# Ch. 9 Scalar Quantization

## Uniform Quantizers

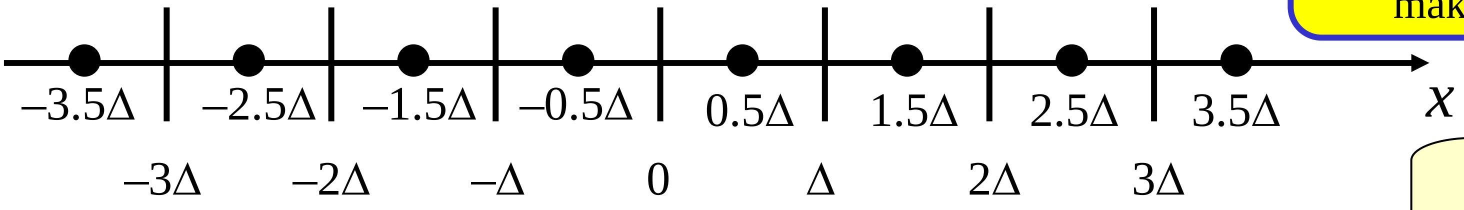
# Characteristics of Uniform Quantizers

Constraining to UQ makes the design easier but performance usually suffers…

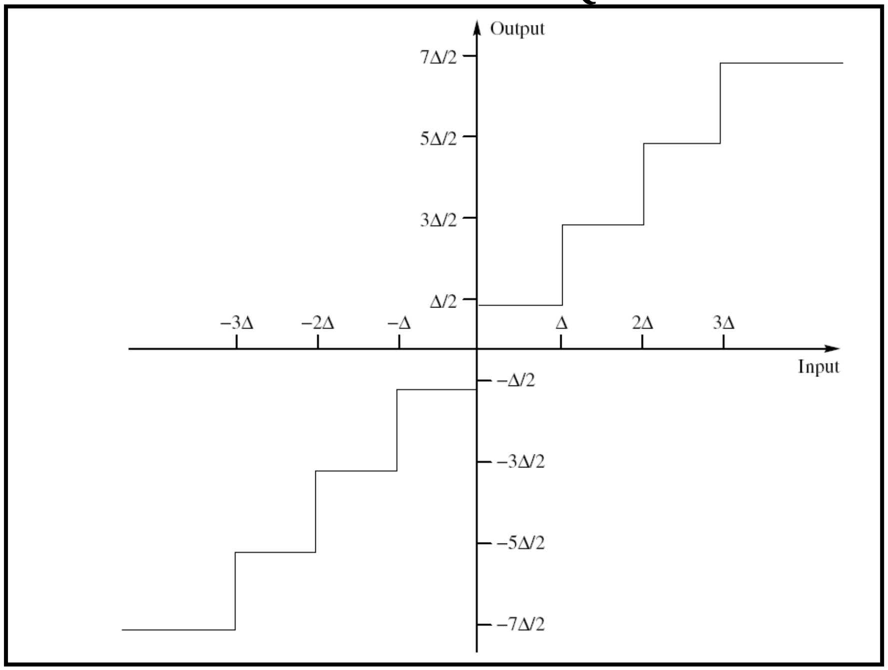
For a uniform quantizer the following two constraints are imposed:

- DBs are equally spaced (Step Size = Δ)
- RLs are equally spaced & centered between the DBs

