Power of Random Processes

Power of a Random Process

Recall : For deterministic signals…
instantaneous power is $x^2(t)$

For a random signal, $x^2(t)$ is a random variable for each time t. Thus there is no single # to associate with “instantaneous power”. To get the Expected Instantaneous Power (i.e., on average) we compute the statistical (ensemble) average of $x^2(t)$:

Power of a Random Process

Often drop the
"Average" terminology

This is also called the
"mean square value"
of the process

Avg. Power of x(t) : $P_X(t) = E\{ x^2(t)\}$

In general, can depend on time.
But we'll see it doesn't for WSS
(& stationary) processes

Relationship of Power to ACF

Recall: $R_X(t, t+\tau) = E\{ x(t)\, x(t+\tau) \}$

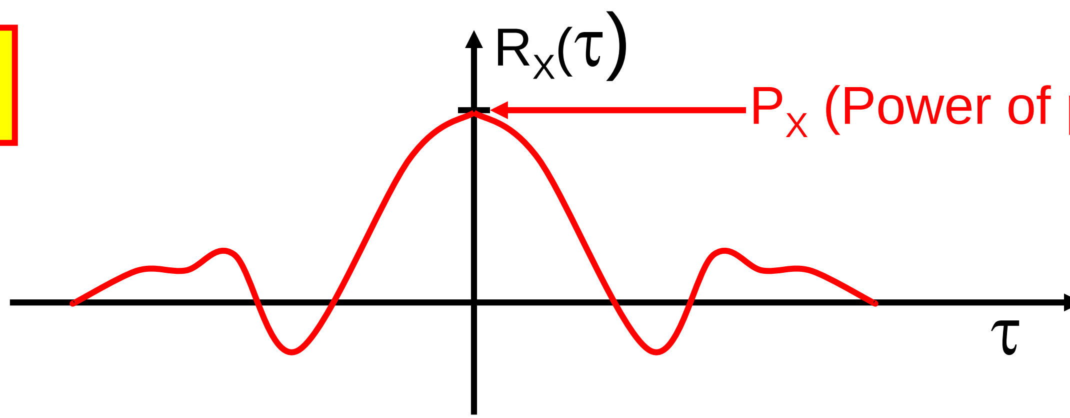
Clearly, setting $\tau = 0$ makes this equal to $P_X(t)$

$$\Rightarrow \mathbf{P_X(t) = R_X(t, t)}$$

If the Process is WSS (or SS) $R_X(t,t) = R_X(0)$

For WSS or SS

$$P_X = R_X(0)$$



Power vs. Variance

Recall Variance: $\sigma_x^2 = E\left\{[x(t) - \bar{x}(t)]^2\right\}$

$= \underbrace{R_x(0)}_{P_X} - \bar{x}^2$ for WSS

$$P_x = \sigma_x^2 + \bar{x}^2$$

"AC" Power (σ_x^2), "DC" Power ($\bar{x}^2$)

Note: If the WSS process is **Zero Mean**, then:

$$P_X = \sigma_x^2$$

Power & Variance are Equal for Zero mean Process

Power Spectral Density of a Random Process

Recall: PSD for Deterministic Signal x(t) :

$$S_x(\omega) = \lim_{T \to \infty} \frac{\left| X_T(\omega) \right|^2}{T}$$

$$\text{where} \quad X_T(\omega) = \int_{-T/2}^{T/2} x(t) e^{-j\omega t} dt$$

Power Spectral Density of a Random Process

For a random Process: each realization (sample function) of process x(t) has different FT and therefore a different PSD.

We again rely on averaging to give the **"Expected" PSD** or **"Average" PSD**…

But… **Usually just call it "PSD"**.

Define PSD for WSS RP

We define PSD of WSS process x(t) to be :

$$S_x(\omega) = \lim_{T \to \infty} E\left\{ \frac{\left| X_T(\omega) \right|^2}{T} \right\} \quad \bigstar$$

This definition isn't very useful for analysis so we seek an alternative form

The **Wiener-Khinchine Theorem** provides this alternative!!!

Weiner- Khinchine Theorem

Let x(t) be a WSS process w/ ACF $R_X(\tau)$ and w/ PSD $S_X(\omega)$ as defined in (★)… Then $R_X(\tau)$ and $S_X(\omega)$ form a FT pair :

$$S_X(\omega) = \mathscr{F}\{ R_X(\tau) \}$$

or Equivalently

$$R_X(\tau) \leftrightarrow S_X(\omega)$$

Proof of WK theorem

By definition : $$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t)\, e^{-j\omega t}\, dt$$

