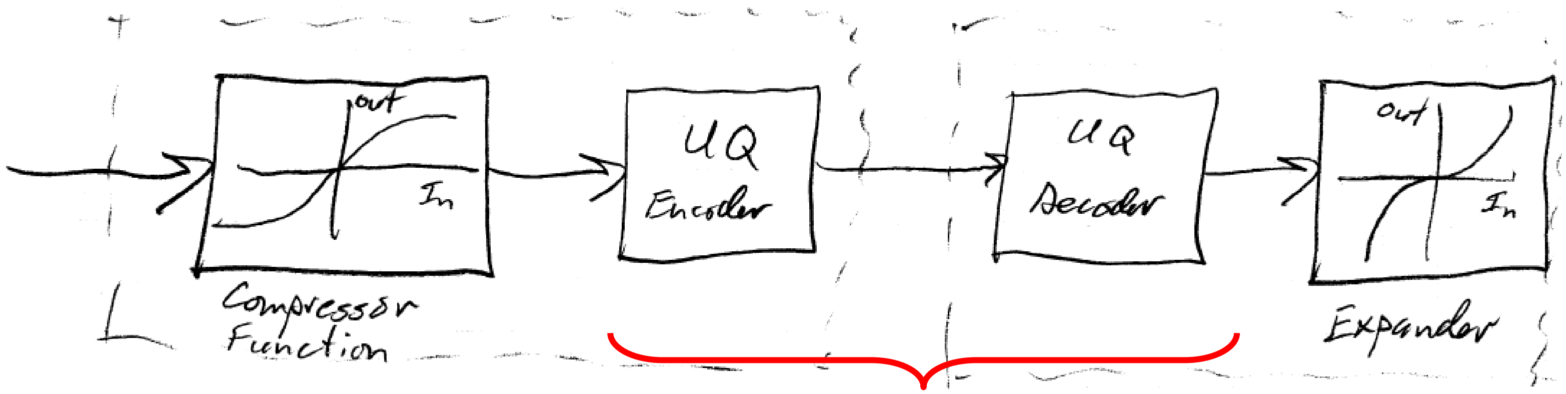
Companded Quantization - Overview

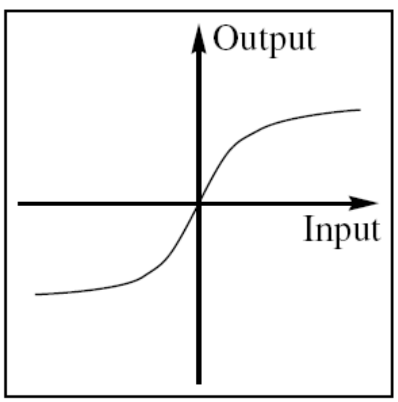
NUQs also suffer PDF Mismatch…

Ideas easily extended from Adaptive UQ ideas → **Adaptive NUQ**

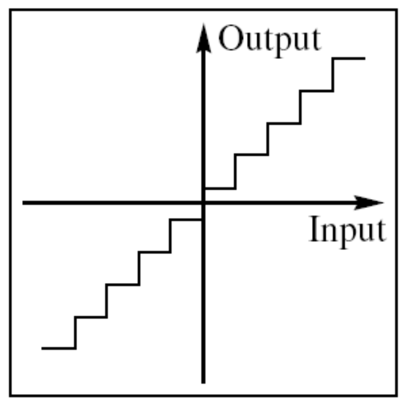
Compander w/ UQ ← We'll discuss this further

Companding:

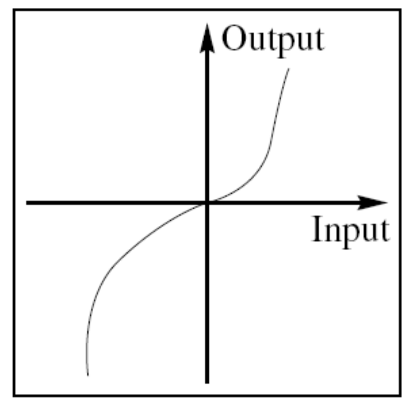




Compressor



Uniform quantizer



Expander

Companded Quantization - Derivation

Goal: Choose compressor function $C(x)$ to give robust performance

Bound the input range: $|x| \leq x_{\max}$

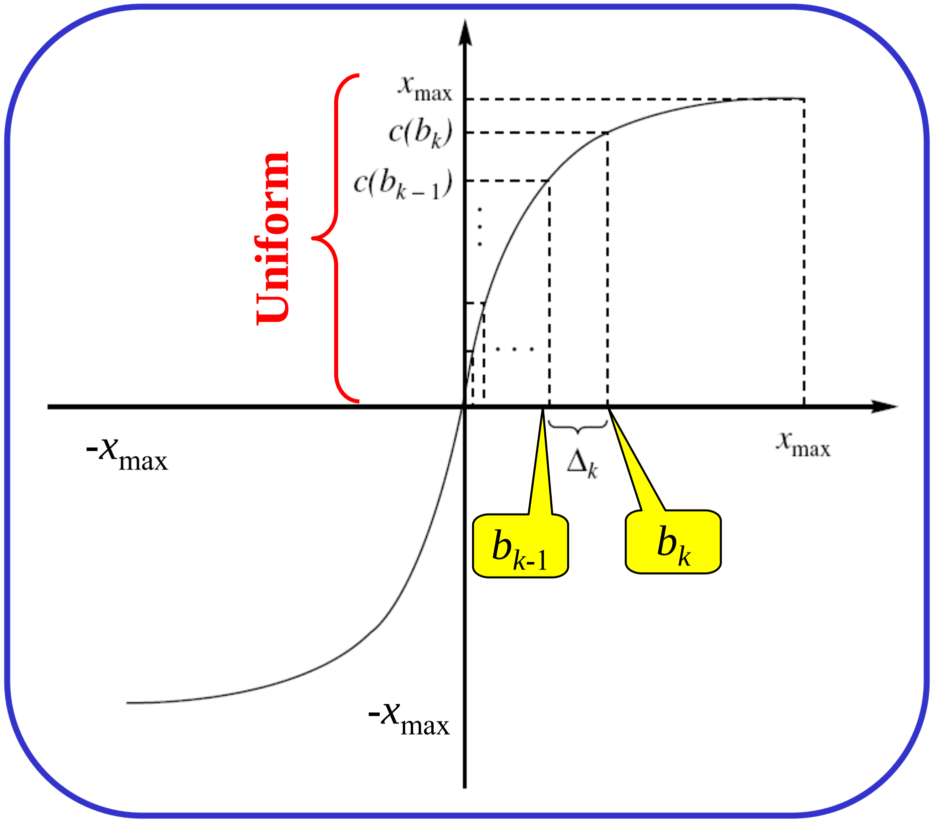
Assume M-Level UQ

If rate of UQ is high enough…

$$\left.\frac{dC(x)}{dx}\right|_{x=y_k} \approx \frac{C(b_k)-C(b_{k-1})}{\Delta_k}$$

$C'(y_k)$

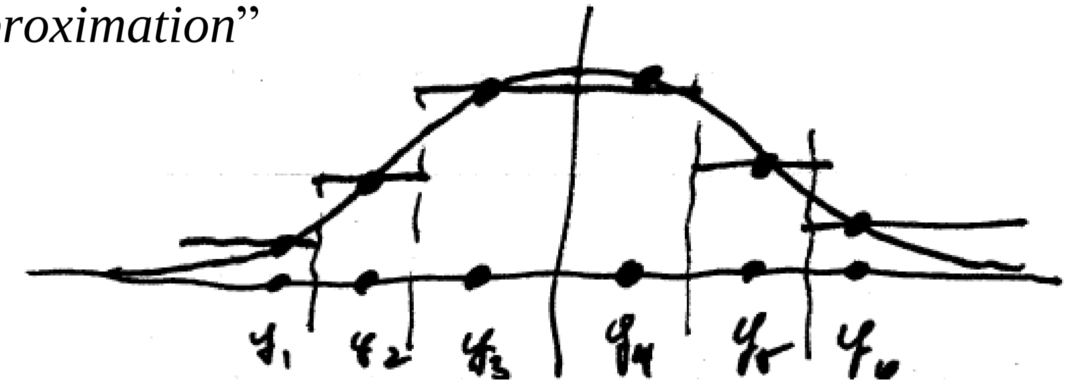
$$= \frac{2x_{\max}/M}{\Delta_k}$$



Solve for Δ_k: $\Delta_k = \dfrac{2x_{\max}}{MC'(y_k)}$ $(\bigstar)$

Now look at MSQE: $$\sigma_q^2 = \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} \left(x - y_i\right)^2 f_X(x)dx$$

