3.7 CRLB for Vector Parameter Case

Vector Parameter: $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_p \end{bmatrix}^T$

Its Estimate: $\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{\theta}_1 & \hat{\theta}_2 & \cdots & \hat{\theta}_p \end{bmatrix}^T$

Assume that estimate is unbiased: $E\left\{\hat{\boldsymbol{\theta}}\right\} = \boldsymbol{\theta}$

For a scalar parameter we looked at its variance…

but for a **vector parameter** we look at its **covariance matrix**:

$$\text{var}\left\{\hat{\boldsymbol{\theta}}\right\} = E\left\{ \left[\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right]\left[\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right]^T \right\} = \mathbf{C}_{\hat{\boldsymbol{\theta}}}$$

For example:

for $\theta = [x \;\; y \;\; z]^{\mathrm{T}}$

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} = \begin{bmatrix} \text{var}(\hat{x}) & \text{cov}(\hat{x}, \hat{y}) & \text{cov}(\hat{x}, \hat{z}) \\ \text{cov}(\hat{y}, \hat{x}) & \text{var}(\hat{y}) & \text{cov}(\hat{y}, \hat{z}) \\ \text{cov}(\hat{z}, \hat{x}) & \text{cov}(\hat{z}, \hat{y}) & \text{var}(\hat{z}) \end{bmatrix}$$

Fisher Information Matrix

For the vector parameter case…

Fisher Info becomes the **Fisher Info Matrix (FIM)** $\mathbf{I}(\theta)$ whose mn^{th} element is given by:

$$\left[\mathbf{I}(\boldsymbol{\theta})\right]_{mn} = -E\left\{\frac{\partial^2 \ln\left[p(\mathbf{x};\boldsymbol{\theta})\right]}{\partial\theta_n\,\partial\theta_m}\right\}, \quad m,n = 1, 2, \ldots, p$$

Evaluate at true value of θ

The CRLB Matrix

Then, under the same kind of regularity conditions,
the **CRLB matrix** is the inverse of the FIM:

$$CRLB = \mathbf{I}^{-1}(\boldsymbol{\theta})$$

So what this means is:

$$\sigma^2_{\hat{\theta}_n} = \left[\mathbf{C}_{\hat{\boldsymbol{\theta}}}\right]_{nn} \geq \left[\mathbf{I}^{-1}(\boldsymbol{\theta})\right]_{nn} \qquad (*)$$

Diagonal elements of Inverse FIM bound the parameter variances, which are the diagonal elements of the parameter covariance matrix

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} = \begin{bmatrix} \mathrm{var}(\hat{x}) & \mathrm{cov}(\hat{x},\hat{y}) & \mathrm{cov}(\hat{x},\hat{z}) \\ \mathrm{cov}(\hat{y},\hat{x}) & \mathrm{var}(\hat{y}) & \mathrm{cov}(\hat{y},\hat{z}) \\ \mathrm{cov}(\hat{z},\hat{x}) & \mathrm{cov}(\hat{z},\hat{y}) & \mathrm{var}(\hat{z}) \end{bmatrix} \qquad \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \mathbf{I}^{-1}(\boldsymbol{\theta})$$

More General Form of The CRLB Matrix

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} - \mathbf{I}^{-1}(\boldsymbol{\theta}) \quad \text{is positive semi-definite}$$

Mathematical Notation for this is:

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} - \mathbf{I}^{-1}(\boldsymbol{\theta}) \geq \mathbf{0} \qquad (\ast\ast)$$