For Real x(t):

$$|X_T(\omega)|^2 = X_T(-\omega)\, X_T(\omega)$$

$$= \int_{-T/2}^{T/2} x(t_1)\, e^{j\omega t_1} dt_1 \cdot \int_{-T/2}^{T/2} x(t_2)\, e^{-j\omega t_2} dt_2$$

$$= \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} x(t_1)\, x(t_2)\, e^{-j\omega(t_2 - t_1)} dt_1\, dt_2$$

Proof of WK theorem(cont'd)

Thus :

$$S_x(\omega) = \lim_{T\to\infty} E\left\{ \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} x(t_1)\, x(t_2)\, e^{-j\omega(t_2 - t_1)} dt_1\, dt_2 \right\}$$

Move E {.} inside integrals :

$$S_x(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} R_x(t_2 - t_1)\, e^{-j\omega(t_2 - t_1)} dt_1\, dt_2$$

Proof of WK theorem(cont'd)

We were almost there… **BUT** we have one-too-many integrals.

For convenience define:

$$\phi(t_2 - t_1) = R_X(t_2 - t_1)\, e^{-j\omega(t2 - t1)}$$

★★ $$S_x(\omega) = \lim_{T\to\infty} \frac{1}{\mathrm{T}} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \phi(t_2 - t_1)\, dt_1\, dt_2$$

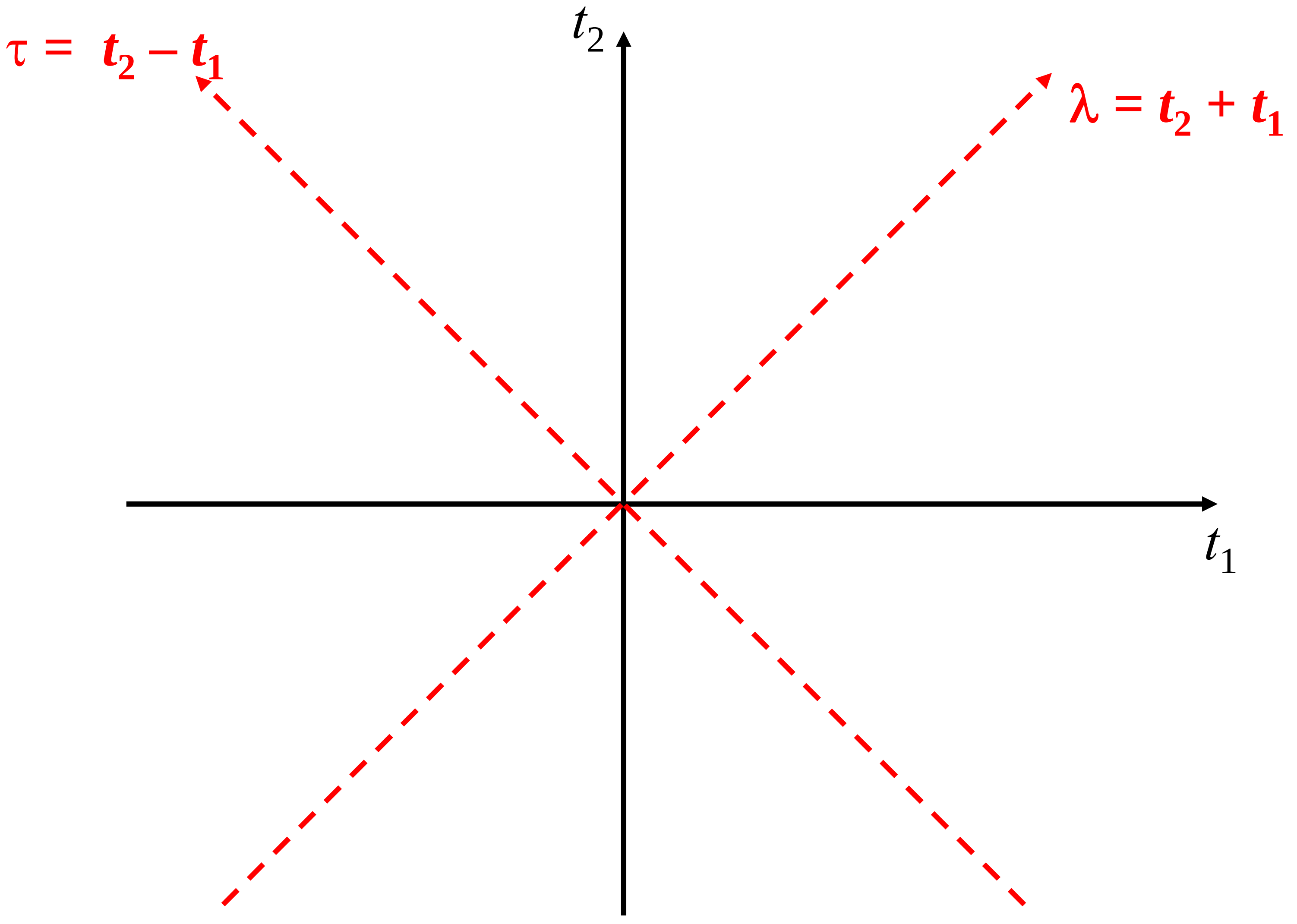
Proof of WK theorem (cont'd)

Change of Variables = Change of Axes

Instead of integrating **"Row-by-Row"** as in ★★ we integrate **"Diagonal-by-Diagonal"**.