For a given Rate…
Only One Choice to
make in Design

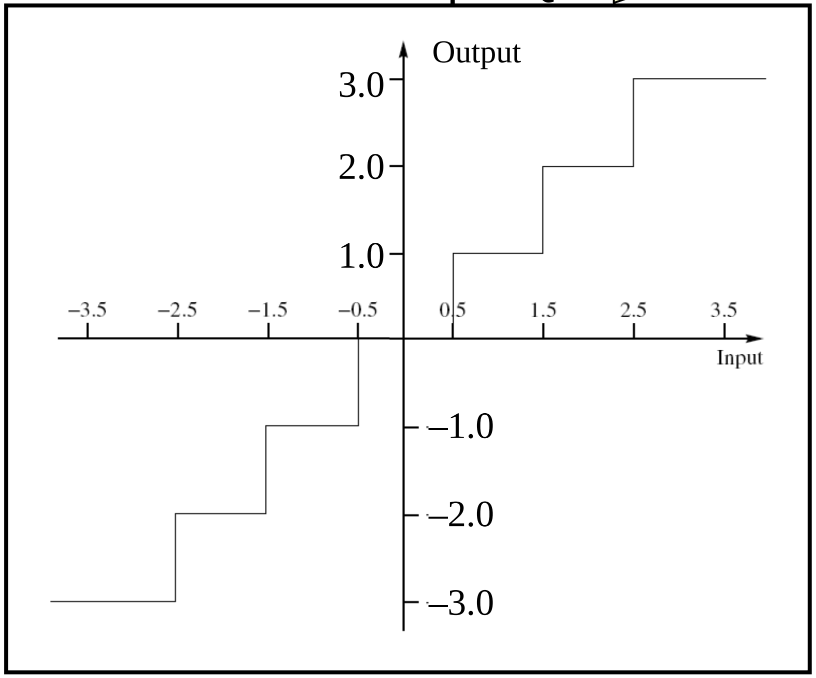


Mid-Riser SQ



This Fig in the
book has an error

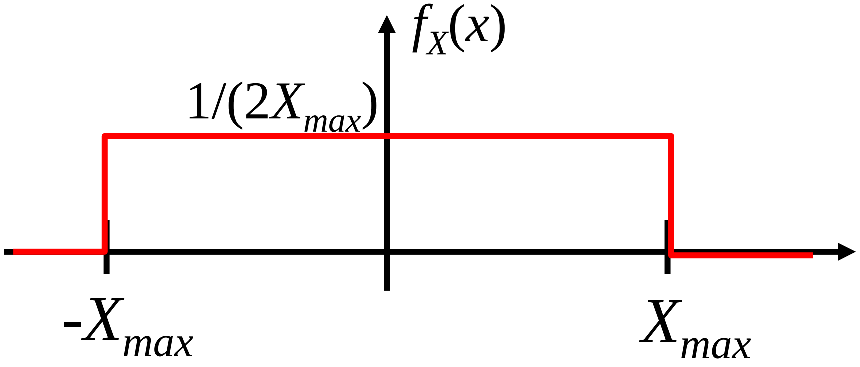
Mid-Step SQ



# PDF-Optimized Uniform Quantizers

The idea here is: Assuming that you know the PDF of the samples to be quantized… design the quantizer's step so that it is optimal for that PDF.

## For Uniform PDF



Want to uniformly quantize an RV $X \sim U(-X_{max}, X_{max})$

Assume that desire $M$ RLs for $R = \lceil \log_2(M) \rceil$

➔ $M$ equally-sized intervals having $\Delta = 2X_{max}/M$

Distortion is: $\sigma_q^2 = \int_{-X_{\max}}^{X_{\max}} \left[x - Q(x)\right]^2 f_X(x)dx$

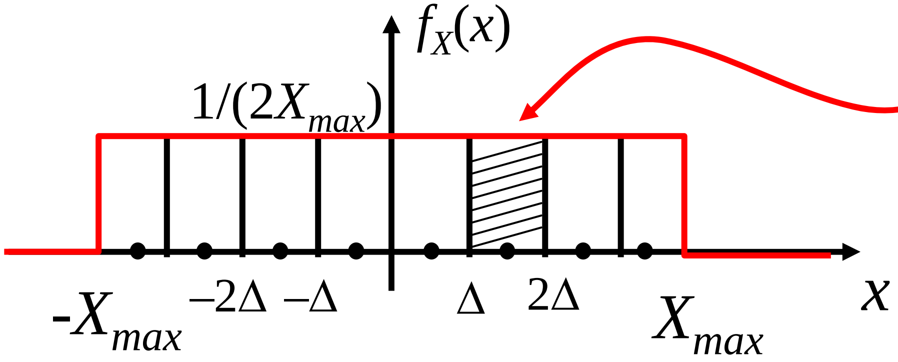
RLs

$$= 2\sum_{i=1}^{M/2} \underbrace{\int_{(i-1)\Delta}^{i\Delta} \left(x-(i-\tfrac{1}{2})\Delta\right)^2 \left[\frac{1}{2X_{\max}}\right] dx}$$

DLs

Now b y exploiting the structure…

Each of these integrals is identical!



Using $\Delta = 2X_{max}/M$

$$\sigma_q^2 = \cancel{2}\frac{M}{\cancel{2}} \int_{-\Delta/2}^{\Delta/2} q^2 \left[\frac{1}{2X_{\max}}\right] dq \quad = \int_{-\Delta/2}^{\Delta/2} q^2 \frac{1}{\Delta} dq$$

So… the result is:

$$\sigma_q^2 = \int_{-\Delta/2}^{\Delta/2} q^2 \frac{1}{\Delta} dq \quad \Rightarrow \quad \boxed{\sigma_q^2 = \frac{\Delta^2}{12}}$$

To get SQR, we need the variance (power) of the signal…

Since the signal is uniformly dist. we know from Prob. Theory that

$$\sigma_X^2 = \frac{(2X_{\max})^2}{12} = \frac{\Delta^2 M^2}{12}$$

$$\Rightarrow \quad SQR(dB) = 10\log_{10}\left[\frac{\sigma_X^2}{\sigma_q^2}\right] = 10\log_{10}\left[M^2\right]$$

For *n*-bit
Quantizer:
M = $2^n$

$$= 20\log_{10}\left[2^n\right]$$

$$= 6.02n \quad dB$$

**6 dB per bit**

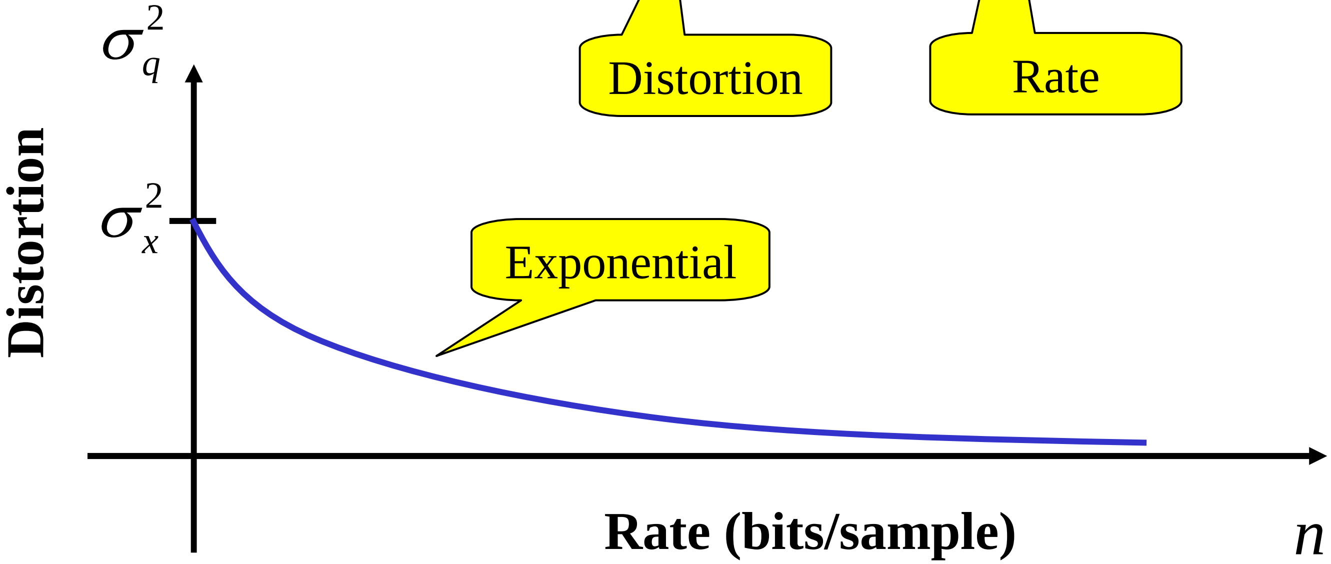
# Rate Distortion Curve for UQ of Uniform RV

