Random Process Examples

Ex. #1: D-T White Noise

Let x[k] be a sequence of RV's where…

each RV x[k] in the sequence is

uncorrelated with all the others:

$$E\{ x[k]\, x[m] \} = 0 \qquad \text{for } k \neq m$$

This DEFINES a DT White Noise

Also called "Uncorrelated Process"

Physically, uncorrelated means that knowing x[k] provides no insight into what value x[m] (for m≠k) will be likely to take (roll a die; the value you get provides no insight into what you expect to get on any future roll)

Ex. #1: D-T White Noise

Recall: Gaussian process are very common, so…

Also, let x[k] be **Gaussian** with Zero mean and Variance σ^2

Define RP's PDF; also called "Normal"

$X[k] \sim N(0,\sigma^2)$

Mean — Variance

PDF : $p(x;k) = \frac{1}{2\pi\sigma} e^{\left(\frac{-x^2}{2\sigma^2}\right)}$

Note: Does not depend on *k*
➔ at least the 1st order PDF is time invariant

Ex. #1: D-T White Noise

TASK : We have a model…. Find the mean, ACF, and check if WSS (also find variance of process)

MEAN of Process : **$E\{ x[k] \} = 0$ CONSTANT**

By definition !

ACF: **$R_X(k_1,k_2)=E\{ x[k_1] . x[k_2] \}$**

By our definition of white noise,

….this is 0 if $k_1 \neq k_2$

Ex. #1: D-T White Noise

Now for $k_1=k_2=k$:

$$R_x(k_1,k_2)\big|_{k=k_1=k_2} = E\{x^2[k]\} = \sigma^2$$

By definition of variance for zero-mean case

Thus,

$$R_x(k_1,k_2) = \begin{cases}\sigma^2, & k_1 = k_2 \\ 0, & k_1 \neq k_2\end{cases}$$

$$= \sigma^2\delta[k_1 - k_2]$$

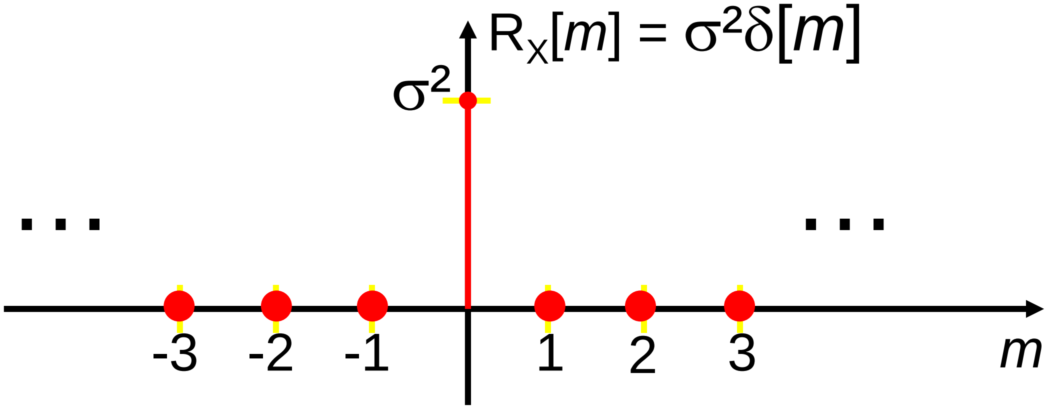
$k_1 - k_2$ = *m* (like τ for cont-time ACF)

ACF for DT White RP

★★ $$\Rightarrow R_x(m) = \sigma^2\delta[m]$$ ★★

Ex. #1: D-T White Noise

ACF displays lack of correlation between any pair of any time instants:



Now since we have constant mean and ACF depends only on $m = k_2 - k_1$ $\Rightarrow$ **WSS**

Ex. #1: D-T White Noise

<u>Variance</u>

$$\sigma_x^2 = R_x[0] - \bar{x}^2$$

=0

$$= R_x[0]$$

$$= \sigma^2$$

For this case:

Variance of the **<u>Process</u>** = Variance of the **<u>RV</u>**

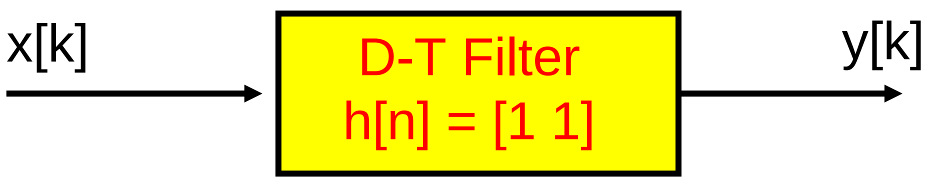
Ex. #2: Filtered D-T RP

Start with White RP x[k] in previous example

Recall : Zero Mean Process

$$R_X[m] = \sigma^2 \delta[m]$$

$\Rightarrow$ **WSS**



$y[k] = \underbrace{x[k] + x[k-1]}$

Two–Tap FIR filter
Taps = [1 1]

Ex. #2: Filtered D-T RP

TASK: Is y[k] WSS?

$\Rightarrow$ need to find mean & ACF

MEAN: Using filter output expressions gives

$$E\{y[k]\} = E\{x[k] + x[k-1]\}$$
$$= E\{x[k]\} + E\{x[k-1]\}$$

$\Rightarrow$ E {y[k]} = 0

Ex. #2: Filtered D-T RP

ACF :

Plug in Eq. for output

$$R_y(k_1,k_2) = E\{y[k_1]y[k_2]\}$$

$$= E\{(x[k_1] + x[k_1 - 1])(x[k_2] + x[k_2 - 1])\}$$

$$= \underbrace{E\{x[k_1]x[k_2]\}}_{R_x(k_2-k_1)} + \underbrace{E\{x[k_1]x[k_2 - 1]\}}_{R_x(k_2-k_1-1)}$$

$$+ \underbrace{E\{x[k_1 - 1]x[k_2]\}}_{R_x(k_2-k_1+1)} + \underbrace{E\{x[k_1 - 1]x[k_2 - 1]\}}_{R_x((k_2-1)-(k_1-1))}$$

Ex. #2: Filtered D-T RP

$$R_y(k_1,k_2) = \underbrace{R_x(k_2 - k_1)}_{\sigma^2\delta[m]} + \underbrace{R_x(k_2 - k_1 - 1)}_{\sigma^2\delta[m-1]}$$

$$+ \underbrace{R_x(k_2 - k_1 + 1)}_{\sigma^2\delta[m+1]} + \underbrace{R_x\big((k_2 - 1) - (k_1 - 1)\big)}_{\sigma^2\delta[m]}$$