Let:
$$\begin{cases} \tau = t_2 - t_1 \\ \lambda = t_1 + t_2 \end{cases} \quad \Longrightarrow \quad \begin{cases} t_2 = (\tau + \lambda)/2 \\ t_1 = (\lambda - \tau)/2 \end{cases}$$

Proof of WK theorem(cont'd)



Proof of WK theorem(cont'd)

From Calculus III

Use **Jacobian** result for 2-D change of variables

$$J = \begin{vmatrix} \dfrac{\partial t_2}{\partial \tau} & \dfrac{\partial t_2}{\partial \lambda} \\[2ex] \dfrac{\partial t_1}{\partial \tau} & \dfrac{\partial t_1}{\partial \lambda} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\[2ex] -\dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{2}$$

Proof of WK theorem(cont'd)

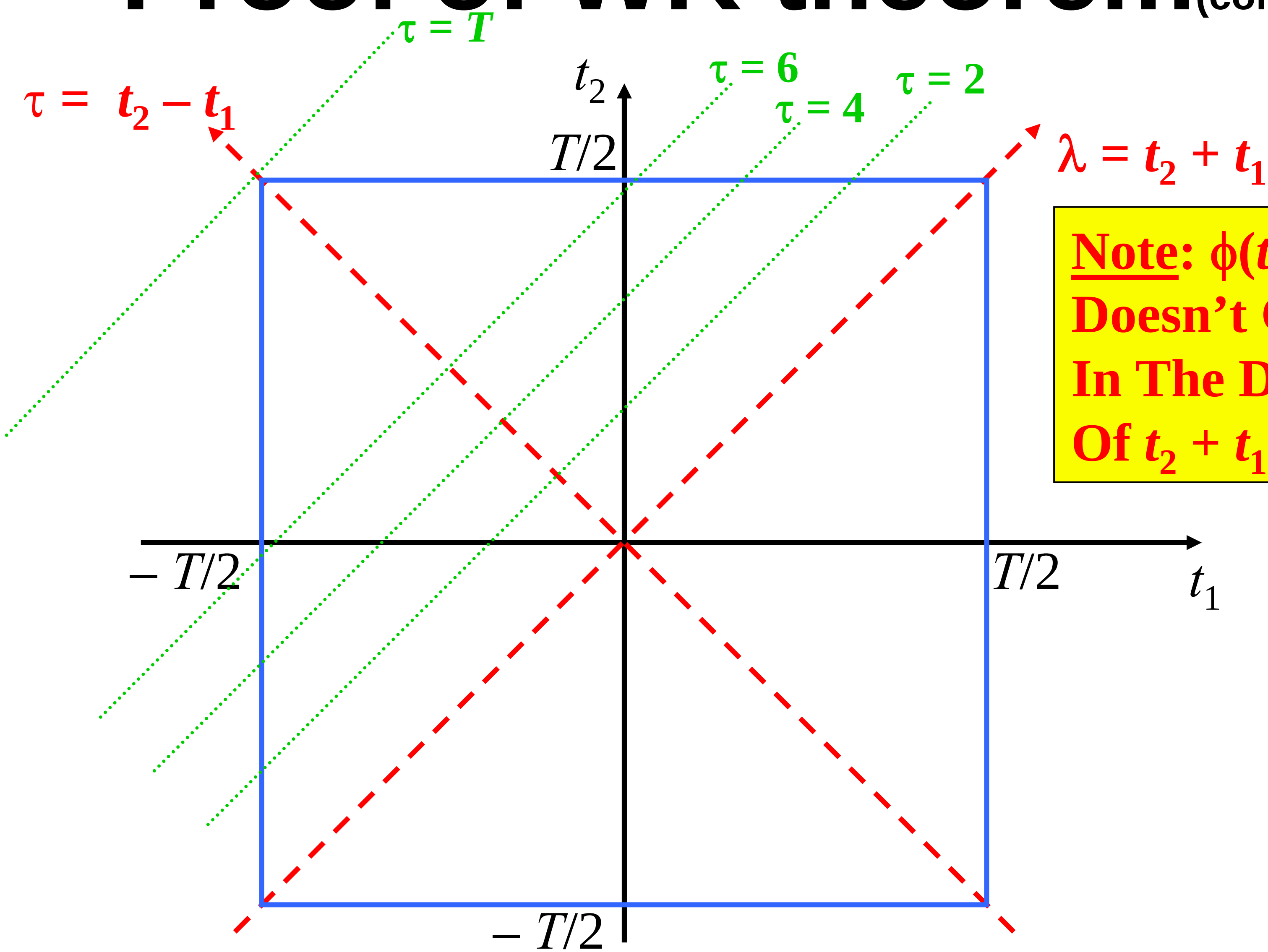
$$S_x(\omega) = \lim_{T\to\infty} \frac{1}{\mathrm{T}} \int_{?}^{?} \int_{?}^{?} \phi(\tau) \frac{1}{2} d\tau d\lambda$$

J

Q: As τ ranges over -T≤τ≤T
how does λ range?