$$\sigma_q^2 = \frac{\Delta^2}{12} = \frac{4X_{\max}^2}{12M^2} = \frac{X_{\max}^2}{3}2^{-2n}$$

$$\boxed{\sigma_q^2 = \frac{X_{\max}^2}{3}2^{-2n}}$$

Distortion

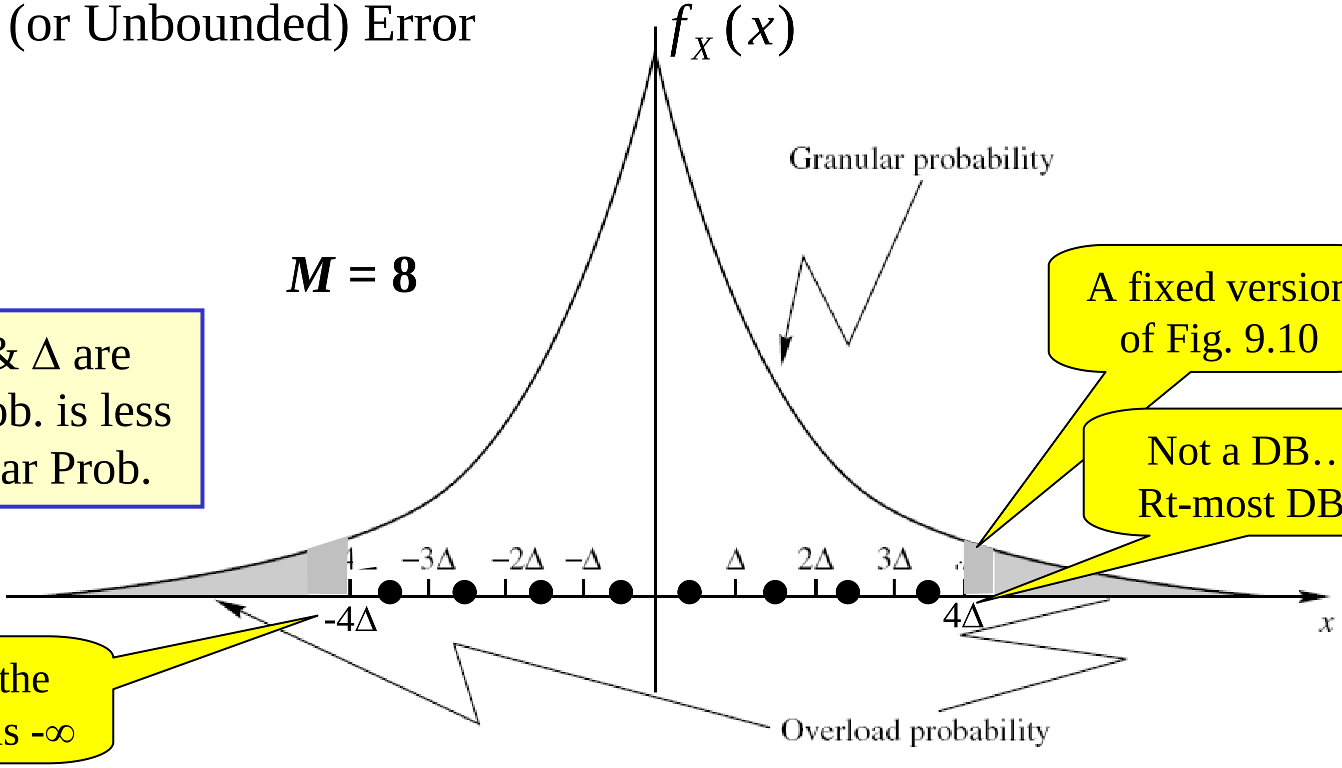
Rate



# PDF-Optimized Uniform Quantizers

## For Non-Uniform PDF

- We are mostly interested in Non-Uniform PDFs whose domain is not bounded.
- For this case, the PDF must decay asymptotically to zero…
- So… we can't cover the whole infinite domain with a finite number of Δ-intervals!
- **We have to choose a Δ & *M* to achieve a desired MSQE**
- Need to balance to types of errors:
  - – Granular (or Bounded) Error
  - – Overload (or Unbounded) Error



***M* = 8**

Typically, *M* & Δ are such that OL Prob. is less than the Granular Prob.

