# Random Processes

# Random Process

**<u>Other Names</u>** : Random Signal
Stochastic Process

A Random Process is an extension of the concept of a Random variable (RV)

**<u>Simplest View</u>** : A Random Process is a RV that is a Function of time

Could be **Discrete-time** or **Continuous-time**

# Examples of Discrete-Time Random Processes

1. x[k] = value on the $k^{th}$ roll of two dice
2. x[k] = lottery payout on day k
3. x[k] = temperature at noon on day k
4. x[k] = sequence of 1's and 0's in a binary file (also a coin flip)

# Examples of Continuous-Time Random Processes

1. X(t) = temperature at time t
2. X(t) = speed of an Aircraft at time t
3. X(t) = thermal noise voltage on resistor
   at time t

# Two Views of a Random Process

## 1) Viewing RP as a sequence of RV's

At each instant of time view x(t) as a RV:

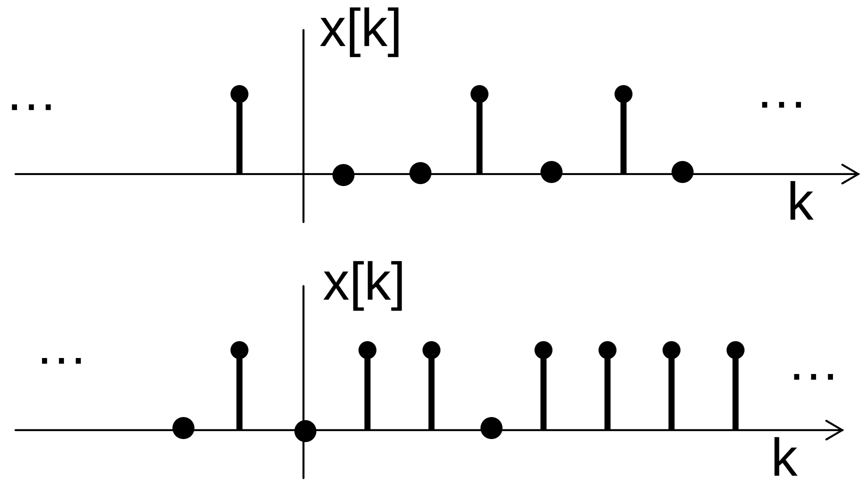
**DT Example #1**: at each k you roll 2 dice and get a random value

**CT Example #3**: at each time t you measure the thermal noise voltage to get a value

# Two Views of a Random Process

**2) <u>Alternate view</u>**: RP is a collection of functions of Time…the one you get is determined by some single "Probabilistic Event"

Ex: Random Bits – Here are just 2 possible functions for this process:



# Two Views of a Random Process

Each of the possible waveforms is called a "Realization" or "Sample Function" of the RP.

The collection of all possible realizations of a RP is called the "Ensemble" of the process