A: For each τ, λ must be restricted to stay inside original square – **see Figure on next Chart**

Proof of WK theorem(cont'd)



<u>Note</u>: $\phi(t_2 - t_1)$ Doesn't Change In The Direction Of $t_2 + t_1$ Axis

Proof of WK theorem (cont'd)

$\tau = t_2 - t_1$ → $t_2 = \tau + t_1$, $t_1 = t_2 - \tau$

$\lambda = t_2 + t_1$ ⇒

$\lambda = (\tau + t_1) + t_1 = 2t_1 + \tau$

$\lambda = t_2 + (t_2 - \tau) = 2t_2 - \tau$

When $\tau > 0$ we need (From Axes Figure):

$t_2 \leq T/2$, $t_1 \geq -T/2$ ⇒ $\lambda \leq T - \tau$, $\lambda \geq -T + \tau$

Works out similarly for $\tau < 0$

Proof of WK theorem(cont'd)

So…
$$S_x(\omega) = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \phi(\tau) \left[\int_{-T+\tau}^{T-\tau} d\lambda \right] \frac{1}{2} d\tau$$

$$= \lim_{T \to \infty} \int_{-T}^{T} \phi(\tau) \left(1 - \frac{\tau}{T} \right) d\tau$$

$$= \int_{-\infty}^{\infty} \phi(\tau)\, d\tau$$

$$= \mathfrak{F}\left\{ R_x(\tau) \right\}$$

<End of Proof>

Some Properties of PSD & ACF

(1) The PSD is an even function of ω for a real process x(t)

proof : since each sample function is real valued then we know that $|x(\omega)|$ is even.

$\Rightarrow$ $|x(\omega)|^2$ is even: (even x even = even)

$\Rightarrow$ So is $S_X(w)$ (this is clear from ★)

Some Prop. of PSD & ACF

(2) $S_X(\omega)$ is real-valued and ≥ 0.

proof : again from (★) – since $|x(w)|^2$ is real-valued & ≥ 0, so is $S_X(w)$

(3) $R_X(\tau)$ is an even function of τ

$$\begin{aligned} R_x(-\tau) &= E\{x(t)x(t-\tau)\} \\ &= E\{x(\sigma)x(\sigma+\tau)\} \\ &= R_x(\tau) \end{aligned}$$

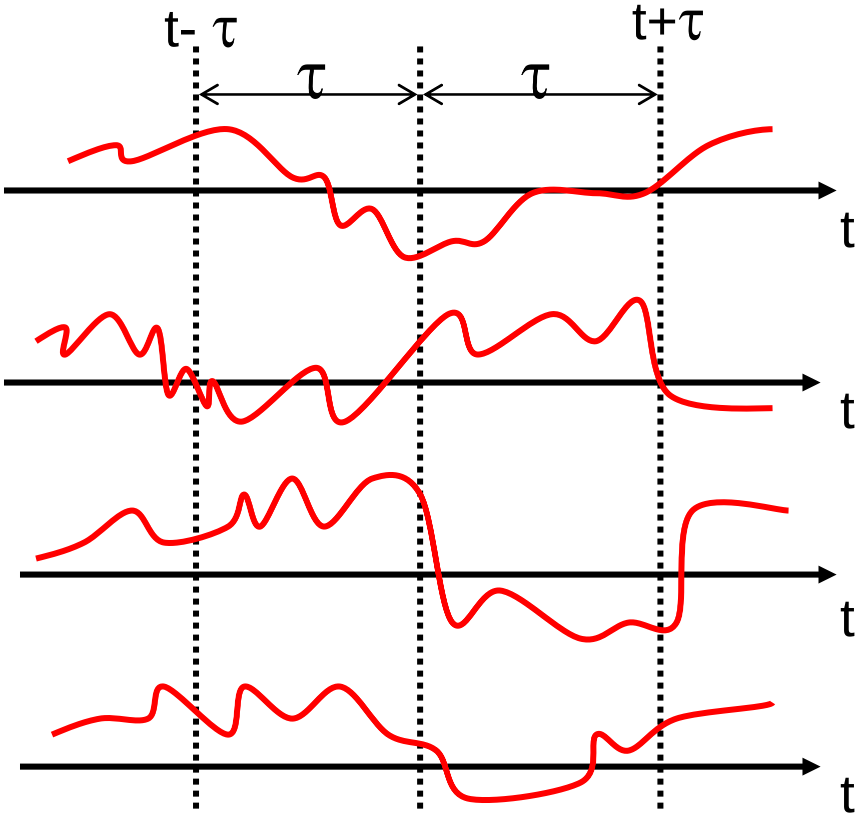
Also follows from IFT{ Real & Even } = Real & Even