A fixed version of Fig. 9.10

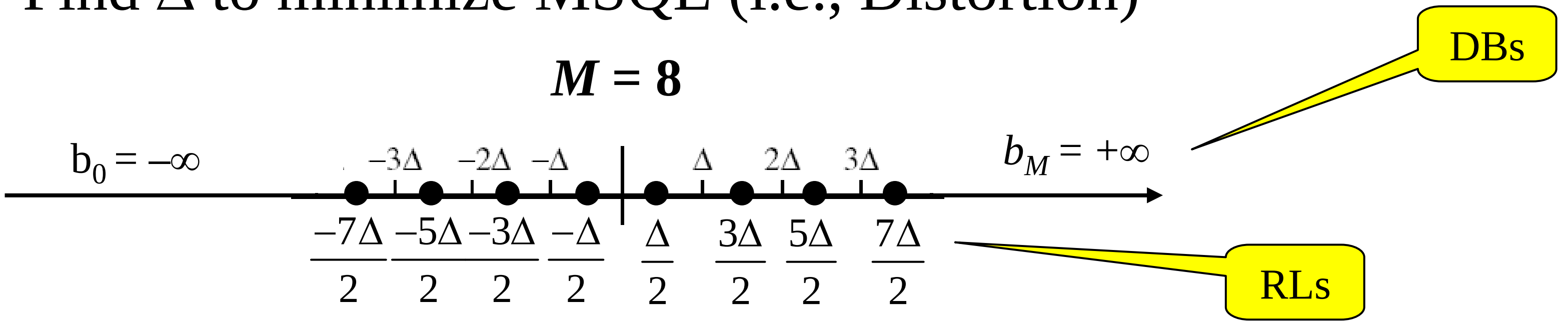
Not a DB… the Rt-most DB is ∞

Not a DB… the left-most DB is -∞

**Goal**: Given $M$ (i.e., given the Rate $n$… typically $M = 2^n$)

Find Δ to minimize MSQE (i.e., Distortion)

**$M$ = 8**



Write distortion as a function of Δ and then minimize w.r.t. Δ:

$$\sigma_q^2 = \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} \left(x - y_i\right)^2 f_X(x)dx$$

Granular

By Assumed Symmetry on PDF

$$= \left[ 2\sum_{i=1}^{M/2} \int_{(i-1)\Delta}^{i\Delta} \left(x - \left(1 - \tfrac{1}{2}\right)\Delta\right)^2 f_X(x)dx \right]$$

Overload

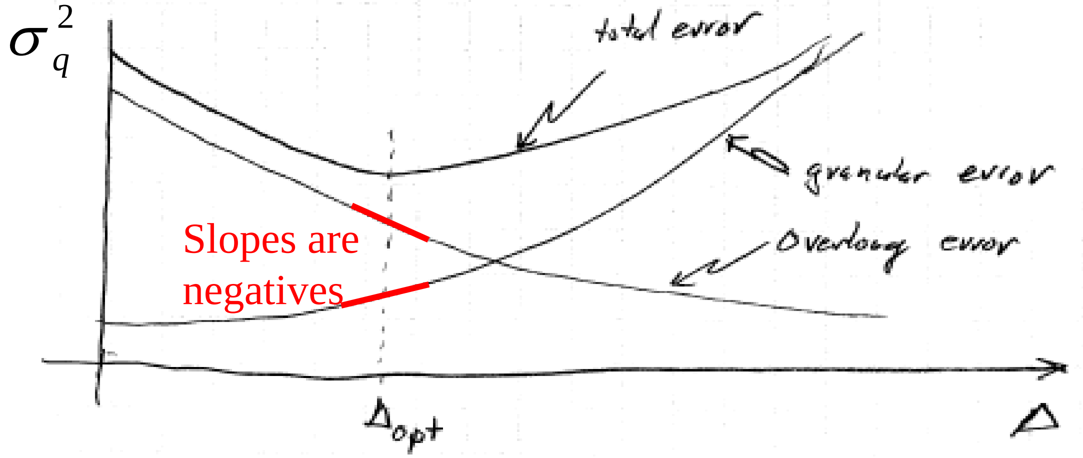
$$+ 2\int_{\frac{M}{2}\Delta}^{\infty} \left(x - \left(\tfrac{M}{2} - \tfrac{1}{2}\right)\Delta\right)^2 f_X(x)dx$$

Now to minimize… take derivative & set to zero: $\dfrac{d\sigma_q^2}{d\Delta} = 0$

Gets complicated & messy to do analytically… <u>solve numerically</u>!

This balances the granular & overload effects to minimize distortion…

**↑Δ ➔ ↓Overload… but… ↑Granular**



Slopes are negatives

$$\min\{ f_1(x) + f_2(x) \}$$

$$\Rightarrow \frac{df_1(x)}{dx} + \frac{df_2(x)}{dx} = 0$$

$$\Rightarrow \frac{df_1(x)}{dx} = -\frac{df_2(x)}{dx}$$

*Distributions that have heavier tails tend to have larger step sizes…*

How practical are these quantizers?

- Useful when source really does adhere to designed-for-pdf
- Otherwise have degradation due to mismatch
  - – Right PDF, Wrong Variance
  - – Wrong PDF Type