Approximate PDF with step-wise function… this is accurate if *M* is large enough: "*High-Rate Approximation*"



$$\sigma_q^2 \approx \sum_{i=1}^{M} f_X(y_i) \underbrace{\int_{b_{i-1}}^{b_i} \left(x - y_i\right)^2 dx}_{=\Delta_i \sigma_{UQ}^2(\Delta_i) = \Delta_i^3/12} = \frac{1}{12}\sum_{i=1}^{M} f_X(y_i)\Delta_i^3$$

Known from (★)

$$= \frac{1}{12}\sum_{i=1}^{M} f_X(y_i)\left(\frac{2x_{\max}}{MC'(y_k)}\right)^2 \Delta_i = \frac{x_{\max}^2}{3M^2} \underbrace{\sum_{i=1}^{M} \frac{f_X(y_i)}{\left[C'(y_k)\right]^2}\Delta_i}_{\text{Approx. as integral}}$$

Leave one here for integral differential

Approx. as integral

The result is…

$$\sigma_q^2 \approx \frac{x_{\max}^2}{3M^2}\int_{-x_{\max}}^{x_{\max}} \frac{f_X(x)}{[C'(x)]^2}\,dx$$

"The Bennett Integral"

Can we choose $C(x)$ to make this variance independent of the shape of $f_X(x)$???

Can we make the SQR entirely independent of $f_X(x)$????

Let's see what happens if we choose $C'(x)$ such that

α is a constant

$$C'(x) = \frac{x_{\max}}{\alpha|x|}$$

Slope of $C(x)$
- is always positive
- $\rightarrow 0$ as $|x| \rightarrow \infty$
- $\rightarrow \infty$ @ $x=0$

Then…

$$\sigma_q^2 \approx \frac{\alpha^2}{3M^2}\underbrace{\int_{-x_{\max}}^{x_{\max}} x^2 f_X(x)\,dx}_{=\sigma_x^2} \quad \Rightarrow MSQE \sim \sigma_x^2$$

$$SQR = \frac{\sigma_x^2}{\sigma_q^2} \approx \frac{3M^2}{\alpha^2}$$

Choosing $C(x)$ this way makes SQR constant regardless of PDF type and variance!!!

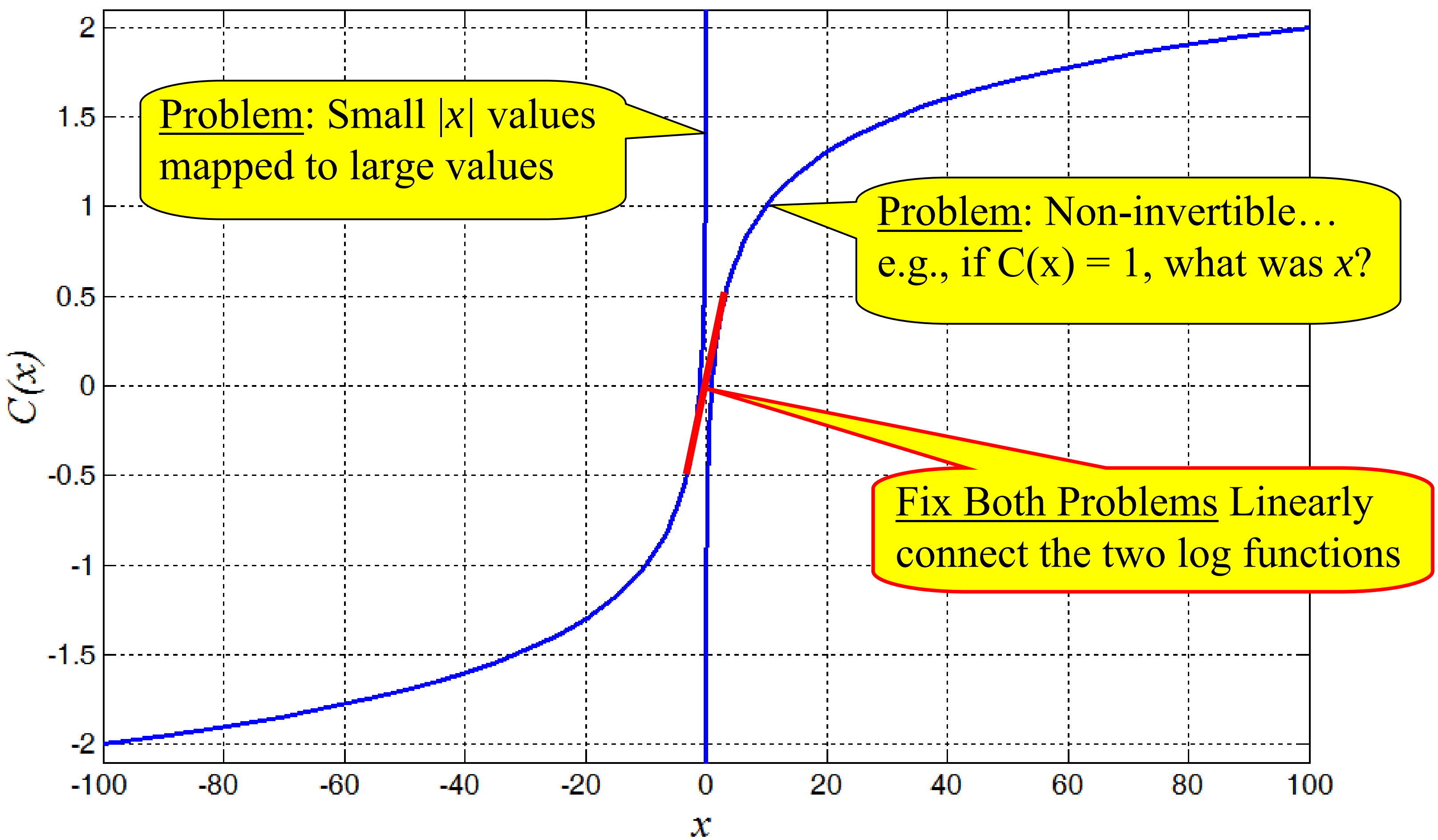
Can we actually *find* such a $C(x)$??