<Property of the FT>

Graphical View of Prop #3

For WSS, ACF does not depend on absolute time only relative time

Doesn't matter if you look forward by τ or look backward by τ



Computing Power from PSD

From it's name – Power Spectral Density – we know what to expect :

$$P_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\, d\omega$$

Well … this part is not obvious!!

So let's Prove this !!!!

Computing Power from PSD

We Know:

$$R_x(\tau) = \mathfrak{F}^{-1}\{S_x(\omega)\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, e^{j\omega\tau}\, d\omega$$

$P_X = R_X(0)$

$$P_X = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, \underbrace{e^{j\omega 0}}_{=1}\, d\omega\right]$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega) d\omega \quad \text{Q.E.D.!}$$

Units of PSD Function

$$P_x = \int_{-\infty}^{\infty} S_x(\omega)\,\frac{d\omega}{2\pi} \quad \Rightarrow \quad S_x(\omega)$$

Watts

Hz

[Watts / Hz]

Using Symmetry of $S_X(w)$

$$P_x = 2\left[\frac{1}{2\pi}\int_0^{\infty} S_x(\omega)\,d\omega\right]$$

Integrate only from 0 → ∞

Double to account for the -∞ → 0 part of integral

PSD for DT Processes

Not much changes – mostly, just use DTFT instead of CTFT!!

$$S_X(\Omega) = \text{DTFT}\{ R_X[m] \}$$

Periodic in Ω with period 2π

$$P_X = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(\Omega)\, d\Omega$$

Need only look at $-\pi \leq \Omega < \pi$

Big Picture of PSD & ACF

Narrow ACF ⟷ Broad PSD

Less Correlated Sample-to-sample ⟷ Process exhibits Rapid fluctuations

i.e. **High Frequencies** have **Large** power content

Broad ACF ⟷ Narrow PSD

More Correlated Sample-to-sample ⟷ Process exhibits Slow fluctuations

i.e. **High Frequencies** have **Small** power content

<<See “Big Picture: Filtered RP” Charts in “V-3 RP Examples” >>

White Noise

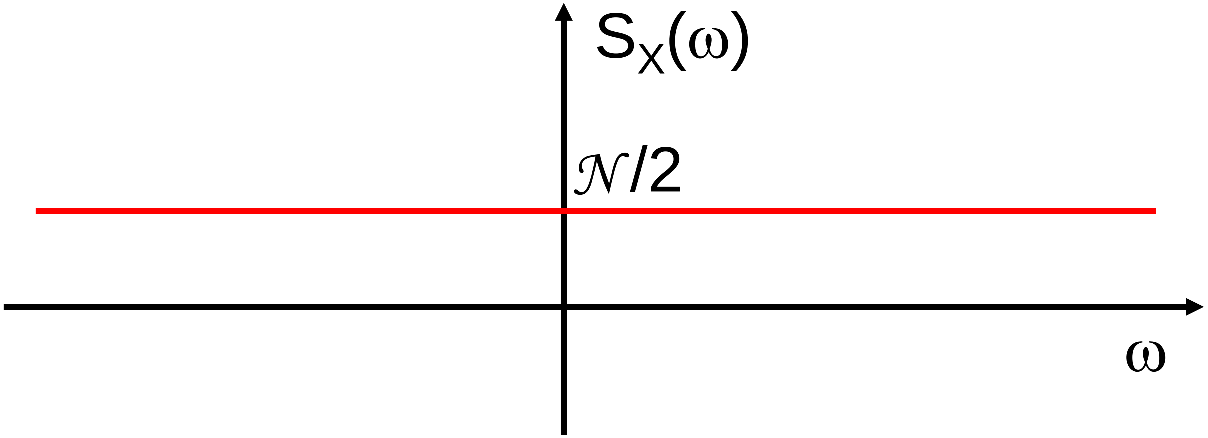
The term **"White Noise"** refers to a WSS process whose **PSD is flat** over all frequencies

White Noise Has Broadest Possible PSD

C-T White Noise

$$S_X(\omega) = \frac{\mathcal{N}}{2} \quad \forall \omega$$

Convention to use this form (i.e. w/ division by 2)



C-T White Noise

<u>NOTE</u> : C-T white noise has **<u>infinite Power</u>** :

$$\int_{-\infty}^{\infty} \mathcal{N}/2 \, d\omega \rightarrow \infty$$

Can't **really** exist in practice but still a **very** useful <u>Model for Analysis of Practical Scenarios</u>

C-T White Noise

Q : what is the ACF of C-T white Noise ?