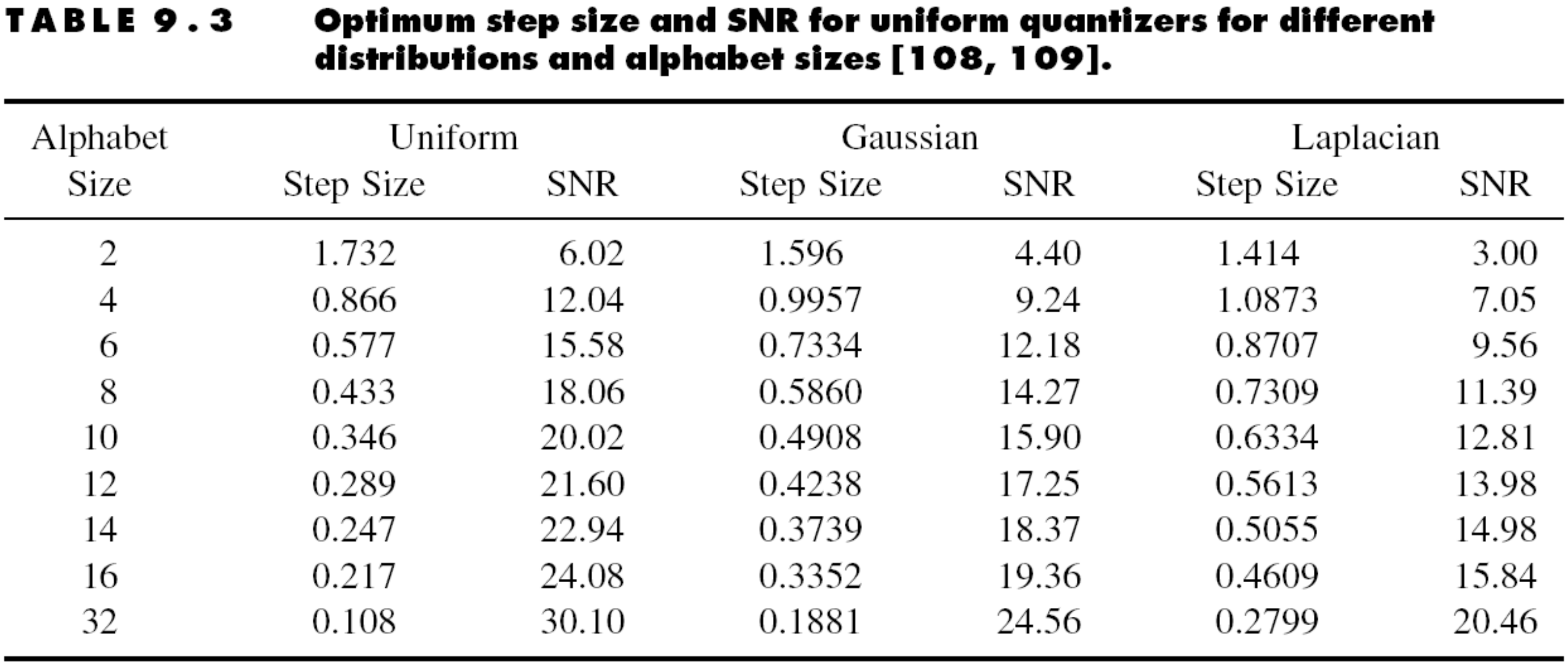


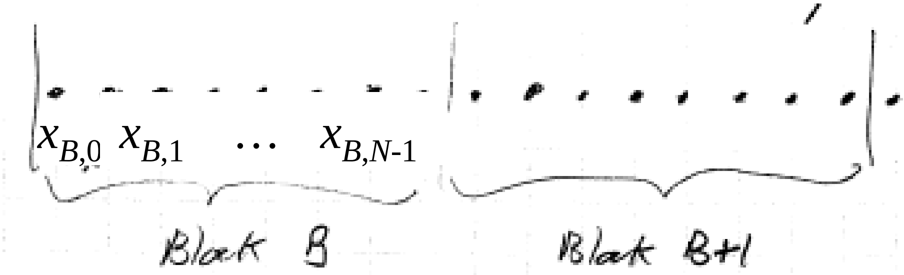
**TABLE 9.3** **Optimum step size and SNR for uniform quantizers for different distributions and alphabet sizes [108, 109].**

| **# of Bits** | Alphabet Size | Uniform | | Gaussian | | Laplacian | |
|---|---|---|---|---|---|---|---|
| | | Step Size | SNR | Step Size | SNR | Step Size | SNR |
| 1 | 2 | 1.732 | 6.02 | 1.596 | 4.40 | 1.414 | 3.00 |
| 2 | 4 | 0.866 | 12.04 | 0.9957 | 9.24 | 1.0873 | 7.05 |
| | 6 | 0.577 | 15.58 | 0.7334 | 12.18 | 0.8707 | 9.56 |
| 3 | 8 | 0.433 | 18.06 | 0.5860 | 14.27 | 0.7309 | 11.39 |
| | 10 | 0.346 | 20.02 | 0.4908 | 15.90 | 0.6334 | 12.81 |
| | 12 | 0.289 | 21.60 | 0.4238 | 17.25 | 0.5613 | 13.98 |
| | 14 | 0.247 | 22.94 | 0.3739 | 18.37 | 0.5055 | 14.98 |
| 4 | 16 | 0.217 | 24.08 | 0.3352 | 19.36 | 0.4609 | 15.84 |
| 5 | 32 | 0.108 | 30.10 | 0.1881 | 24.56 | 0.2799 | 20.46 |