The form for function $C(x)$ that has the correct derivative is

$$C(x) = A + \frac{x_{\max}}{\alpha}\,\mathrm{sgn}(x)\ln(|x|)$$

Eq. (9.52) in 3rd Ed. Text (and (8.52) in 2nd Ed.) is not quite correct..

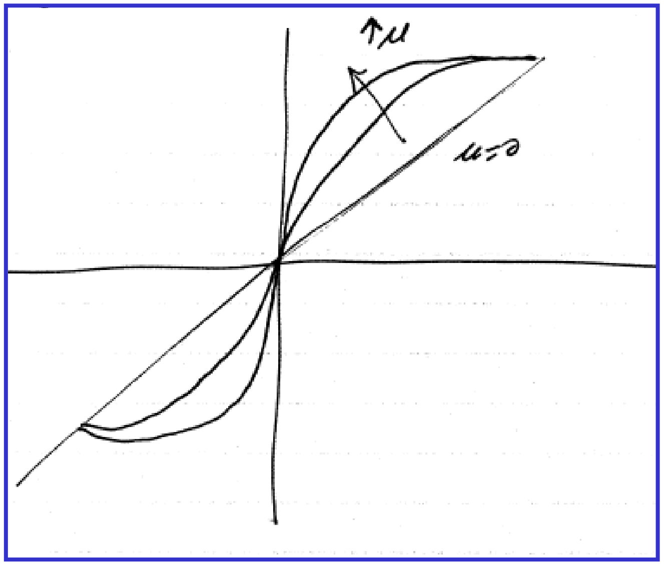
<*A is a constant we can choose as* $A = 0$>



There are two common functions used to enact this approximation:

μ-Law (used in N. America & Japan phone systems)

$$C(x) = x_{max} \frac{\ln\left(1 + \mu \frac{|x|}{x_{max}}\right)}{\ln(1+\mu)} \text{sgn}(x)$$



μ = 255 is the standard

- **A-Law (used in phone systems elsewhere)**

$$C(x) = \begin{cases} \dfrac{A|x|}{\ln(1+A)} \text{sgn}(x), & 0 \le \dfrac{|x|}{x_{max}} \le \dfrac{1}{A} \\ x_{max} \dfrac{1 + \ln\left(\frac{A|x|}{x_{max}}\right)}{\ln(1+A)} \text{sgn}(x), & \dfrac{1}{A} \le \dfrac{|x|}{x_{max}} \le 1 \end{cases}$$