Of the two views, #1 is the most useful

i.e. a sequence of RV's

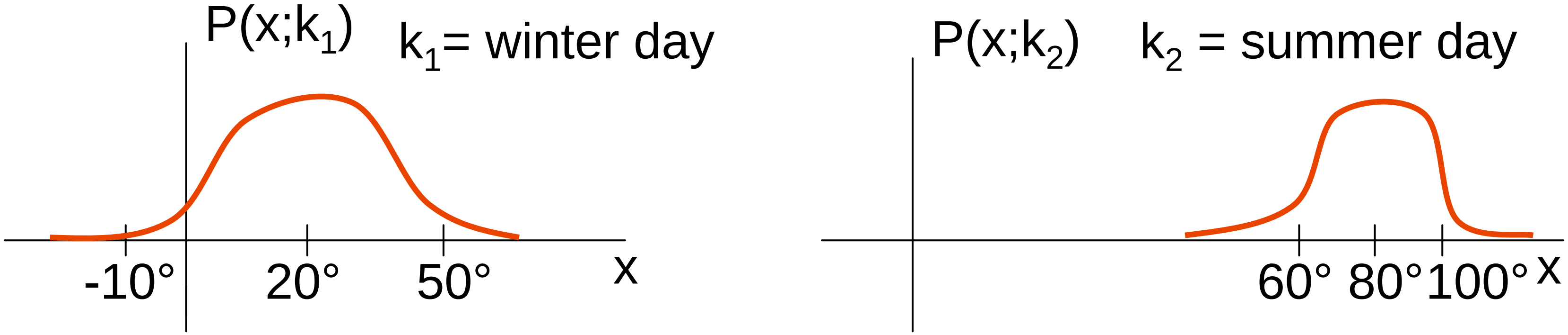
# Random Processes

How do we probabilistically characterize a RP?

**View #1 provides one clue!**

At each time $t_i$ we have a RV and it can be described by a PDF: $P(x;t_i)$. It is possible that the PDF changes with time.

**DT Example #3 – temp @ noon on day k.**



# Random Processes

So to describe a RP, we need P(x;t) for all t!

Is that enough? NO!

**RECALL**: To describe two RV's you need a PDF for each and need to know how they are related

$\Rightarrow$how they are Correlated $\Rightarrow$need their joint PDF!

But we have more than 2 RV's in our RP!

$\Rightarrow$To completely describe a RP we need all possible joint PDF's over the Time Points:

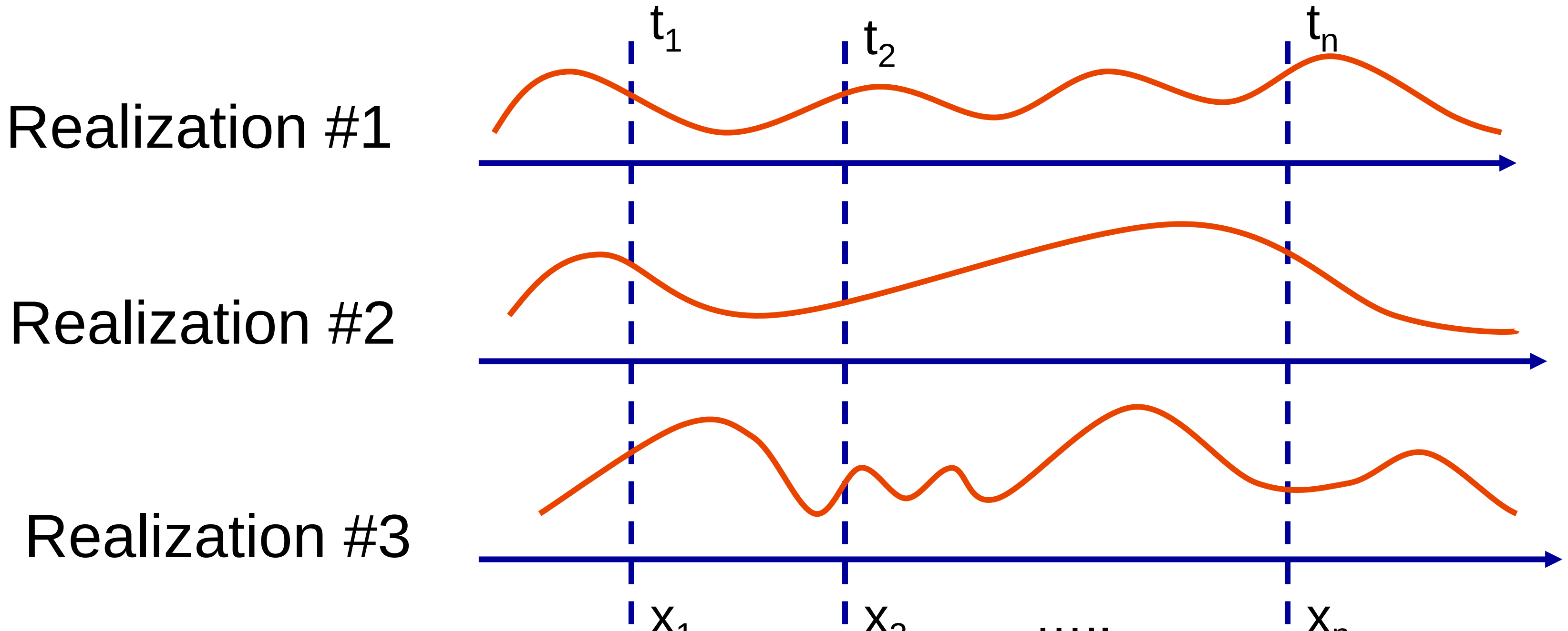
# Random Processes

To **Completely** Describe a RP…

Need $P(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots t_n)$

For <u>every</u> choice of $t_1, \ldots, t_n$

For <u>all</u> n (up to $\infty$ !)



# Random Process

This complete description of RP is virtually impossible to use for practical applications!
Usually make do with **1st and 2nd order PDF's:**

**P(x;t) & P(x1,x2; t1,t2)**

**Q: What do the 1st and 2nd order PDF's tell us ?**

**Ans #1** 1st order P(x;t) tells, as a function of time, what values are likely and unlikely to occur

P(x;t) can be characterized (but not necessarily completely) by two parameters: Mean & Variance

# Random Process

1) **Mean or Average or "Expected Value" of x(t)**

$$E\{x(t)\} = \int_{-\infty}^{\infty} x\, P(x;t)\, dx$$

**Other Notations:**

$$\overline{x(t)} = E\{x(t)\}$$

$$m_x(t) = E\{x(t)\}$$

Shows "center of concentration" of possible values of x(t) as a Function of time in general.

# Random Process

2) **Variance** of X(t)

$$\sigma^2_{x(t)} = E\left\{ \underbrace{\left[x(t) - \overline{x}(t)\right]^2}_{\text{Deviation from mean}} \right\}$$

**Deviation from mean**

$$= \int_{-\infty}^{\infty} \left[x - \overline{x}(t)\right]^2 P(x;t)\, dx$$

Shows a measure of expected range of variation around the mean, as a function of time in general

