Multiple Random Processes & Relationships
Multiple Random Variables

When you studied RV’s you considered how two (or more) RV’s were related

$X = \text{GPA of Engineering Student}$

$Y = \text{Starting Salary of Eng. Student}$

Shows a statistical relationship between $X$ & $Y$

Here: $X$ & $Y$ are correlated

Correlation allows prediction (but not perfectly) of $Y$ given a specific value of $X$
Multiple Random Processes

Want to do a similar thing with Random Processes!!

For two Random Processes $x(t)$ & $y(t)$ define:

\[
\text{Cross-Correlation Function} = R_{xy}(t_1, t_2) = \mathbb{E}\{x(t_1) y(t_2)\}
\]

Measures statistical similarity between $x(t)$ at $t_1$ …
& $y(t)$ at $t_2$
… as a function of $t_1$ and $t_2$
Jointly WSS Processes

We are most often interested in case when $x(t)$ & $y(t)$ are both WSS and are also Jointly WSS, which happens if:

$$R_{xy}(t_1, t_2) = E\{x(t_1) y(t_2)\} = R_{xy}(t_2 - t_1)$$

Thus, Jointly WSS processes have a cross-correlation function that depends only on relative time:

$$R_{xy}(\tau) = E\{x(t) y(t+\tau)\}$$
Uncorrelated Jointly WSS RP’s

Jointly WSS RP’s $x(t)$ and $y(t)$ are said to be:

“Uncorrelated” if

$$R_{XY}(\tau) = E\{x(t)y(t+\tau)\} = E\{x(t)\} E\{y(t)\}$$

“Orthogonal” if

$$R_{XY}(\tau) = 0$$

This is a special case of Uncorrelated w/ the extra condition of:

$E\{x(t)\} = 0$ and/or $E\{y(t)\} = 0$

“Independent” if

For any $t_1$ & $t_2$

The RVs $x(t_1)$ and $y(t_2)$ are independent

NOTE: Independence $\Rightarrow$ Uncorrelated

But… Uncorrelated $\nRightarrow$ Independence
Independent vs. Uncorrelated

When they arise from different generating mechanisms…
… We often assume they are either:

- independent
- uncorrelated

Ex: - Speech is often modeled as a RP
- Thermal Noise in electronic systems

Often assume \( s(t) \) and \( n(t) \) are either

- **Independent**
- **Uncorrelated**

Which one? The **weakest assumption** that lets you do your analysis!
Speech + Noise Example

Assume $s(t)$ and $n(t)$ are both WSS with zero means. Assume $n(t)$ is white noise. Assume $s(t)$ and $n(t)$ are uncorrelated with each other ($\Rightarrow$ orthogonal since zero means).

Find the PSD of $x(t)$

**Solution**

Approach: Find ACF $R_x(\tau)$ & take FT to get PSD
Speech + Noise Example (cont.)

\[ R_x(\tau) = E\{x(t)x(t+\tau)\} = E\{[s(t) + n(t)][s(t+\tau) + n(t+\tau)]\} \]

\[ = E\{s(t)s(t+\tau)\} + E\{n(t)s(t+\tau)\} \]

\[ R_s(\tau) \]

\[ = 0 \quad \text{Assumed Orthog!} \]

\[ + E\{s(t)n(t+\tau)\} + E\{n(t)n(t+\tau)\} \]

\[ = 0 \quad \text{Assumed Orthog!} \]

\[ R_n(\tau) = \frac{\mathcal{N}}{2}\delta(\tau) \]

\[ = R_s(\tau) + \frac{\mathcal{N}}{2}\delta(\tau) \quad \Leftrightarrow \quad S_x(\omega) = S_s(\omega) + \frac{\mathcal{N}}{2} \]
Speech + Noise Example (cont.)

\[ S_x(\omega) = S_s(\omega) + \frac{\mathcal{N}}{2} \]

**Insight from Example:** Our assumption of uncorrelated and zero-mean processes resulted in a simple and very usable form for the PSD.

**Without that assumption:**
- difficult or impossible to analyze
- more complicated result may be harder to interpret

**Modeling Trade-Off:** Want a model that is…
- simple enough to give insight but not so simple it is “wrong”
- complex enough to be “right” but not so complex it gives no insight
RPs Through LTI Systems

We already saw that passing DT white noise through a FIR filter reshapes the ACF and PSD.