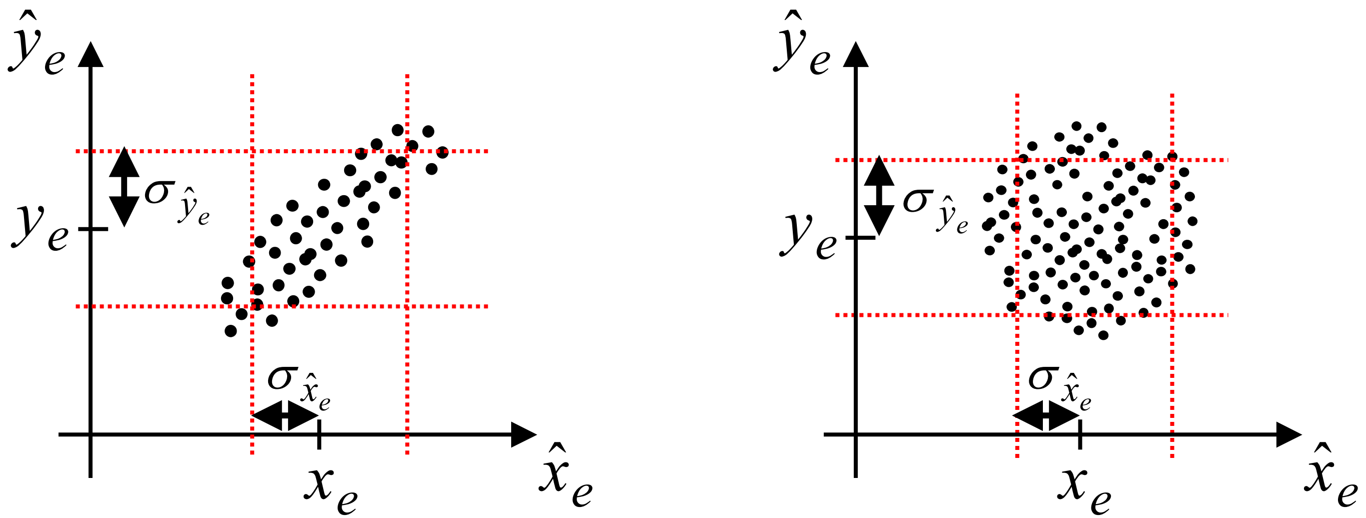
<u>Note</u>: property #5 about p.d. matrices on p. 573 states that $(\ast\ast) \Rightarrow (\ast)$

CRLB Off-Diagonal Elements Insight

Not In Book

Let $\theta = [x_e \;\; y_e]^T$ represent the 2-D x-y location of a transmitter (emitter) to be estimated.

Consider the two cases of "scatter plots" for the estimated location:



Each case has the same variances… but location accuracy characteristics are very different. $\Rightarrow$ This is the effect of the off-diagonal elements of the covariance

Should consider effect of off-diagonal CRLB elements!!!

CRLB Matrix and Error Ellipsoids

Not In Book

Assume $\hat{\boldsymbol{\theta}} = [\hat{x}_e \quad \hat{y}_e]^T$ is 2-D Gaussian w/ zero mean and a cov matrix $\mathbf{C}_{\hat{\boldsymbol{\theta}}}$

Only For Convenience

Then its PDF is given by:

$$p(\hat{\boldsymbol{\theta}}) = \frac{1}{(2\pi)^N \sqrt{|\mathbf{C}_{\hat{\boldsymbol{\theta}}}|}} \exp\left[-\frac{1}{2} \hat{\boldsymbol{\theta}}^T \mathbf{C}_{\hat{\boldsymbol{\theta}}}^{-1} \hat{\boldsymbol{\theta}} \right]$$

Quadratic Form!!
(recall: it's scalar valued)

Let : $\mathbf{C}_{\hat{\boldsymbol{\theta}}}^{-1} = \mathbf{A}$ for ease

So the "equi-height contours" of this PDF are given by the values of $\hat{\boldsymbol{\theta}}$ such that:

$$\hat{\boldsymbol{\theta}}^T \mathbf{A} \hat{\boldsymbol{\theta}} = k$$

Some constant

Note: **A** is symmetric so $a_{12} = a_{21}$ …because any cov. matrix is symmetric and the inverse of symmetric is symmetric

What does this look like? $a_{11}\hat{x}_e^2 + 2a_{12}\hat{x}_e\hat{y}_e + a_{22}\hat{y}_e^2 = k$

An Ellipse!!! (Look it up in your calculus book!!!)