# Random Process

**Example**: let x[k] be temperature at noon on $k^{th}$ day (at a particular fixed location).

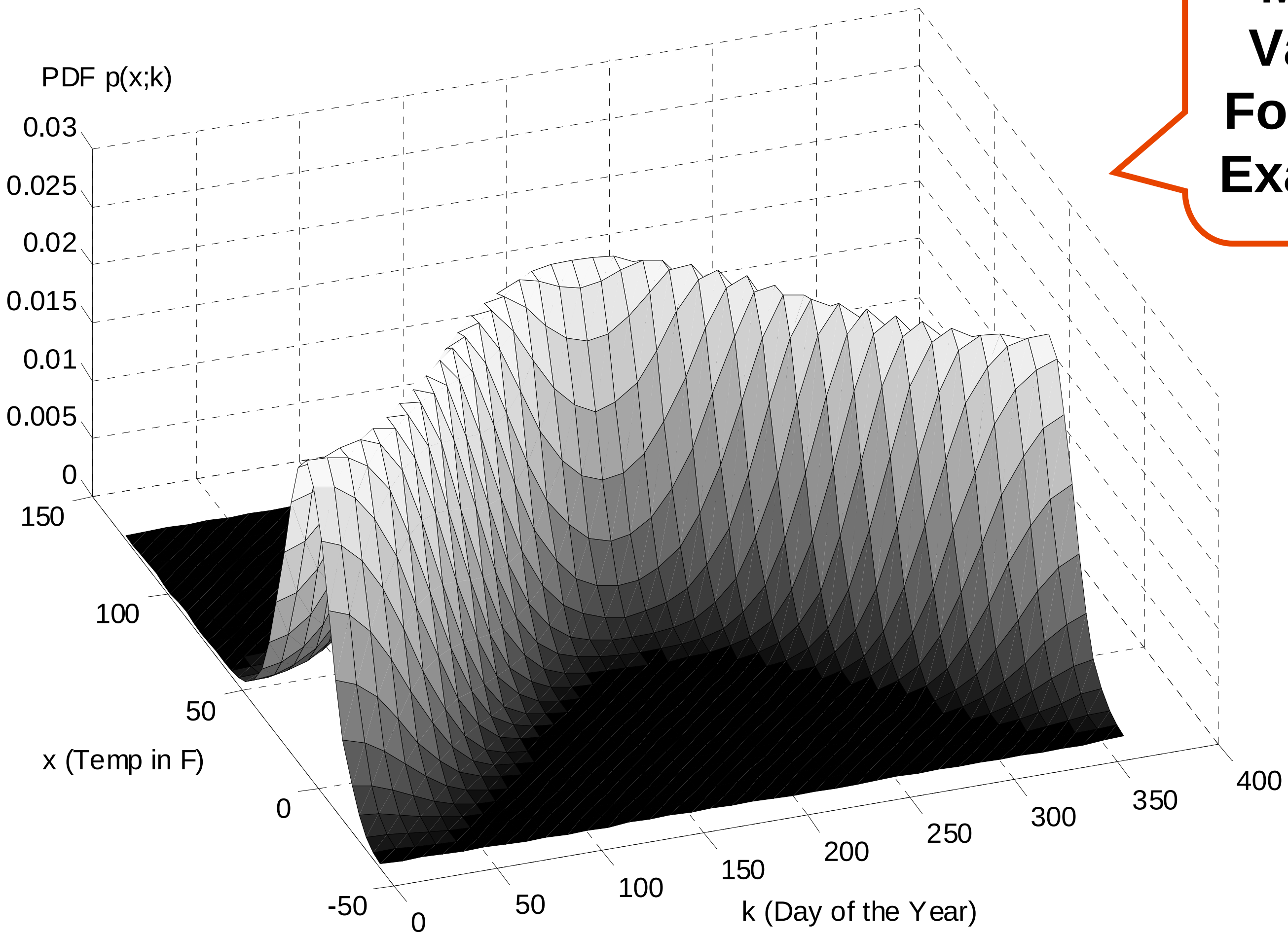
At each day k we view the ensemble of temperature realizations x[k] as occurring over a collection of "parallel universes" and the values occur with probability according to the PDF P(x;k)

# Random Processes

This ensemble view is theoretical and is used to model reality – don't confuse these **<u>theoretical ensemble averages</u>** with the **<u>empirical averages</u>** used in data analysis.