Here we learn the General Theory:
(Extremely useful for Modeling Practical RP's)

Input RP
WSS w/ $R_x(\tau)$
$S_x(\omega)$

LTI System
Impulse Response $h(t)$
Frequency Response $H(\omega) = \mathcal{F}\{h(t)\}$

Output RP

What does it look like?
RPs & LTI Systems: Results

To describe output RP $y(t)$ we look at its:

(i) Mean
(ii) ACF and
(iii) PSD

Results First (Proof Later)

(i) Mean: \[ E\{y(t)\} = H(0)E\{x(t)\} \]

**Comment**: Means are viewed as the DC Value of a RP – it makes sense that the Filter’s DC Response, $H(0)$, transfers “input-DC” to “output-DC”
RPs & LTI Systems: Results

(ii) ACF: \[ R_y(\tau) = h(\tau) \cdot h(-\tau) \cdot R_x(\tau) \]

Comments:
(1) Implicit in this is “WSS into LTI gives WSS out”
(2) The “second-order” dependence on \( h(.) \) comes from the ACF being a “second-order” characteristic
(3) ACF is a time-domain characteristic so it makes sense that convolution is involved.
RPs & LTI Systems: Results

(iii) PSD:

\[ S_y(\omega) = |H(\omega)|^2 S_x(\omega) \]

Comments: (1) Again, 2nd-order dependence on \( H(\omega) \) comes from PSD being a 2nd-order characteristic.

(2) PSD is a Frequency-domain characteristic so it makes sense that the frequency response \( H(\omega) \) is involved.
RPs & LTI Systems: Proof

**MEAN**

\[ E\{y(t)\} = E\left\{ \int_{-\infty}^{\infty} h(\alpha) x(t - \alpha) \, d\alpha \right\} \]

\[= \int_{-\infty}^{\infty} h(\alpha) E\{x(t - \alpha)\} \, d\alpha \]

\[= \bar{x} \int_{-\infty}^{\infty} h(\alpha) \, d\alpha \]

**Use Convolution to get output from input**

Since \( E\{\cdot\} \) is an integration, this is like changing order of integration.
Note: \( h(.) \) is not random so it gets pulled outside \( E\{\cdot\} \).

Since \( x(t) \) is WSS

\[ H(0) = \left[ \int_{-\infty}^{\infty} h(t)e^{j\omega t} \, dt \right]_{\omega=0} \]
RPs & LTI Systems: Proof

**ACF:**

\[ R_y(\tau) = E\{y(t)y(t+\tau)\} \]

\[
= E \left\{ \int_{-\infty}^{\infty} h(\alpha)x(t-\alpha)d\alpha \left[ \int_{-\infty}^{\infty} h(\beta)x(t+\tau-\beta)d\beta \right] \right\} \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)E\{x(t+\tau-\beta)x(t-\alpha)\}d\alpha d\beta \\
= h(\tau)*h(-\tau)*R_x(\tau)
\]

**PSD:** Follows from the ACF result and two applications of the convolution–FT property
RPs & LTI Systems: In vs Out

Now, for this system:
Q: How is the output correlated with the input?

A: Compute their Cross-correlation Function:

**Result:**
\[ R_{xy}(\tau) = h(\tau) * R_x(\tau) \]

**Proof:**
\[
R_{xy}(\tau) = E \left\{ x(t) \int_{-\infty}^{\infty} h(\beta)x(t + \tau - \beta) \, d\beta \right\} \\
= \int_{-\infty}^{\infty} h(\beta)E\left\{ x(t)x(t + \tau - \beta) \right\} d\beta \\
= R_x(\tau) \]
Earlier we looked at figures showing how five different (but similar) filters impact the output ACF. Recall that in those examples the input was D-T white noise $\Rightarrow R_x[m] = \sigma^2 \delta[m]$. Thus the output ACF’s are just the convolution: $\sigma^2 h[m] * h[-m]$.

The filters in the previous case all had rectangular impulse responses, which when convolved like this give the triangular ACF’s shown in the previous figures.

**Note also**: rectangular FIR filters are low-pass filters whose cut-off frequency gets lower as the filter length increases.
Ex: Filtered White Noise

Thus, since

\[ S_y(\Omega) = |H(\Omega)|^2 S_x(\Omega) \]

= $N/2$ for White Noise

PSD’s of processes that are outputs of longer rectangle filters have narrower PSD’s

Wide ACF  \iff\  Narrow ACF

Narrow PSD  \iff\  Broad PSD

Process has slow fluctuations

Process has fast fluctuations