Recall: If $a_{12} = 0$, then the ellipse is aligned w/ the axes & the a_{11} and a_{22} control the size of the ellipse along the axes

Note: $a_{12} = 0 \Rightarrow \quad \mathbf{C}_{\hat{\boldsymbol{\theta}}}^{-1} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \Rightarrow \mathbf{C}_{\hat{\boldsymbol{\theta}}} = \begin{bmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{bmatrix}$

$\Rightarrow \quad \hat{x}_e$ & $\hat{y}_e$ are uncorrelated

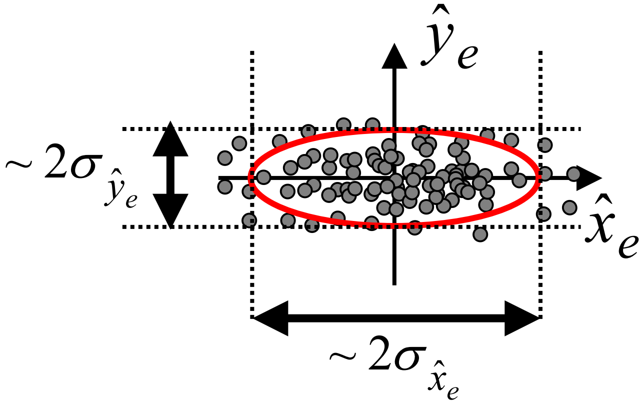
Note: $a_{12} \neq 0 \quad \Rightarrow \quad \hat{x}_e$ & $\hat{y}_e$ are correlated

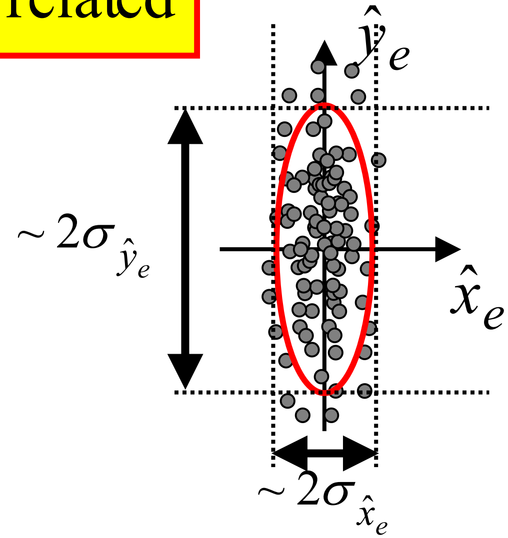
$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} = \begin{bmatrix} \sigma_{\hat{x}_e}^2 & \sigma_{\hat{x}_e\hat{y}_e} \\ \sigma_{\hat{y}_e\hat{x}_e} & \sigma_{\hat{y}_e}^2 \end{bmatrix}$$