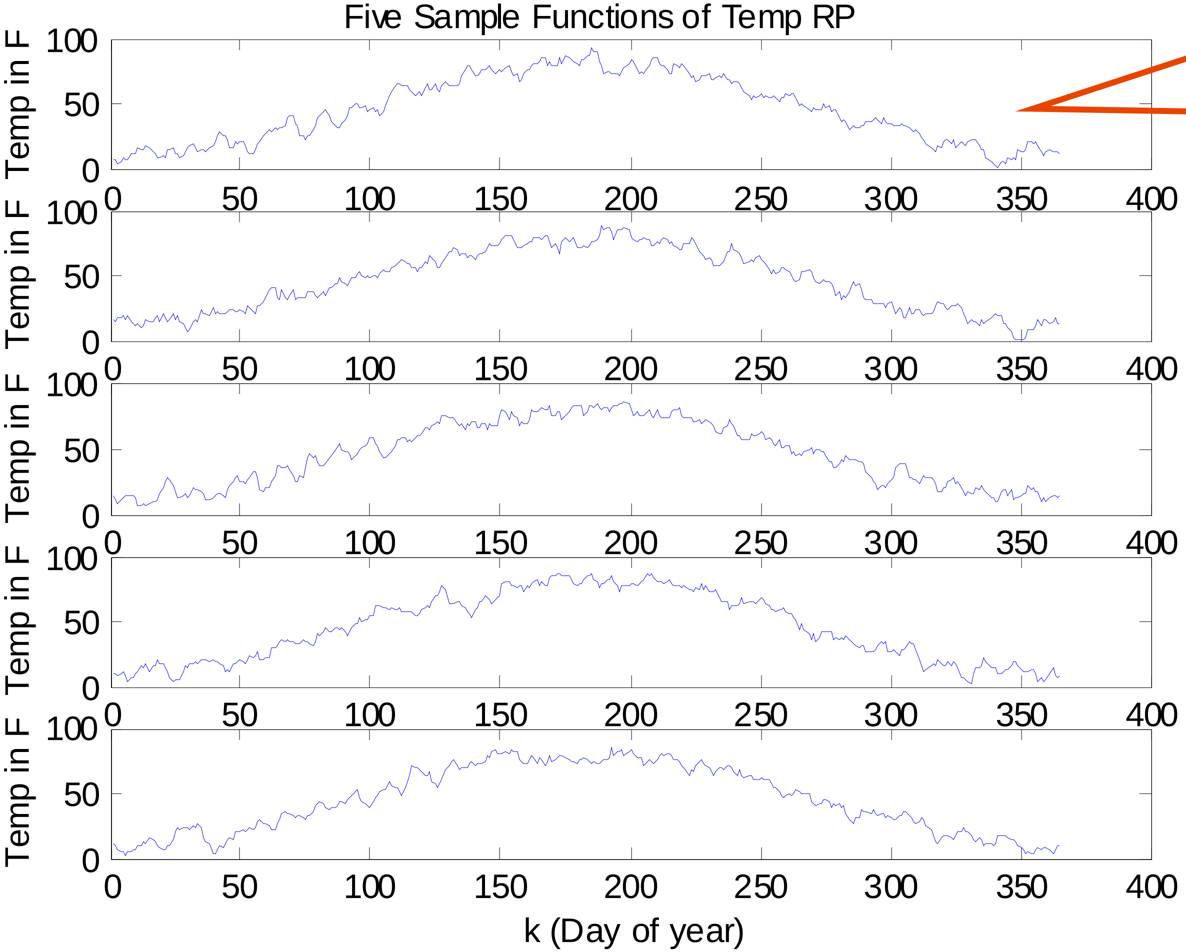
**Test Average =** $\frac{1}{N}\sum_{i=1}^{N} score\ (i)$