What SQR does μ-Law give? To answer, use μ-Law function in Bennett Integral:

$$SQR = \frac{3M^2}{\left[\ln(1+\mu)\right]^2} \; \frac{1}{1 + \dfrac{2\overline{|\tilde{x}|}}{\mu\sigma_{\tilde{x}}^2} + \dfrac{1}{\mu^2\sigma_{\tilde{x}}^2}}$$

$$\tilde{x} = \frac{x}{x_{\max}} \qquad \overline{|\tilde{x}|} = E\left\{|\tilde{x}|\right\}$$

For large μ: $$SQR \approx \frac{3M^2}{\left[\ln(1+\mu)\right]^2}$$ ***Constant!***

Note the Trade-Off:

- Large μ improves robustness by de-emphasizing these terms
- But… large μ reduces the SQR level that is achieved

From: Modern Digital and Analog Comm. syst. by B.P. Lathi

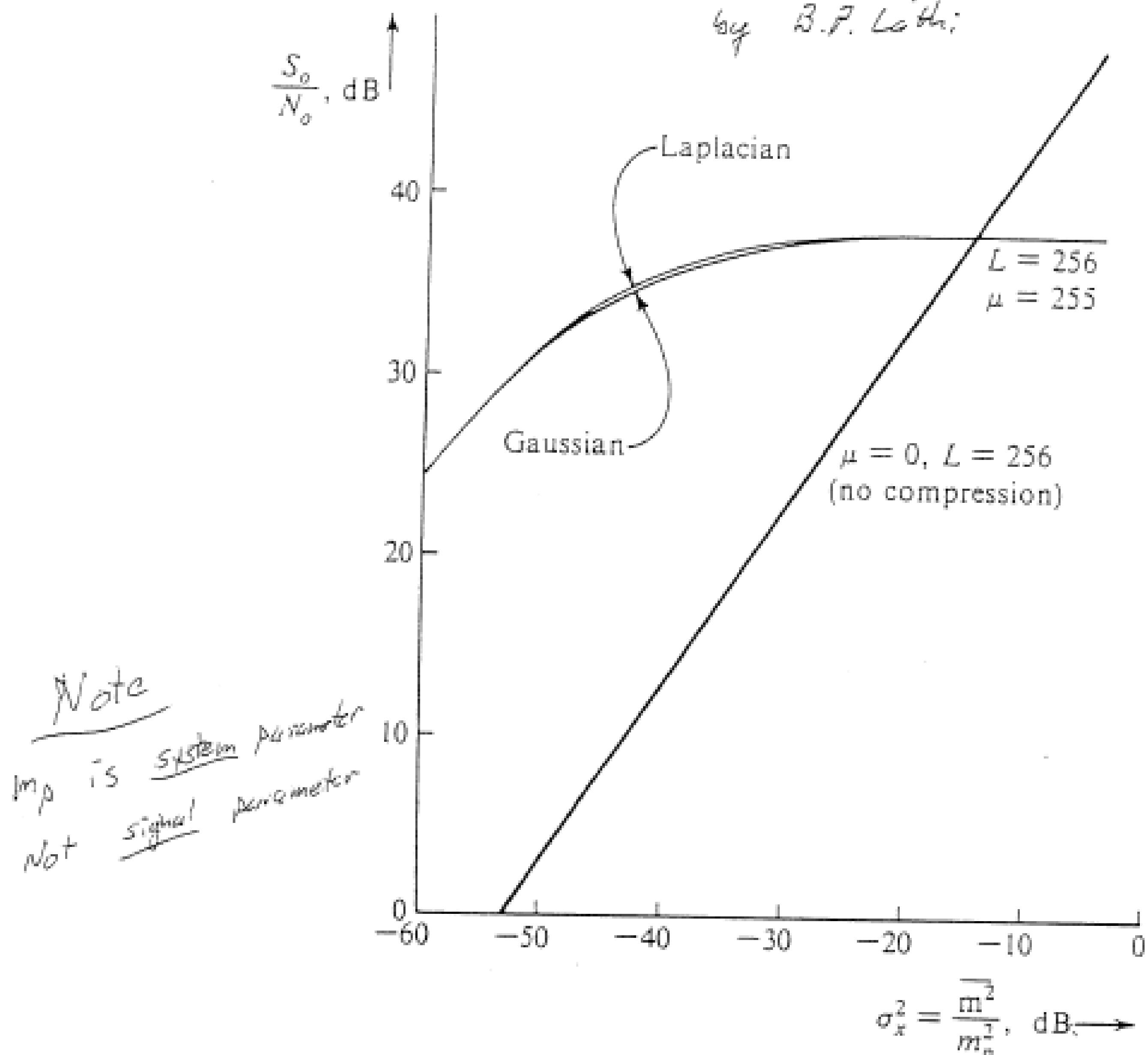


Figure 6.17 PCM performance with and without companding.