Error Ellipsoids and Correlation

Not In Book

$if \quad \hat{x}_e \;\&\; \hat{y}_e$ are uncorrelated





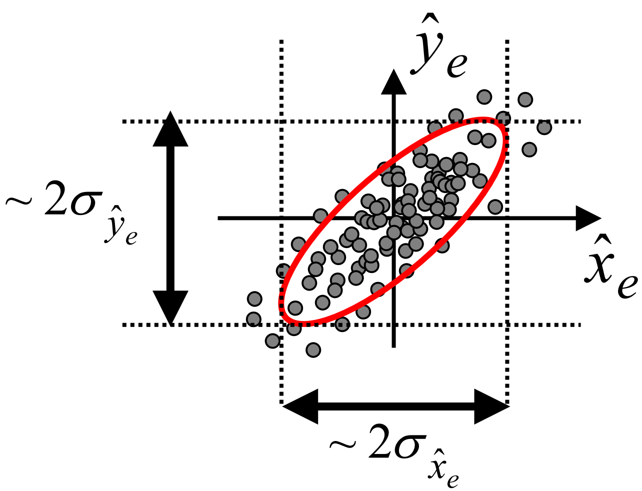
Choosing *k* Value

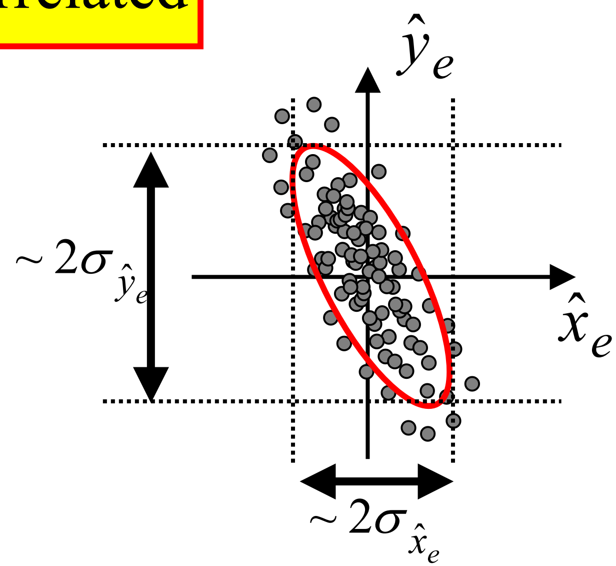
For the 2-D case…

$$k = -2 \ln(1-P_e)$$

where P_e is the prob. that the estimate will lie inside the ellipse

$if \quad \hat{x}_e \;\&\; \hat{y}_e$ are correlated





See posted paper by Torrieri

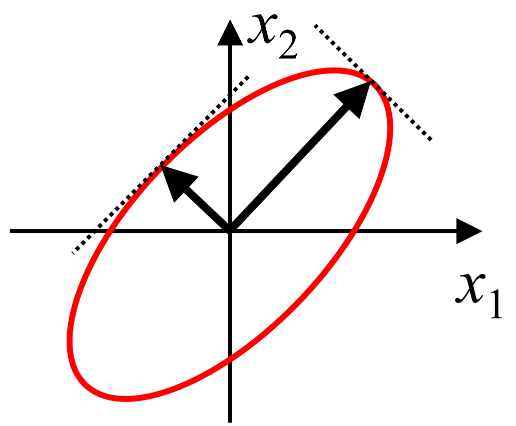
Ellipsoids and Eigen-Structure

Not In Book

Consider a symmetric matrix $\mathbf{A}$ & its quadratic form $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$

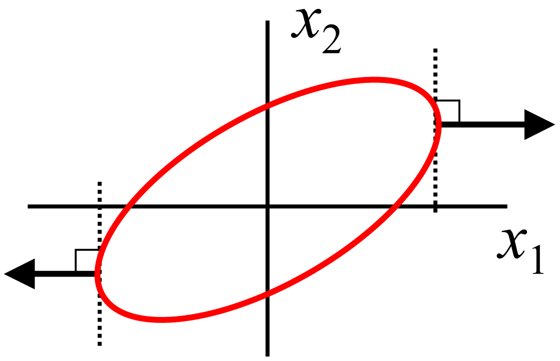
$\Rightarrow$ Ellipsoid: $\mathbf{x}^{T}\mathbf{A}\mathbf{x} = k$ or $\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = k$

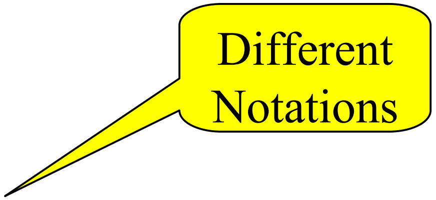
Principle Axes of Ellipse are orthogonal to each other… and are orthogonal to the tangent line on the ellipse:



Theorem: The principle axes of the ellipsoid $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = k$ are eigenvectors of matrix **A.**

Proof: From multi-dimensional calculus: gradient of a scalar-valued function $\phi(x_1, \ldots, x_n)$ is orthogonal to the surface:





$$grad\ \phi(x_1, \ldots, x_n) = \nabla_{\mathbf{x}} \phi(\mathbf{x}) = \frac{\partial \phi(\mathbf{x})}{\partial \mathbf{x}} =$$

$$= \begin{bmatrix} \frac{\partial \phi}{\partial x_1} & \cdots & \frac{\partial \phi}{\partial x_n} \end{bmatrix}^T$$

