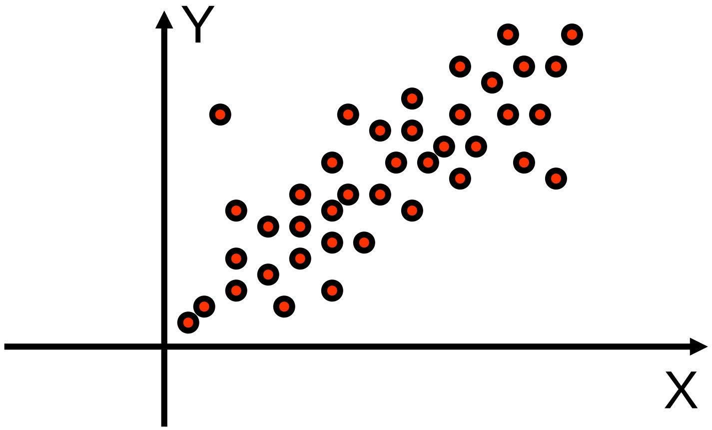
Multiple Random Processes & Relationships

Multiple Random Variables

When you studied RV's you considered how two (or more) RV's were related

X = GPA of Engineering Student

Y = Starting Salary of Eng. Student



- Shows a **statistical relationship** between X & Y
- Here: X & Y are **correlated**
- **Correlation allows prediction** (but not perfectly) of Y given a specific value of X

Multiple Random Processes

Want to do a similar thing with Random Processes!!

For two Random Processes x(t) & y(t) define:

Cross-Correlation Function $= R_{xy}(t_1,t_2)$
$= E\{x(t_1)\ y(t_2)\}$

Measures statistical similarity
between x(t) at t_1 …
& y(t) at t_2
… as a function of t_1 and t_2

Jointly WSS Processes

We are most often interested in case when x(t) & y(t) are both WSS and are also Jointly WSS, which happens if:

$$R_{xy}(t_1,t_2) = E\{x(t_1)\, y(t_2)\}$$

$$= R_{xy}(\underbrace{t_2 - t_1}_{=\tau})$$

Thus, Jointly WSS processes have a cross-correlation function that depends only on relative time:

$$R_{xy}(\tau)=E\{x(t)\, y(t+\tau)\}$$

Uncorrelated Jointly WSS RP's

Jointly WSS RP's x(t) and y(t) are said to be:

"Uncorrelated" if

$$R_{XY}(\tau) = E\{x(t)y(t+\tau)\} = E\{x(t)\}\,E\{y(t)\}$$

"Orthogonal" if

$$R_{XY}(\tau) = 0$$

This is a special case of Uncorrelated w/ the extra condition of:

$E\{x(t)\} = 0$ and/or $E\{y(t)\} = 0$

"Independent" if

For <u>any</u> t_1 & t_2
The RVs $x(t_1)$ and $y(t_2)$ are independent

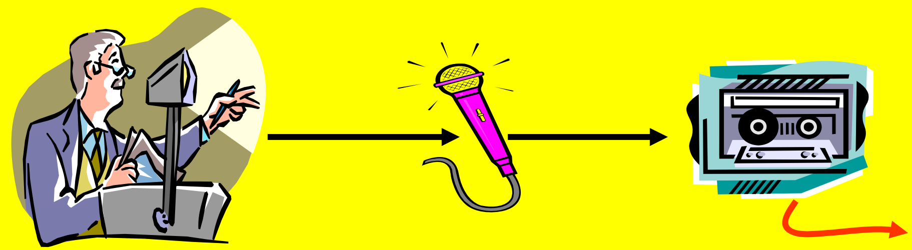
<u>NOTE</u>: Independence ⇨ Uncorrelated

But… Uncorrelated ⇏ Independence

Independent vs. Uncorrelated

When they arise from different generating mechanisms…
… We often assume they are either:
independent or uncorrelated

Ex: - Speech is often modeled as a RP
- Thermal Noise in electronic systems



Speech Noise

$x(t)=s(t)+h(t)$

Often assume s(t) and n(t) are either
Independent or **Uncorrelated**

Which one? The weakest assumption that lets you do your analysis!

Speech + Noise Example

Assume s(t) and n(t) are **both WSS with zero means**.
Assume n(t) is **white noise** .
Assume s(t) and n(t) are **uncorrelated** with each other
($\Rightarrow$ orthogonal since zero means).

