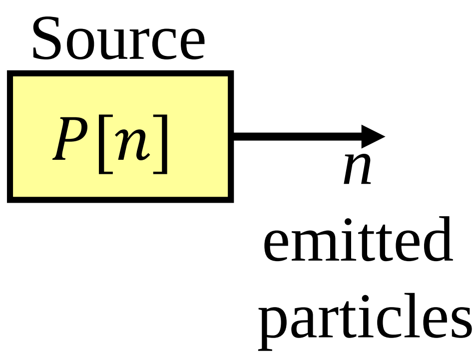
Bayesian Ex. – Imperfect Geiger Counter†

A radioactive source emits n radioactive particles, where n is random.

Our job is to estimate how many particles were emitted.



A common model for # of times something occurs is the Poisson distribution

For Bayesian estimation we need a ***prior probability*** for n.

Suppose we've determined it is Poisson w/ known parameter λ:

$$P[n] = e^{-\lambda}\frac{\lambda^n}{n!}, \quad n \geq 0$$

$$E\{n\} = \lambda \qquad \text{var}\{n\} = \lambda$$

Parm.

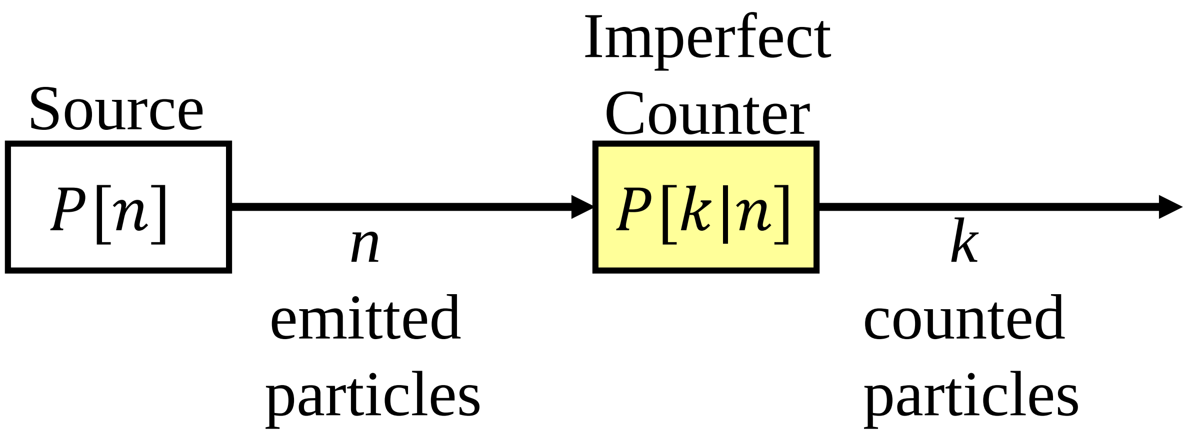
Standard Results for Poisson

†Based on pp. 287 – 290 of L. Scharf, *Statistical Signal Processing: Detection, Estimation, and Time Series Analysis*, Addison-Wesley, 1991. Which itself was based on a 1958 book by E. T. Jaynes!

But suppose we have an imperfect Geiger counter…

It misses some particles → Let p be the prob of detecting a particle.

So we only count $k \leq n$ particles with cond. prob. of $P[k|n]$



Binomial Dist. is the classic result for "*k* successes out of *n* tries with prob of success of *p*"

This $P[k|n]$ is the classic binomial distribution:

$$P[k \mid n] = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n$$

$$E\{k \mid n\} = np$$

$$\text{var}\{k \mid n\} = np(1-p)$$

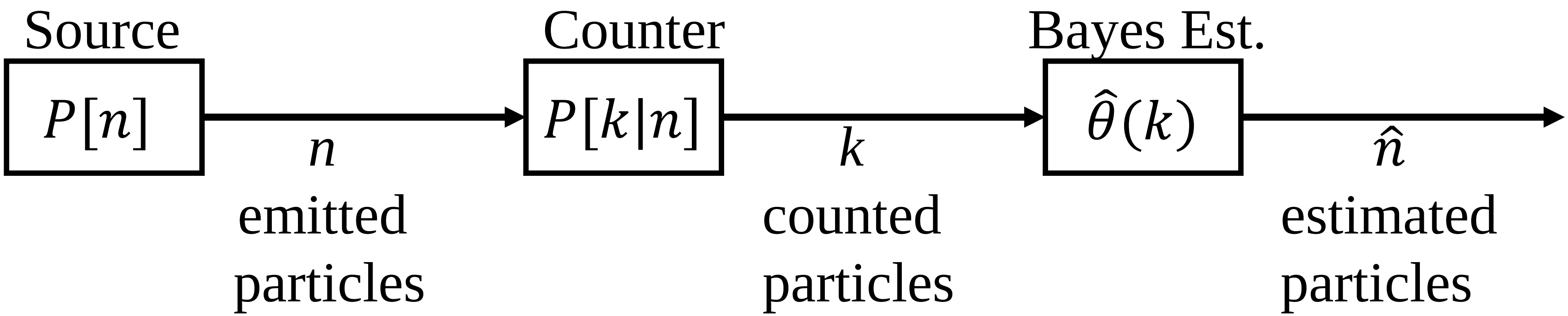
Data

Parm.

Standard Results for Binomial

We could just accept the count k…

Or… devise a Bayesian estimator: map observed k into estimate $\hat{n}$



Regardless of which Bayesian estimation form we use, we need the ***posterior probability*** for n.

$$P[n \mid k] = \frac{P[k,n]}{P[k]} = \frac{P[k \mid n]P[n]}{P[k]}$$

Bayes' Rule!

Parm.

Data

So we need to determine all this stuff!

Need to analyze: $$P[n\,|\,k]=\frac{P[k,n]}{P[k]}=\frac{P[k\,|\,n]P[n]}{P[k]}$$

First, the numerator – we have both of the parts so plug in:

$$P[k,n]=P[k\,|\,n]P[n] \quad\Rightarrow\quad P[k,n]=\binom{n}{k}p^k(1-p)^{n-k}e^{-\lambda}\frac{\lambda^n}{n!};\quad 0\le k\le n$$

Second, the denominator – we have the joint prob and need to sum it to get the marginal on *k*:

$$P[k]=\sum_{n=k}^{\infty}P[k,n]=\sum_{n=k}^{\infty}\binom{n}{k}p^k(1-p)^{n-k}e^{-\lambda}\frac{\lambda^n}{n!} \quad\Rightarrow\quad P[k]=e^{-\lambda p}\frac{(\lambda p)^k}{k!}$$

How?

Thinking: Get into a power series! → $=\lambda^k\lambda^{n-k}$

$$P[k]=\sum_{n=k}^{\infty}\left(\frac{n!}{k!(n-k)!}\right)p^k(1-p)^{n-k}e^{-\lambda}\frac{\lambda^n}{n!}=\frac{e^{-\lambda}p^k\lambda^k}{k!}\sum_{n=k}^{\infty}\left(\frac{1}{(n-k)!}\right)[\lambda(1-p)]^{n-k}$$

$$=\frac{e^{-\lambda}p^k\lambda^k}{k!}e^{\lambda(1-p)}\quad=\frac{e^{-\lambda p}(\lambda p)^k}{k!}$$

Use $e^x=1/0!+x/1!+x^2/2!+x^3/3!+\cdots$

So now we have what we need to form $P[n|k]$:

$$P[n\,|\,k]=\frac{P[k,n]}{P[k]}=\frac{\left(\dfrac{n!}{k!(n-k)!}\right)p^k(1-p)^{n-k}\,e^{-\lambda}\,\lambda^n/n!}{e^{-\lambda p}(\lambda p)^k/k!}$$

… which gives us the posterior distribution we need!

$$P[n\,|\,k]=\frac{1}{(n-k)!}\left[\lambda(1-p)\right]^{n-k}e^{-\lambda(1-p)};\quad n\geq k\geq 0$$