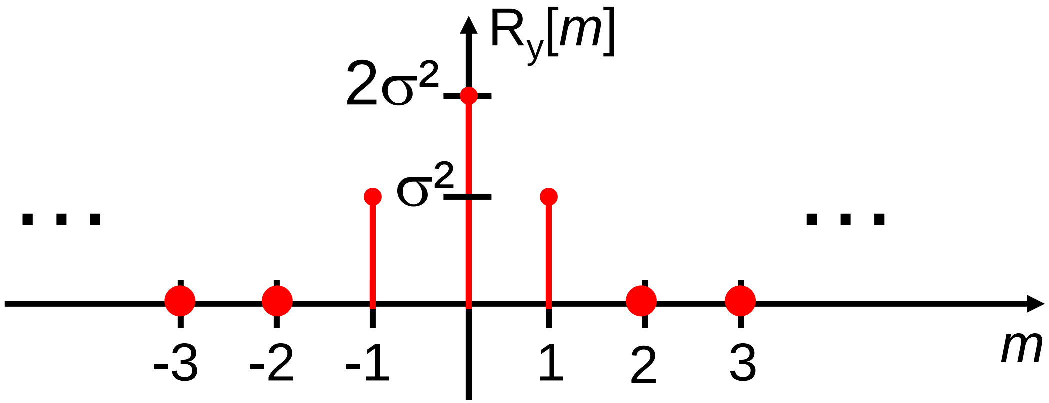
where $m = k_2 - k_1$

y[k] is WSS

ACF for 2-Tap Filtered White RP

☆☆ $\Rightarrow R_Y(m) = \sigma^2[2\delta[m] + \delta[m-1] + \delta[m+1]]$ ☆☆

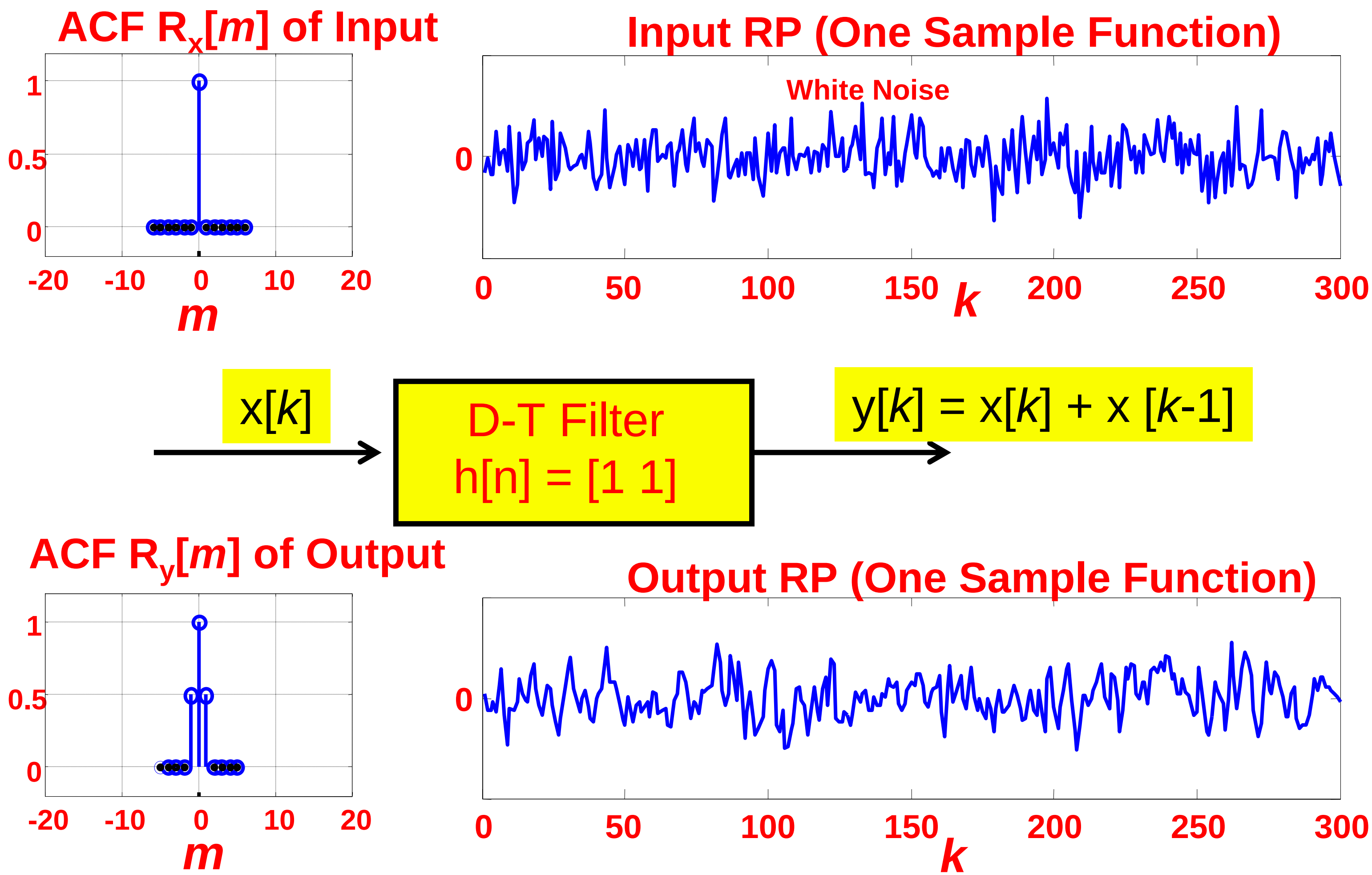
Ex. #2: Filtered D-T RP



Note : Filter introduces correlation between adjacent samples - but still no correlation for samples 2 or more samples apart (for this filter)