# Adaptive Uniform Quantizers

We use adaptation to make UQ robust to Variance Mismatch

## Forward-Adaptive Quantization



- Collect a Block of *N* Samples
- Estimate Signal Variance in the *B*th Block

$$\hat{\sigma}_x^2(B) = \frac{1}{N}\sum_{i=0}^{N-1} x_{B,i}^2$$

Error in Book

- Normalize the Samples in the *B*th Block

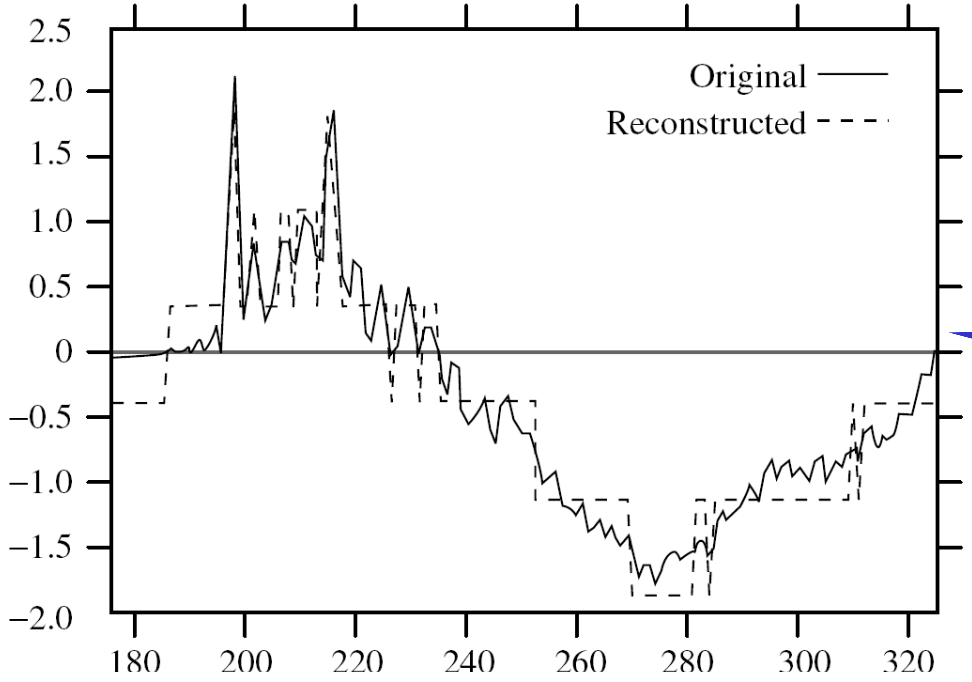
$$\hat{x}_{B,i} = x_{B,i} / \sqrt{\hat{\sigma}_x^2(B)}$$

- Quantize Normalized Samples
  - Always Use Same Quantizer: Designed for Unit Variance
- Quantize Estimated Variance
  - Send as “Side Info”

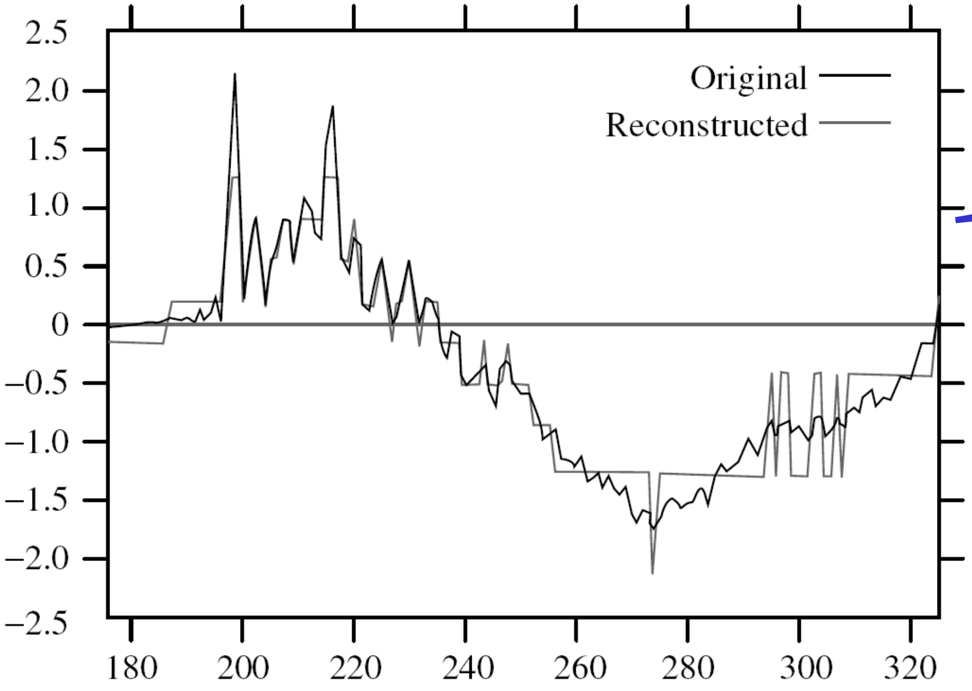
**Design Issues**

- Block Size
  - Short ➔ captures changes, ↑SI
  - Long ➔ misses changes, ↓SI
- # of bits for SI… 8 bits is “typical”

# Quantization of 16-bit Speech



16-Bit Original
vs.
3-bit Fixed Quantizer



16-Bit Original
vs.
3-bit Forward-Adapt Quant.

## Another Application

Synthetic Aperture Radar (SAR) often uses “Block Adaptive Quantizer (BAQ)”

# Alternative Method for Forward-Adaptive Quant.

“Block-Shifted Adaptive Quantization (BSAQ)”

- Given digital samples w/ large # of bits (*B* bits)
- In each block, shift all samples up by *S* bits
  - *S* is chosen so that the MSB of largest sample in block is “filled”
- Truncate shifted samples to $b < B$ bits
- Side Info = $S$ (use $\lceil \log_2(B) \rceil$ bits)

**Original 8 bit samples**

000000000000
000000000000
000000100000
100100001001
111111111111
101011001110
010001001010
010110010010

**Shifted Up by 2 Bits**

000000100000
100100001001
111111111111
101011001110
010001001010
010110010010

Coded S = 010

**Truncated to 3 Bits**

000000100000
100100001001
111111111111

To Decode: Shift down, fill zeros above & below

000000000000
000000000000
000000100000
100100001001
111111111111
000000000000
000000000000
000000000000

# Backward-Adaptive Quantization

There are some downsides to Forward AQ:

- Have to send Side Information… reduces the compression ratio
- Block-Size Trade-Offs… Short ➔ captures changes, ↑SI
- Coding Delay… can't quantize any samples in block until see whole block

Backward-Adaptation Addresses These Drawbacks as Follows:

- Monitor which quantization cells the past samples fall in
  - Increase Step Size if outer cells are too common
  - Decrease Step Size if inner cells are too common

Because it is based on past *quantized* values,

- no side info needed for the decoder to synchronize to the encoder
  - At least when no transmission errors occur
- no delay because current sample is quantized based on past samples
- So in principle block size can be set based on rate of signal's change

How many past samples to use? How are the decisions made?