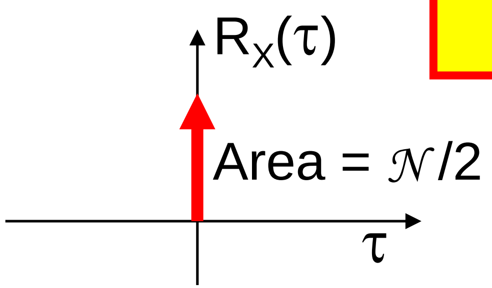
A: Take the IFT of the flat PSD :

$$R_x(\tau) = \mathfrak{F}^{-1}\{\mathcal{N}/2\}$$

$$= \frac{\mathcal{N}}{2}\delta(\tau)$$

Delta function !

Narrowest ACF



$x(t_1)$ & $x(t_2)$ are <u>uncorrelated</u> for <u>any</u> $t_1 \neq t_2$

Also….

$$P_X = R_X(0) = \mathcal{N}/2\delta(0) \rightarrow \infty$$

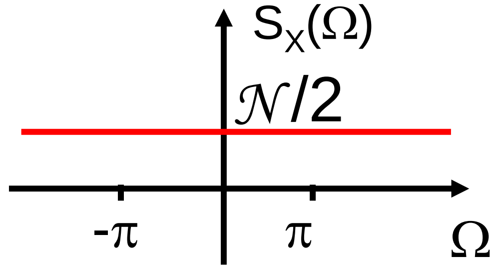
Infinite Power.. It Checks!

D-T White Noise

PSD is:

$S_X(\Omega) = \mathcal{N}/2 \;\forall\; \Omega$

…but focus on $\Omega \in [-\pi, \pi]$



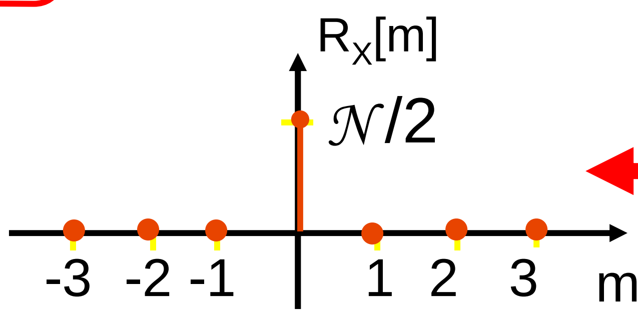
Broadest Possible PSD

ACF is:

$R_X[m] = \text{IDTFT}\{\mathcal{N}/2\}$

$= \mathcal{N}/2\,\delta[m]$

Delta sequence



Narrowest ACF

$x[k_1]$ & $x[k_2]$ are uncorrelated for <u>any</u> $k_1 \neq k_2$

D-T White Noise

Note:

$$P_x = R_x[0] = \frac{\mathcal{N}}{2} \quad \text{watts}$$

$$P_x = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathcal{N}}{2}\,d\Omega = \frac{\mathcal{N}}{2} \quad \text{watts}$$

D-T White Noise has **Finite Power**
(unlike C-T White Noise)

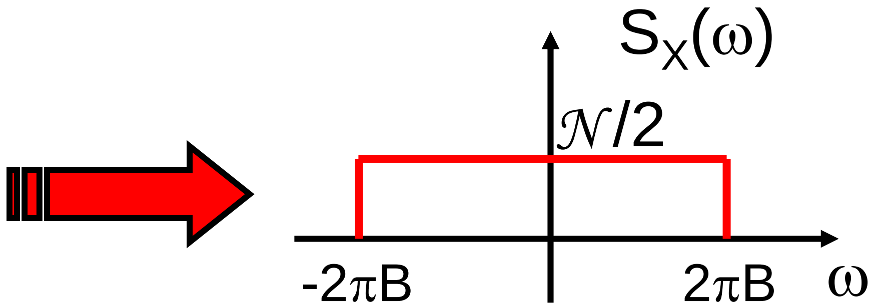
Examples of PSD

Example 1: "BANDLIMITED WHITE NOISE"

This looks like white noise within some bandwidth but its PSD is zero outside that bandwidth – hence the name.