From: Digital Comm: Fund. and Appl. by B. Sklar

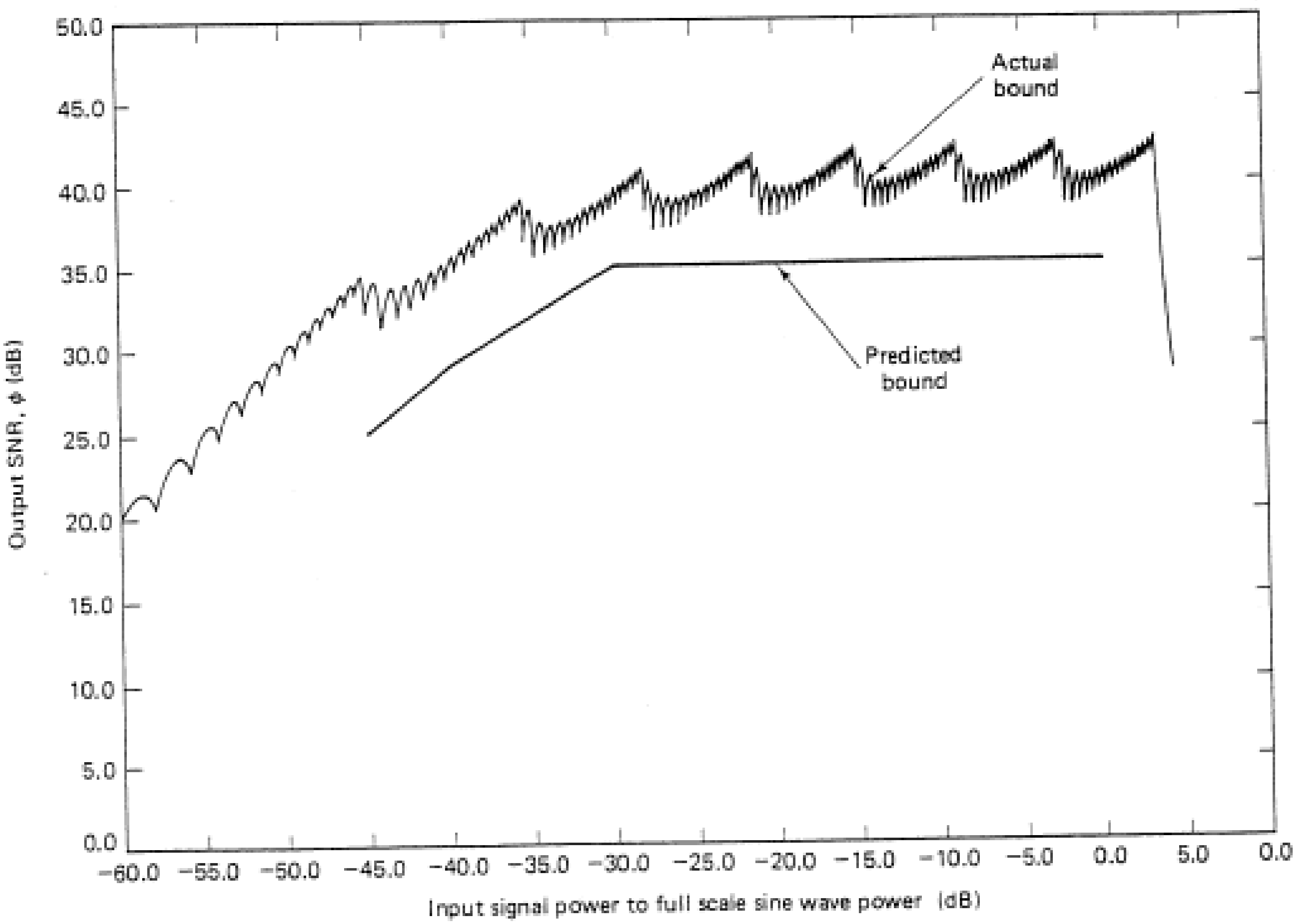


Figure 11.15 Predicted and measured SNR for a μ-law quantizer.