Jayant provided simple answers!!

# Jayant Quantizer (Backward-Adaptive)

- Use *single* most recent output
- If it…
  - is in outer levels, increase Δ … or… is in inner levels, decrease Δ
- Assign each interval a multiplier: $M_k$ for the $k^{\text{th}}$ interval
- Update Δ according to: $\Delta_n = M_{l(n-1)}\Delta_{n-1}$

New Δ

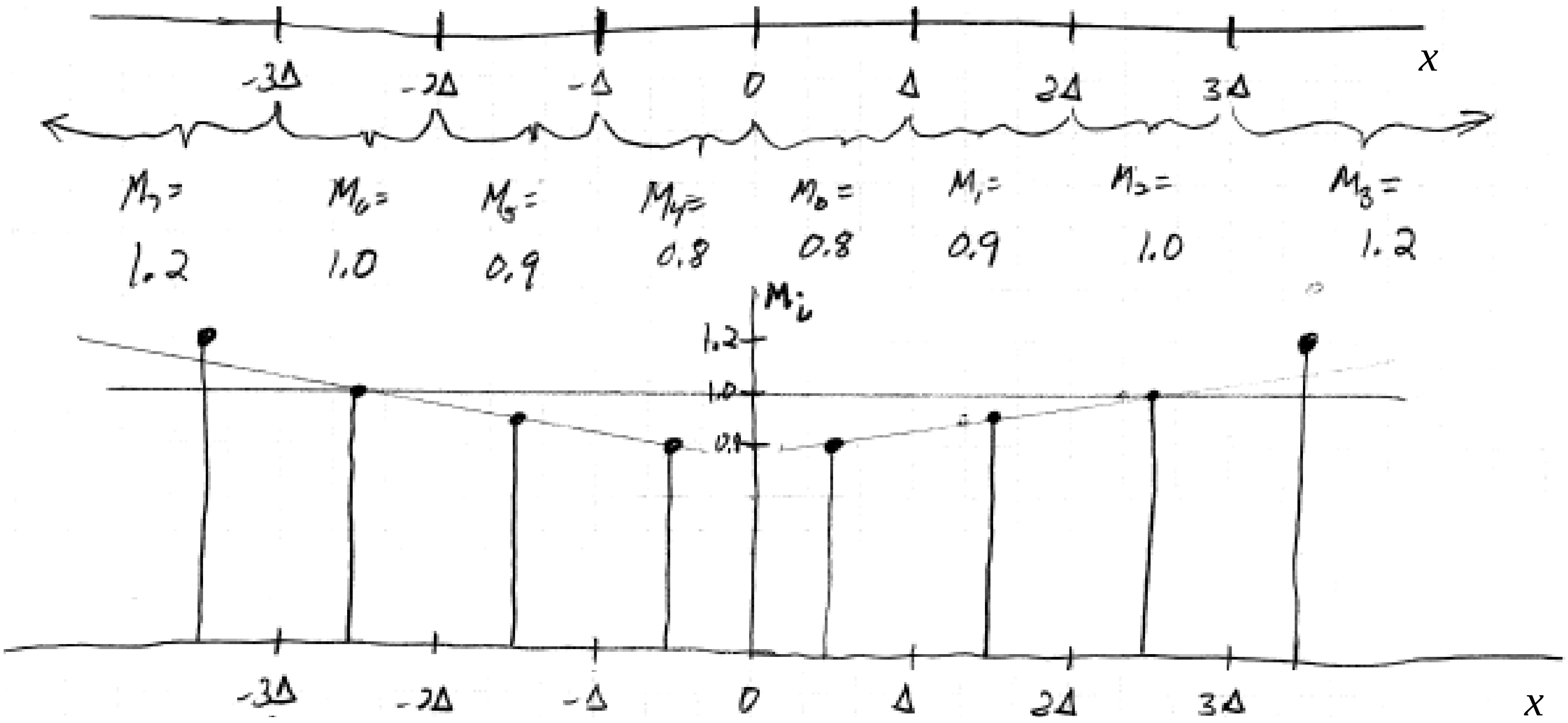
last sample's level index

Old Δ

- Multipliers for outer levels are > 1 (Multipliers are symmetric)

  "  "  inner  "  "  < 1
- Specify $\Delta_{min}$ & $\Delta_{max}$ to avoid "going too far"

# Example of Jayant Multipliers



# How Do We Pick the Jayant Multipliers?

***IF*** we knew the PDF & designed for the correct Δ…

Then we want the multipliers to have no effect after, say, *N* samples:

$$M_3 M_6 M_4 M_1 \cdots M_4 M_5 \approx 1 \quad (\bigstar)$$

Sequence of multipliers for some observed *N* samples…

\# of Levels

Use = "for design"

Let $n_k$ = # of times $M_k$ is used… Then (★) becomes $\prod_{k=0}^{M} M_k^{n_k} = 1$