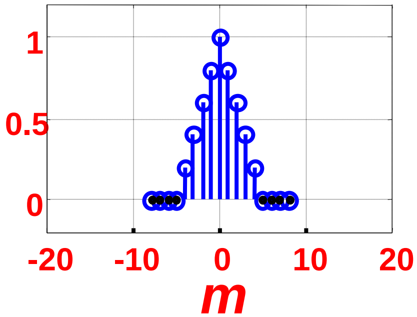
Big Picture: Filtered RP

Filters can be used to change the correlation structure of a RP:

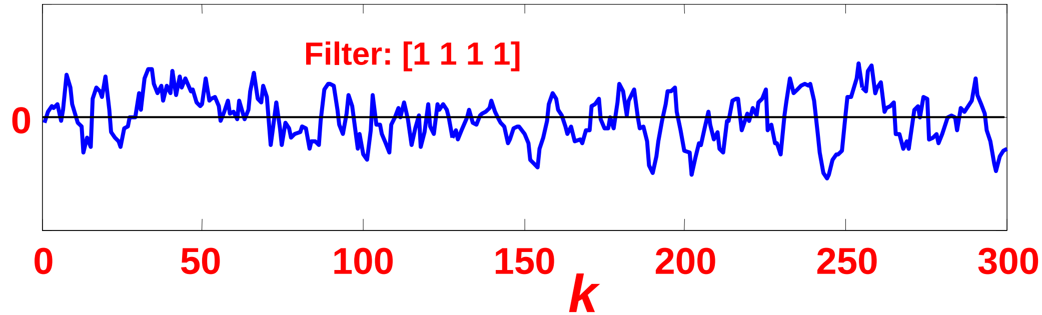


Big Picture: Filtered RP (cont)

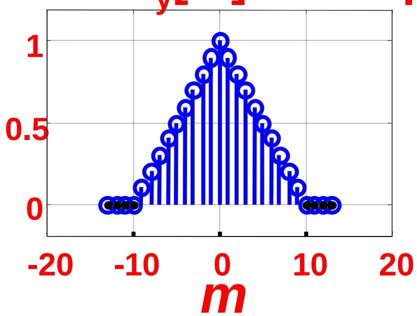
ACF $R_y[m]$ of Output



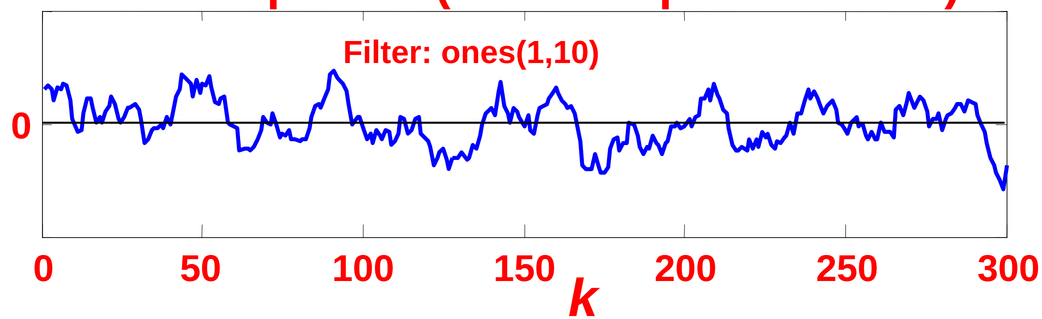
Output RP (One Sample Function)