See handout posted on Blackboard on Gradients and Derivatives

For our quadratic form function we have:

$$\phi(\mathrm{x}) = \mathrm{x}^T \mathrm{A}\,\mathrm{x} = \sum_i \sum_j a_{ij} x_i x_j \quad \Rightarrow \quad \frac{\partial \phi}{\partial x_k} = \sum_i \sum_j a_{ij} \frac{\partial (x_i x_j)}{\partial x_k} \qquad (\clubsuit)$$

Product rule: $$\frac{\partial (x_i x_j)}{\partial x_k} = \underbrace{\frac{\partial x_i}{\partial x_k}}_{=\delta_{ik} = \begin{cases} 1 & i=k \\ 0 & i \neq k \end{cases}} x_j + x_i \underbrace{\frac{\partial x_j}{\partial x_k}}_{\delta_{jk}} \qquad (\clubsuit\clubsuit)$$

Using ($\clubsuit\clubsuit$) in ($\clubsuit$) gives: $$\frac{\partial \phi}{\partial x_k} = \sum_j a_{jk} x_j + \sum_i a_{ik} x_j$$

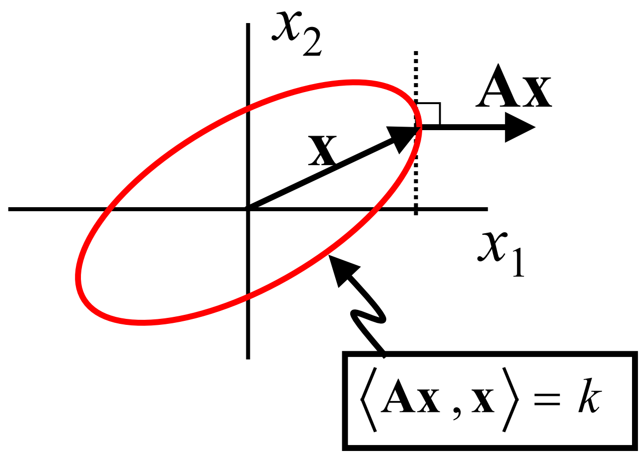
$$= 2 \sum_j a_{kj} x_j$$

By Symmetry:
$a_{ik} = a_{ki}$

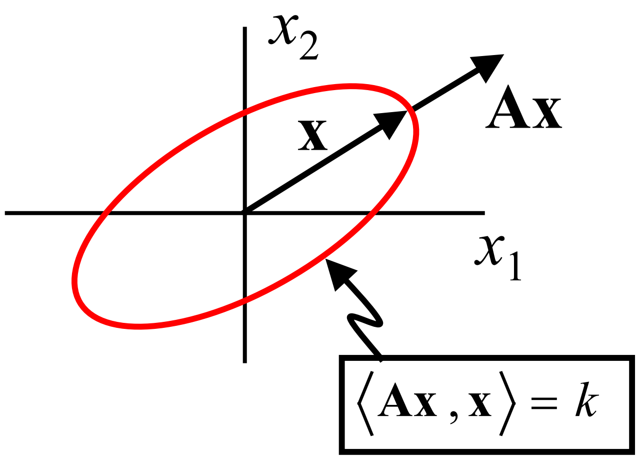
And from this we get:

$$\boxed{\nabla_{\mathbf{x}} (\mathbf{x}^T \mathbf{A} \mathbf{x}) = 2\mathbf{A}\mathbf{x}}$$

Since *grad* ⊥ ellipse, this says $\mathbf{Ax}$ is ⊥ ellipse:



When $\mathbf{x}$ is a principle axis, then $\mathbf{x}$ and $\mathbf{Ax}$ are aligned:



$$\mathbf{Ax} = \lambda \mathbf{x}$$

Eigenvectors are Principle Axes!!!