Now taking the $N^{th}$ root gives… $\prod_{k=0}^{M} M_k^{\frac{n_k}{N}} = 1 \Rightarrow \boxed{\prod_{k=0}^{M} M_k^{P_k} = 1}$

where we have used the frequency of occurrence view: $P_k \approx n_k/N$

For a given "*designed-for*" PDF it is possible to find the correct Δ and then find the resulting $P_k$ probabilities… Then this…

**Is a requirement on the multipliers $M_k$ values**

**But… there are infinitely many solutions!!!!**

One way to further restrict to get a unique solution is to require a specific form on the $M_k$:

$$\boxed{M_k = \gamma^{l_k}}$$

Integers (+/-)

Real # >1

$$\Rightarrow \boxed{\prod_{k=0}^{M} \gamma^{l_k P_k} = 1} \Rightarrow \boxed{\gamma^{\left[\sum_{k=0}^{M} l_k P_k\right]} = 1} \Rightarrow \boxed{\sum_{k=0}^{M} l_k P_k = 0} \quad (\bigstar\bigstar)$$

So… if we've got an idea of the $P_k$… $(\bigstar\bigstar)$ ➔ values for the $l_k$

Then use <u>creativity</u> to select γ: $\left\{ \begin{array}{l} \text{Large } \gamma \text{ gives fast adaptation} \\ \text{Small } \gamma \text{ gives slow adaptation} \end{array} \right.$

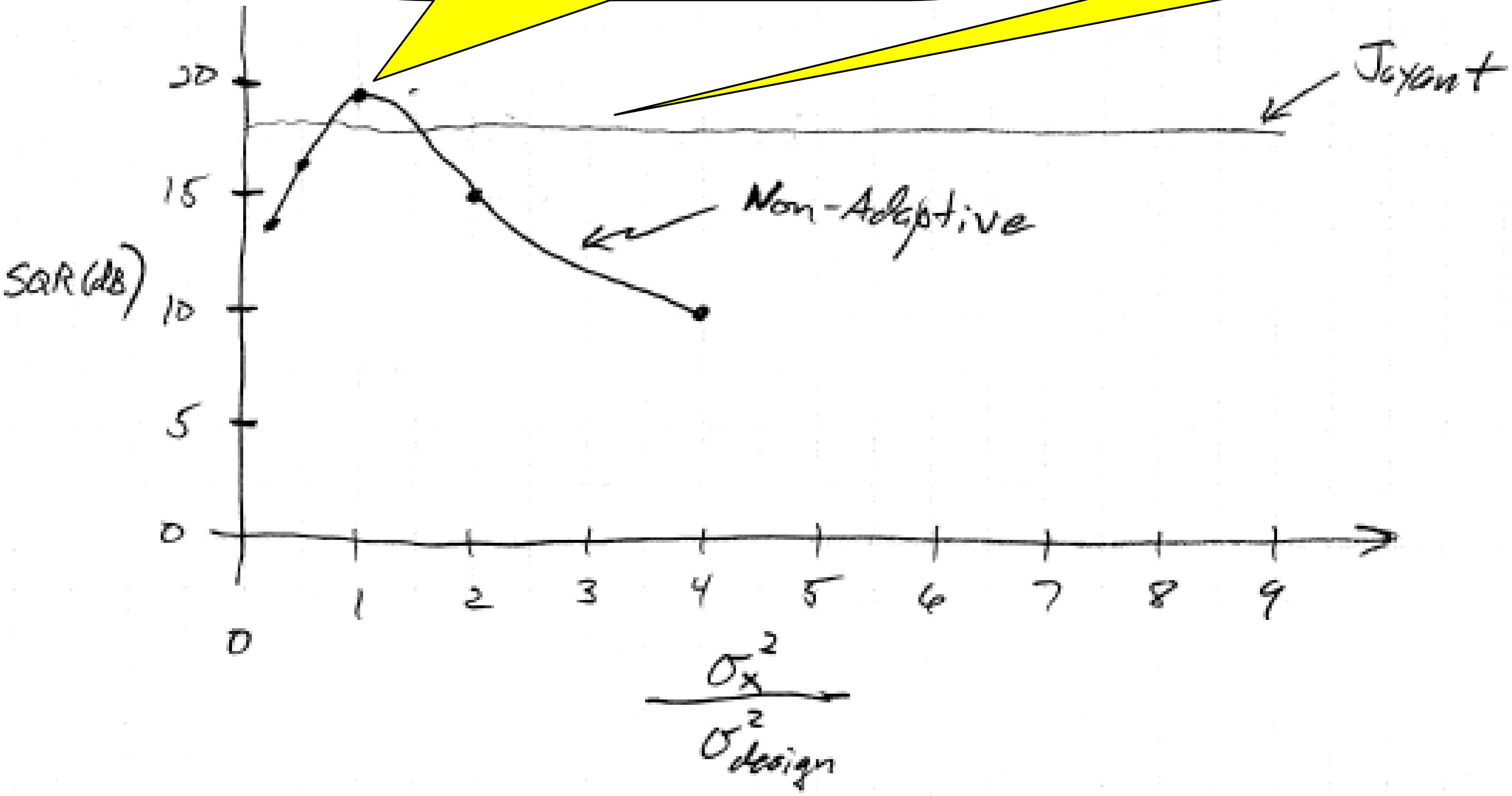
In general we want faster expansion than contraction:

- Samples in <u>outer</u> levels indicate possible overload (potential big overload error)… so we need to expand fast to eliminate this potential
- Samples in <u>inner</u> levels indicate possible underload (granular error is likely too big… but granular is not as dire as overload)… so we only need to contract slowly

# Robustness of Jayant Quantizer

Optimal Non-Adapt is slightly better when perfectly matched

Jayan is significantly better when mismatched



Combining Figs. 9.11 & 9.18