Thus:

$$S_x(\omega) = \mathcal{N}/2 \ \mathrm{rect}\left(\frac{\omega}{4\pi B}\right)$$



And…

$$P_x = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} \mathcal{N}/2 \, d\omega = \frac{1}{2\pi} \cdot \mathcal{N}/2 \cdot (2\pi B + 2\pi B)$$

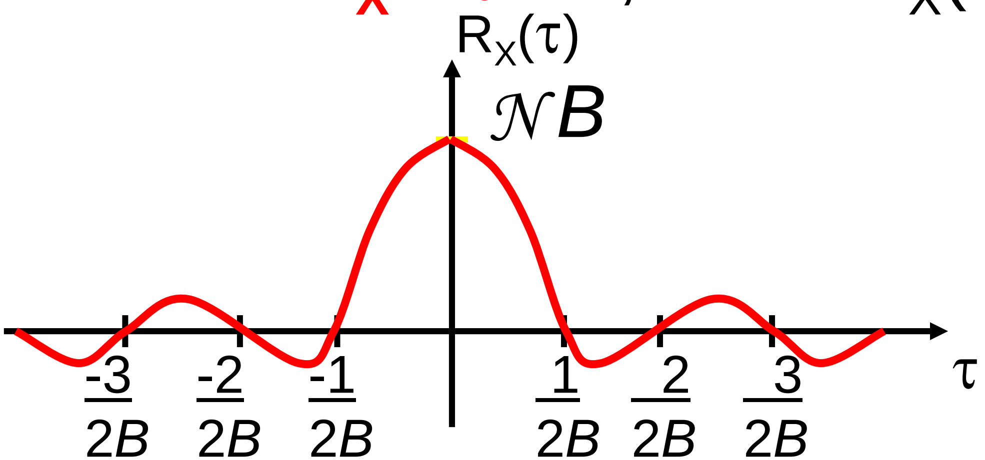
$$= \mathcal{N}B$$

Example: BL White Noise

The ACF (using FT pair for rect and sinc) is:

$$R_X(\tau) = \mathcal{N}B\ \text{sinc}\ (2\pi B\tau)$$

Again we see $P_X = \mathcal{N}B$, since $R_X(0) = \mathcal{N}B$



Note: For $\tau > 1/2B$ …

$x(t)$ & $x(t+\tau)$ are Approximately Uncorrelated

Example #2 of PSD

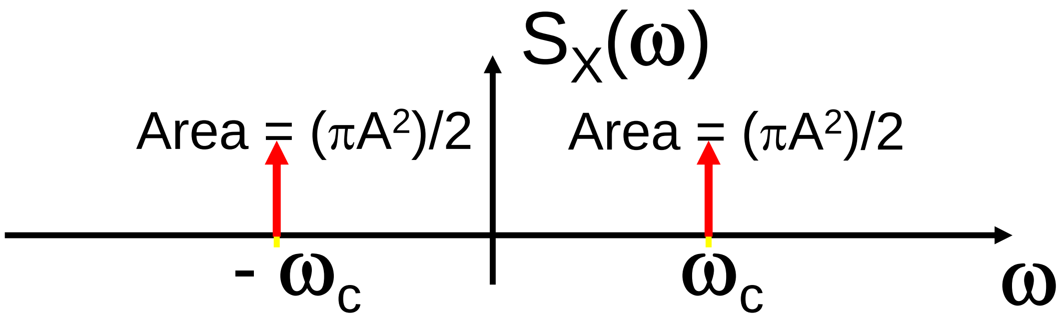
Example 2 **: "SINUSOID WITH RANDOM PHASE"**

$x(t) = A \cos(\omega_C t + \theta)$

We examined this RP before: $R_X(\tau) = \frac{A^2}{2} \cos(\omega_C \tau)$

So using FT Pair for a Cosine gives the PSD:

$$S_X(\omega) = (\pi A^2)/2 \; [\delta(\omega + \omega_C) + \delta(\omega - \omega_C)]$$



Example #2 of PSD

<u>Note</u> : Can get P_X in two ways:

1. $R_x(0) = \frac{A^2}{2}$

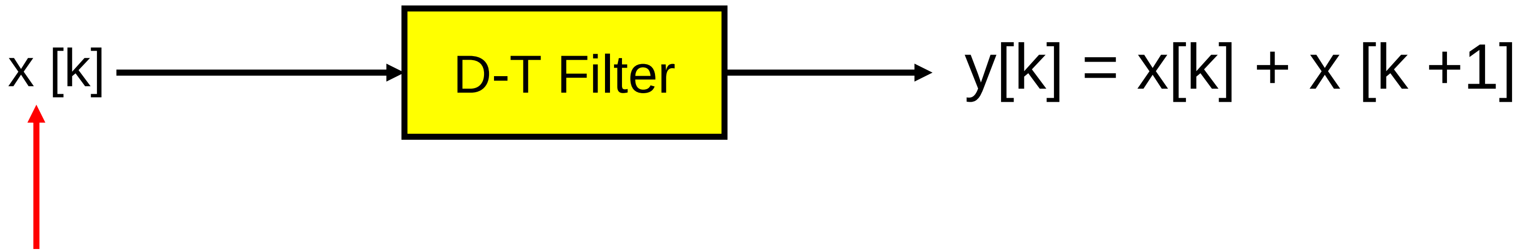
2. $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega = \frac{A^2}{2}$ <Sifting Property>

$$\Rightarrow P_X = \frac{A^2}{2}$$

Example #3 of PSD

Example 3: "FILTERED D-T RANDOM PROCESS"