What is it??? Compare to regular Poisson $\quad P_\lambda[n]=e^{-\lambda}\dfrac{\lambda^n}{n!},\quad n\geq 0$

Looks like a k-shifted Poisson w/ parameter $\lambda(1-p)$!!!!

$$P_{\lambda(1-p)}[n-k]=e^{-\lambda(1-p)}\frac{\left(\lambda(1-p)\right)^{n-k}}{(n-k)!},\quad n-k\geq 0$$

So can use std. Poisson results to get conditional mean and variance:

$$E\{n\,|\,k\}=k+\lambda(1-p)$$

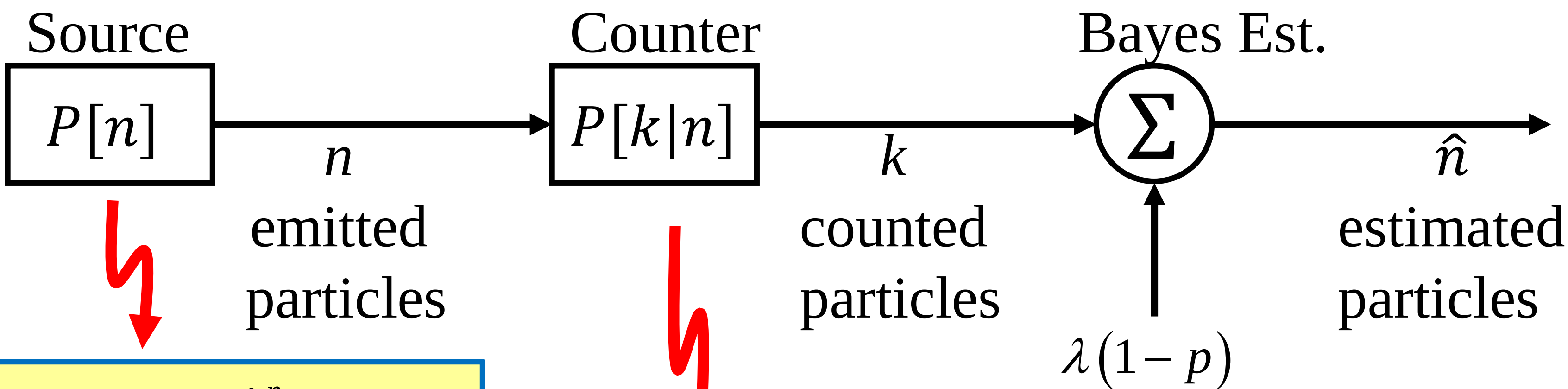
k-shift of prob. func. shifts mean by k

$$\mathrm{var}\{n\,|\,k\}=E\left\{\left(n-E\{n\,|\,k\}\right)^2|\,k\right\}=\lambda(1-p)$$

shift of prob. func. has no effect on variance

So now if we use quadratic Bayes risk, the MMSE estimate is the conditional mean:

$$\hat{n} = E\{n \mid k\} = k + \lambda(1-p)$$



$$P_\lambda[n] = e^{-\lambda}\frac{\lambda^n}{n!}, \quad n \geq 0$$

$$E\{n\} = \lambda$$

Expected # of Particles

$$(1-p)$$

Prob. of Missing a Particle

$$\lambda(1-p)$$

Expected # of Missed Particles

Makes Sense: Add this to count!!

What is the performance of this estimator?

We have general results that say the MMSE estimator…

- Is unbiased: $E_{nk}\{\hat{n}\} = E_n\{n\} = \lambda$
- Has variance = Bmse

"Decomposing Joint Expected Values"

$$Bmse = E_{nk}\left\{(n-\hat{n})^2\right\} = E_k\left\{\underbrace{E_{n|k}\left\{(n - E\{n \mid k\})^2\right\}}\right\} = E_k\left\{\lambda(1-p)\right\} = \lambda(1-p)$$

(variance of $P[n|k]$) $= \lambda(1-p)$

Thus, the performance of this estimator is characterized by

$$E\{\hat{n}-n\} = 0$$

$$\text{var}\{\hat{n}-n\} = Bmse = \lambda(1-p)$$