Find the PSD of x(t)

Solution

Approach: **Find ACF $R_x(\tau)$ & take FT to get PSD**

Speech + Noise Example (cont.)

$$R_x(\tau) = E\{x(t)\,x(t+\tau)\} = E\{[s(t)+n(t)]\,[s(t+\tau)+n(t+\tau)]\}$$

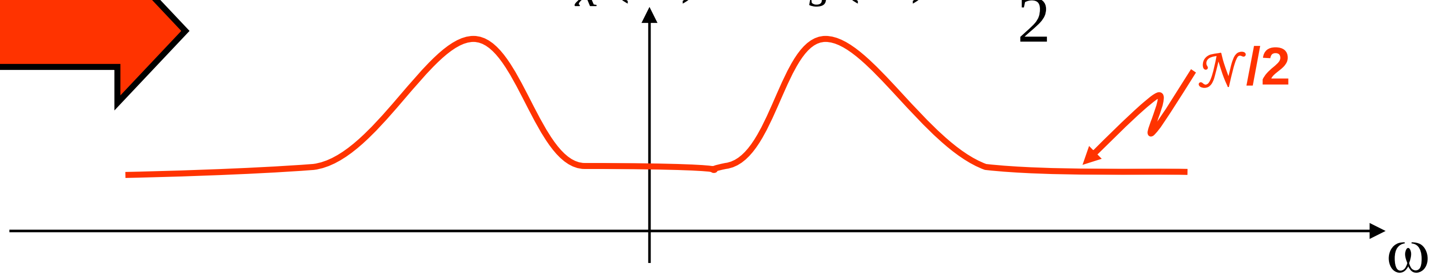
$$= \underbrace{E\{s(t)s(t+\tau)\}}_{R_s(\tau)} + \underbrace{E\{n(t)s(t+\tau)\}}_{=0 \text{ Assumed Orthog!}}$$

$$+ \underbrace{E\{s(t)n(t+\tau)\}}_{=0 \text{ Assumed Orthog!}} + \underbrace{E\{n(t)n(t+\tau)\}}_{R_n(\tau)=\frac{\mathcal{N}}{2}\delta(\tau)}$$

$$= R_s(\tau) + \frac{\mathcal{N}}{2}\delta(\tau) \quad \Leftrightarrow \quad S_x(\omega) = S_s(\omega) + \frac{\mathcal{N}}{2}$$

Speech + Noise Example (cont.)

$$S_x(\omega) = S_s(\omega) + \frac{\mathcal{N}}{2}$$



Insight from Example: Our assumption of uncorrelated and zero-mean processes resulted in a simple and very usable form for the PSD

Without that assumption:

- difficult or impossible to analyze
- more complicated result may be harder to interpret

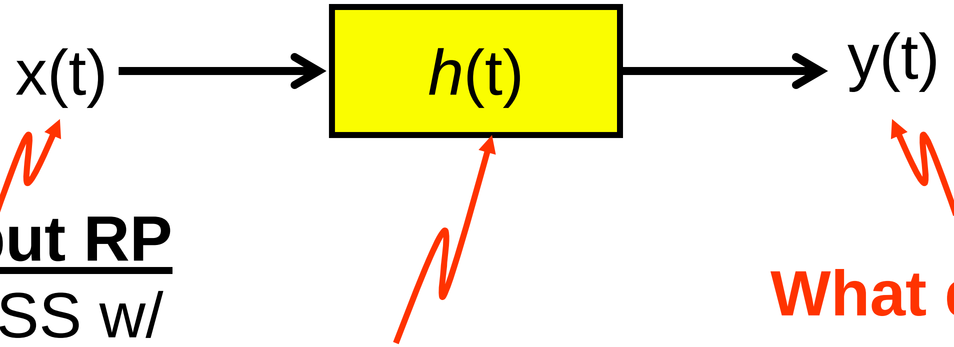
Modeling Trade-Off: Want a model that is….

- simple enough to give insight but not so simple it is "wrong"
- complex enough to be "right" but not so complex it gives no insight

RPs Through LTI Systems

We already saw that passing DT white noise through a FIR filter reshapes the ACF and PSD.