< End of Proof >

<u>Theorem</u>: The length of the principle axis associated with eigenvalue λ_i is $\sqrt{k / \lambda_i}$

<u>Proof</u>: If **x** is a principle axis, then $\mathbf{Ax} = \lambda\mathbf{x}$. Take inner product of both sides of this with **x**:

$$\underbrace{\langle \mathbf{Ax}, \mathbf{x} \rangle}_{=k} = \lambda \langle \mathbf{x}, \mathbf{x} \rangle \qquad \underbrace{\langle \mathbf{x}, \mathbf{x} \rangle}_{=\|\mathbf{x}\|^2} = \frac{k}{\lambda} \quad \Rightarrow \quad \|\mathbf{x}\| = \sqrt{\frac{k}{\lambda}}$$

< End of Proof >

<u>Note</u>: This says that if **A** has a zero eigenvalue, then the error ellipse will have an infinite length principle axis $\Rightarrow$ **<u>NOT GOOD!!</u>**

So… we'll require that all $\lambda_i > 0$
$\Rightarrow \mathbf{C}_{\hat{\boldsymbol{\theta}}}$ must be positive definite

Application of Eigen-Results to Error Ellipsoids

The Error Ellipsoid corresponding to the estimator covariance matrix $\mathbf{C}_{\hat{\boldsymbol{\theta}}}$ must satisfy:

$$\hat{\boldsymbol{\theta}}^T \mathbf{C}_{\hat{\boldsymbol{\theta}}}^{-1} \hat{\boldsymbol{\theta}} = k$$

Note that the error ellipse is formed using the ***inverse*** cov

Thus finding the eigenvectors/values of $\mathbf{C}_{\hat{\boldsymbol{\theta}}}^{-1}$ shows structure of the error ellipse

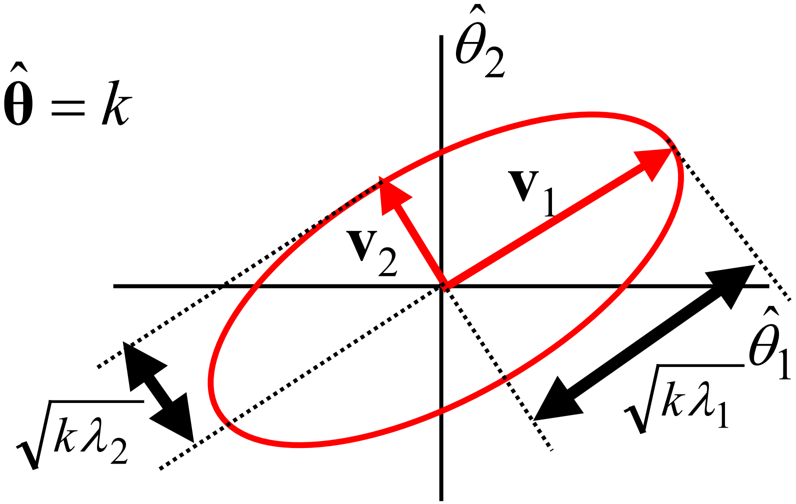
Recall: Positive definite matrix **A** and its inverse $\mathbf{A}^{-1}$ have the

- same eigenvectors
- reciprocal eigenvalues

Thus, we could instead find the eigenvalues of $\mathbf{C}_{\hat{\boldsymbol{\theta}}} = \mathbf{I}^{-1}(\boldsymbol{\theta})$ and then the principle axes would have lengths set by its eigenvalues not inverted

Inverse FIM!!

Illustrate with 2-D case: $\hat{\boldsymbol{\theta}}^T \mathbf{C}_{\hat{\boldsymbol{\theta}}}^{-1} \hat{\boldsymbol{\theta}} = k$



$\mathbf{v}_1$ & $\mathbf{v}_2$
λ_1 & λ_2

Eigenvectors/values for $\mathbf{C}_{\hat{\boldsymbol{\theta}}}$
(not the ***inverse***!)

The CRLB/FIM Ellipse

Can make an ellipse from the CRLB Matrix…
instead of the Cov. Matrix

This ellipse will be the smallest error ellipse that an unbiased estimator can achieve!

We can re-state this in terms of the FIM…

Once we find the FIM we can:

- Find the inverse FIM
- Find its eigenvectors… gives the Principle Axes
- Find its eigenvalues… Prin. Axis lengths are then $\sqrt{k\lambda_i}$