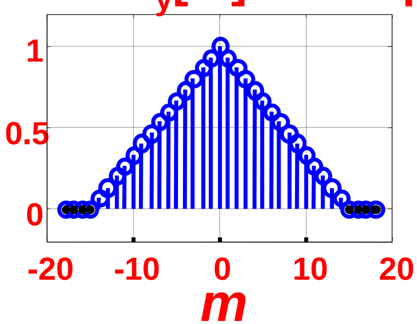
ACF $R_y[m]$ of Output



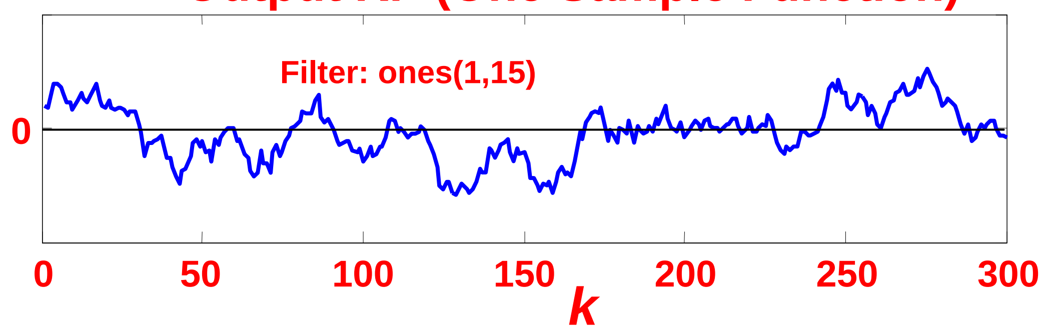
Output RP (One Sample Function)



ACF $R_y[m]$ of Output



Output RP (One Sample Function)



Filtered RPs: Insight

Our study of the ACFs of filtered random processes and the degree of “smoothness” of the sample functions shows the following general result:

Narrow ACF	↔	**Rapid Fluctuations**
Broad ACF	↔	**Slow Fluctuations**

Ex. #3: Exp. Sig. + White Noise

Define a RP as:

$$y[k] = a^k x + w[k]$$

Deterministic Function

a = deter. #

Random Variable

(picked once randomly and then fixed)

White Noise

(get a new random value at each k – uncorrelated from sample-to-sample)

Ex. #3: Exp. Sig. + WN

Further Definition of this RP:

- w[k] is **white noise**
- Each w[k] is a Gaussian RV with $w[k] \sim N(0, \sigma_w^2)$
- X is a Gaussian RV with $X \sim N(0, \sigma_X^2)$
- RV X is independent of each RV w[k]

  $\Rightarrow$ **E{ x w[k]} = E{x} E{w[k]} = 0**
- The # *a* is a deterministic number

Ex. #3: Exp. Sig. + WN

TASK: Is this WSS?

MEAN: $E\{\, y[k] \,\} = a^k E\{X\} + E\{\, w[k] \,\}$ (= 0, = 0)

$= 0$ (Constant)

ACF: $R_y(k,k+m) = E\{\, y[k]\, y[k+m] \,\}$

$= E\,\{(a^k X + w[k])\,(a^{k+m} X + w[k+m])\}$

$= a^{2k+m} E\{X^2\} + a^k E\{X\, w[k+m]\}$

$+ a^{k+m} E\{X\, w[k]\}$

$+ E\,\{w[k]\, w[k+m]\}$

$E\{X^2\} = \sigma_X^2$

= 0 Due to Indep. (for $E\{X\, w[k+m]\}$ and $E\{X\, w[k]\}$)