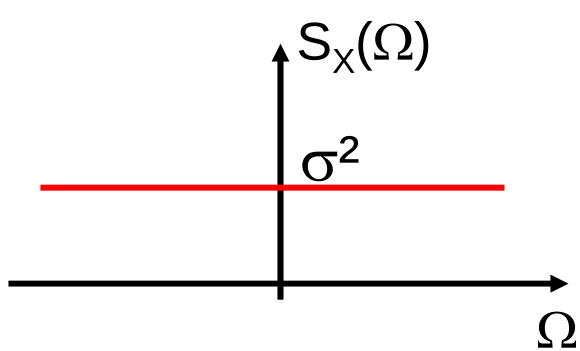
< See Also: Class Notes on "Filtered RPs" >



Zero mean
White noise

$\Rightarrow R_X[m] = \sigma^2 \delta[m]$ (Input ACF)

$\Rightarrow S_X(\Omega) = \sigma^2 \quad \forall \Omega$ (Input PSD)



Example #3 of PSD

For this case we showed earlier that for this filter output the ACF is :

$$R_Y[m] = \sigma^2 \{ 2\delta[m] + \delta[m-1] + \delta[m+1] \}$$

So the Output PSD is:

$$S_Y(\Omega) = \sigma^2 [2 + e^{-j\Omega} + e^{-j\Omega}]$$

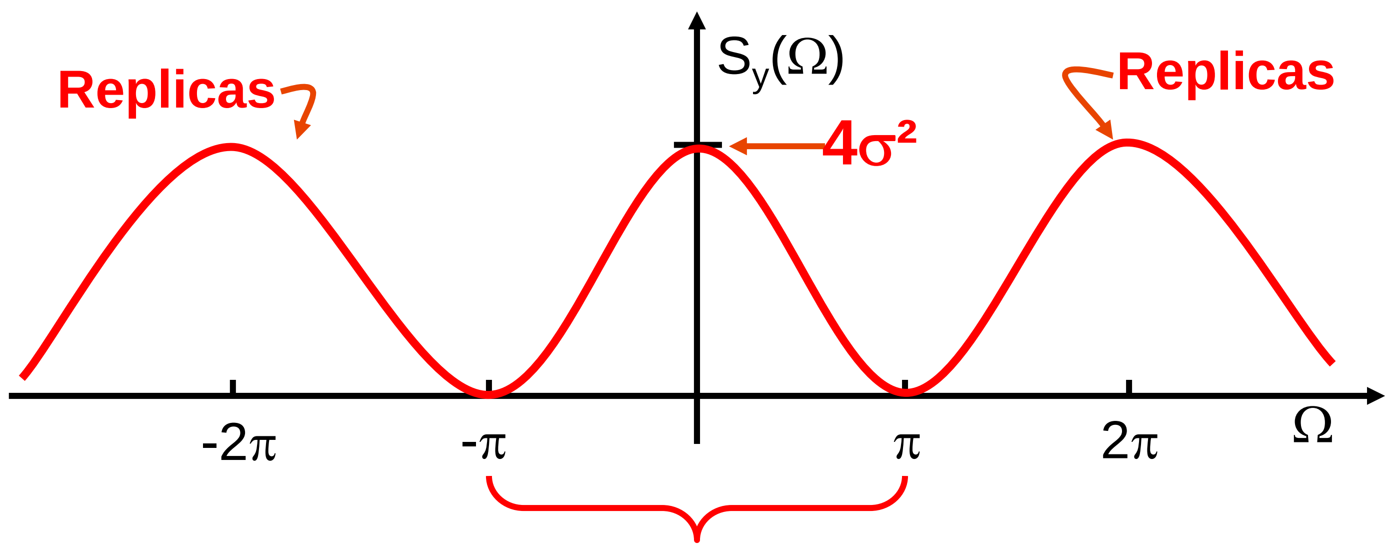
$$= 2\sigma^2 [\cos(\Omega) + 1]$$

Use the result for DTFT of $\delta[m]$ and also time-shift property

$= 2\cos(\Omega)$ By Euler

Example #3 of PSD

$S_Y(\Omega) = 2\sigma^2\,[\cos(\Omega) + 1]$



Remember: Limit View to [$-\pi,\pi$]

<u>General Idea…Filter Shapes Input PSD:</u>

Here it suppresses High Frequency power