Here we learn the <u>General Theory:</u>
(extremely useful for Modeling Practical RP's)



<u>Input RP</u>
WSS w/
$R_x(\tau)$
$S_x(\omega)$

<u>LTI System</u>
Impulse Response $h(t)$
Frequency Response $H(\omega) = \mathfrak{F}\{h(t)\}$

<u>Output RP</u>
What does it look like?

RPs & LTI Systems: Results

To describe output RP y(t) we look at its:

(i) Mean

(ii) ACF and

(iii) PSD

<u>Results First (Proof Later)</u>

(i) Mean:

$$E\{y(t)\} = H(0)E\{x(t)\}$$

<u>Comment</u>: Means are viewed as the DC Value of a RP – it makes sense that the <u>Filter's DC Response, H(0)</u>, transfers "input-DC" to "output-DC"

RPs & LTI Systems: Results

(ii) ACF: $$R_y(\tau) = h(\tau)*h(-\tau)*R_x(\tau)$$

Comments:

(1) Implicit in this is **"WSS into LTI gives WSS out"**

(2) The "second-order" dependence on h(.) comes from the ACF being a "second-order" characteristic

(3) ACF is a time-domain characteristic so it makes sense that convolution is involved.

RPs & LTI Systems: Results

(iii) PSD:

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$

Comments: (1) Again, 2nd-order dependence on H(ω) comes from PSD being a 2nd-order characteristic

(2) PSD is a Frequency-domain characteristic so it makes sense that the frequency response H(ω) is involved.

RPs & LTI Systems: Proof

MEAN

$$E\{y(t)\} = E\left\{ \int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha \right\}$$

Use Convolution to get output from input

$$= \int_{-\infty}^{\infty} h(\alpha) \underbrace{E\{x(t-\alpha)\}}_{\bar{x}} d\alpha$$

Since E{.} is an integration, this is like changing order of integration.
Note: h(.) is not random so It gets pulled outside E{.}.

Since x(t) is WSS

$$= \bar{x} \underbrace{\int_{-\infty}^{\infty} h(\alpha)\, d\alpha}$$

$$H(0) = [\int_{-\infty}^{\infty} h(t) e^{j\omega t} dt]\Big|_{\omega=o}$$

RPs & LTI Systems: Proof

ACF:

$$R_y(\tau) = E\{y(t)y(t+\tau)\}$$

$$= E\left\{ [\int_{-\infty}^{\infty} h(\alpha)x(t-\alpha)d\alpha][\int_{-\infty}^{\infty} h(\beta)x(t+\tau-\beta)d\beta] \right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)\underbrace{E\{x(t+\tau-\beta)x(t-\alpha)\}}_{R_x(\tau+\alpha-\beta)}d\alpha d\beta$$

$$= h(\tau) * h(-\tau) * R_x(\tau)$$

PSD: Follows from the ACF result and two applications of the convolution–FT property

RPs & LTI Systems: In vs Out

Now, for this system:

Q: How is the output correlated with the input ?

A: Compute their Cross-correlation Function:

Result:

$$R_{xy}(\tau) = h(\tau) * R_x(\tau)$$

Proof:

$$R_{xy}(\tau) = E\left\{ x(t) \int_{-\infty}^{\infty} h(\beta) x(t+\tau-\beta)\, d\beta \right\}$$

$$= \int_{-\infty}^{\infty} h(\beta) \underbrace{E\{x(t)x(t+\tau-\beta)\}}_{R_x(\tau-\beta)}\, d\beta$$

Ex: Filtered White Noise

Earlier we looked at figures showing how five different (but similar) filters impact the output ACF.

Recall that in those examples the input was **D-T white noise** $\Rightarrow R_X[m]=\sigma^2\delta[m]$. Thus the output ACF's are just the convolution: $\sigma^2 h[m]*h[-m]$.

The filters in the previous case all had rectangular impulse responses, which when convolved like this give the triangular ACF's shown in the previous figures.

Note also: rectangular FIR filters are low-pass filters whose cut-off frequency gets lower as the filter length increases.

Ex: Filtered White Noise

Thus , Since $S_y(\Omega) = |H(\Omega)|^2 \underbrace{S_x(\Omega)}$

$= \mathcal{N}/2$ for White Noise

PSD's of processes that are outputs of longer rectangle filters have narrower PSD's

Wide ACF ⟷ **Narrow PSD** ⇒ Process has slow fluctuations

Narrow ACF ⟷ **Broad PSD** ⇒ Process has fast Fluctuations