$E\{w[k]\, w[k+m]\} = \sigma_w^2 \delta[m]$

Ex. #3: Exp. Sig. + WN

$$R_y(k, k+m) = \sigma_x^2 \, a^{2k+m} + \sigma_w^2 \, \delta[m]$$

In General: Depends on k not just m

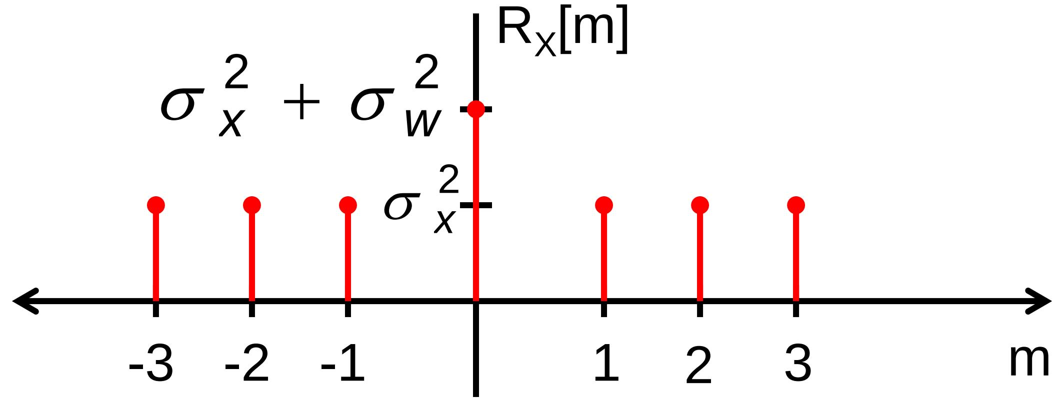
Not WSS !!!

But… If a = ±1 then $a^{2K+M} = a^{2K} a^M = a^M$

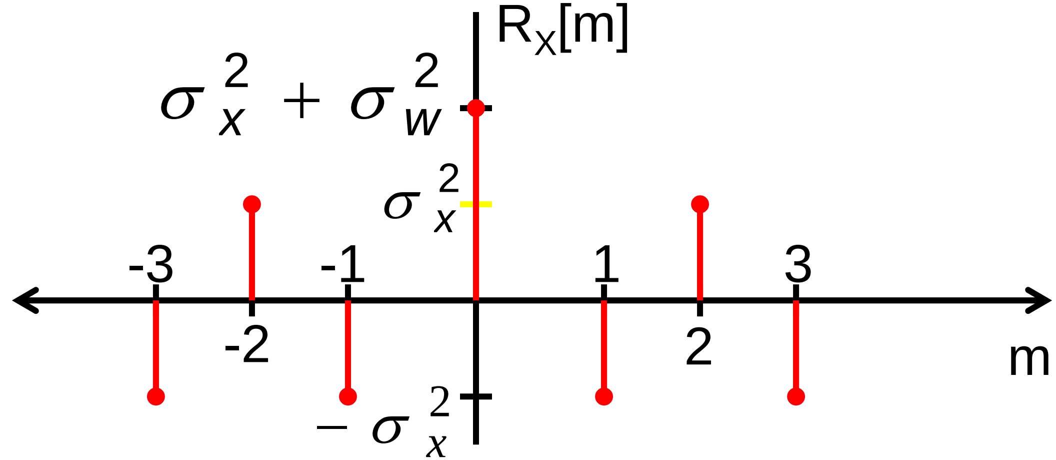
Then it is WSS

Ex. #3: Exp. Sig. + WN

If a = +1



If a = -1



Ex. #4: Weird Func of WN

Let z[k] be white noise with zero mean & variance of

Define x[n] = Z [int(n/2)]

$\text{int}(D_3D_2D_1.d_1d_2d_3d_4) = D_3D_2D_1$

Then
x[0] = z[0]
x[1] = z[0]
x[2] = z[1]
x[3] = z[1]
⋮ ⋮

Always appear in pairs

Ex. #4: Weird Func of WN

TASK: Is this RP WSS ?

MEAN: E { x[n] } = E { z[k] } = 0

$k = n/2$ if n is even
$= (n-1)/2$ if n is odd

$\mathbf{R_X(n, n+m)}$ = E{ x[n] x[n+m] }

If |m| ≥ 2 then x[n] & x[n+m] are two <u>different</u> z[.] values ⇒ uncorrelated

⇒ $R_X(n,n+m) = 0$ for $|m| \geq 2$

Ex. #4: Weird Func of WN

If m = 0 then $R_X(n, n) = E\{z^2[k]\} = \sigma_z^2$

If m = 1

… and n is even: $R_X(n, n+1) = \sigma_z^2$

… and n is odd: $R_X(n, n+1) = 0$

n dependence causes it not to be WSS