# Time-Varying PDF of RP



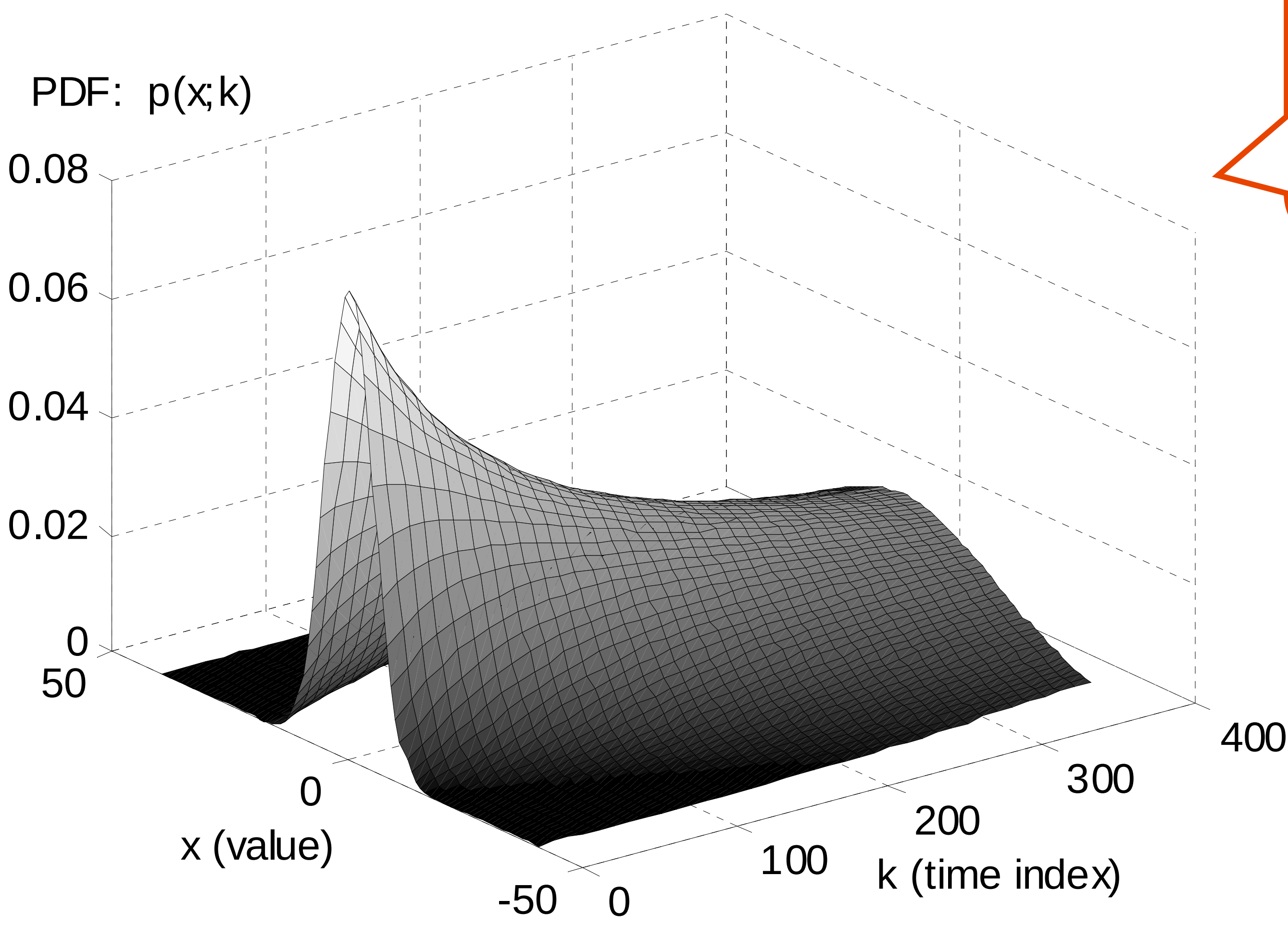

**Mean Varies For This Example**

# Sample Functions of <u>This</u> TV RP



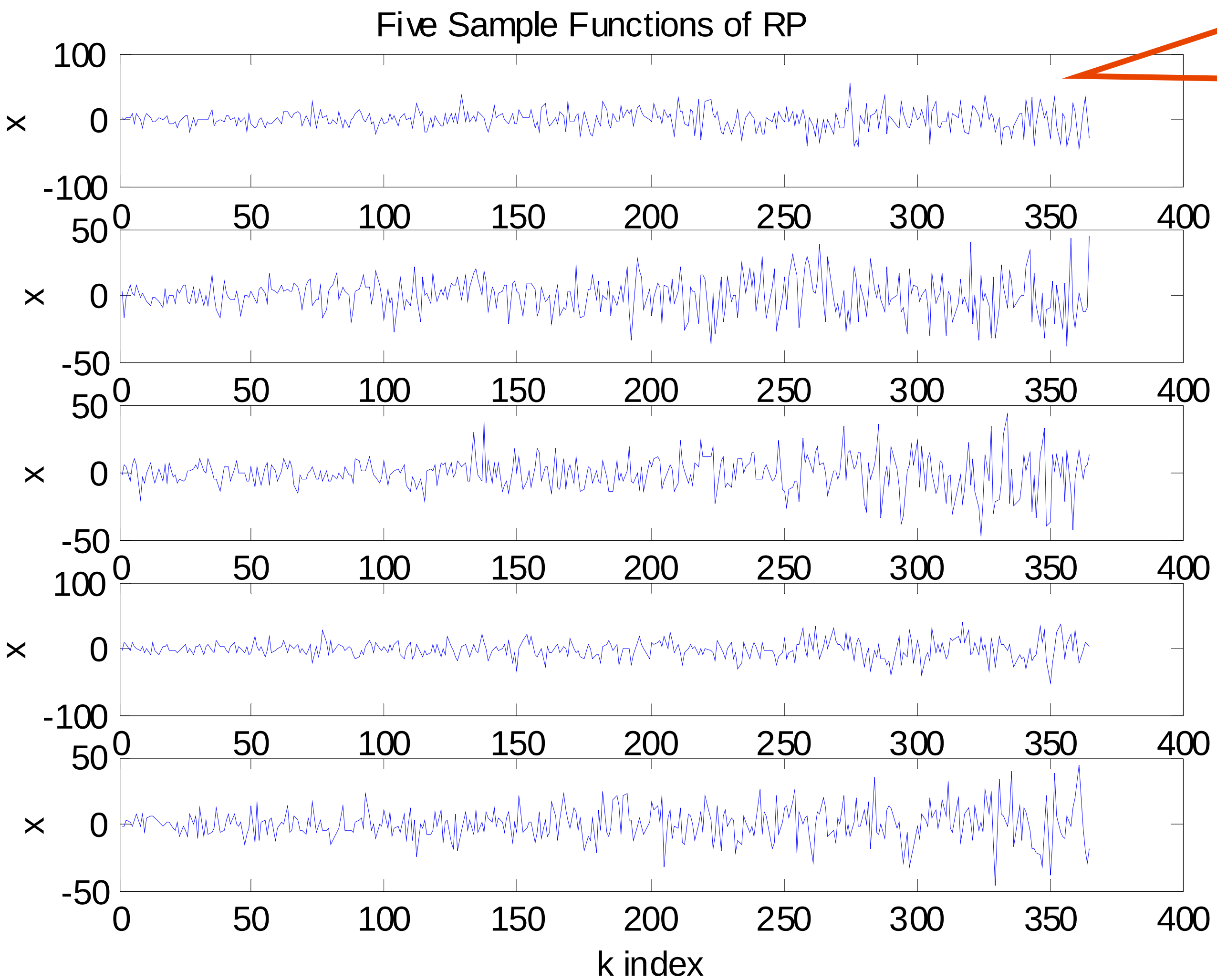

Can See The Varying Mean

# Different Example
# Time-Varying PDF of RP




**Variance Varies For This Example**

# Sample Functions of <u>this</u> TV RP




Can See The Varying Variance

# Random Processes

**Q: What do the 1st and 2nd order PDF's tell us ?**

**Ans #2** → **Recall** : 1st order PDF of RP tells the likelihood of values occurring at each given time

2nd Order PDF Characterizes "Probabilistic Coupling" between RP values at each pair of times $t_1$ and $t_2$

**Example** : What is the probability $x(t_1)$ and $x(t_2)$ are…

…. both Positive?
…. both Negative?
…. of Opposite Signs?

# Random Processes

As with mean and variance for the 1st order PDF, we want something that captures most of the essence of the 2nd order PDF

**Auto correlation function (ACF) of a RP**

$$R_x(t_1, t_2) = \underbrace{E\left\{x(t_1)\, x(t_2)\right\}}$$

**Correlates process at pairs of times $t_1$, $t_2$**

$$R_x(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1\, x_2\, P(x_1, x_2; t_1, t_2)\, dx_1, dx_2$$

# Auto Correlation Function

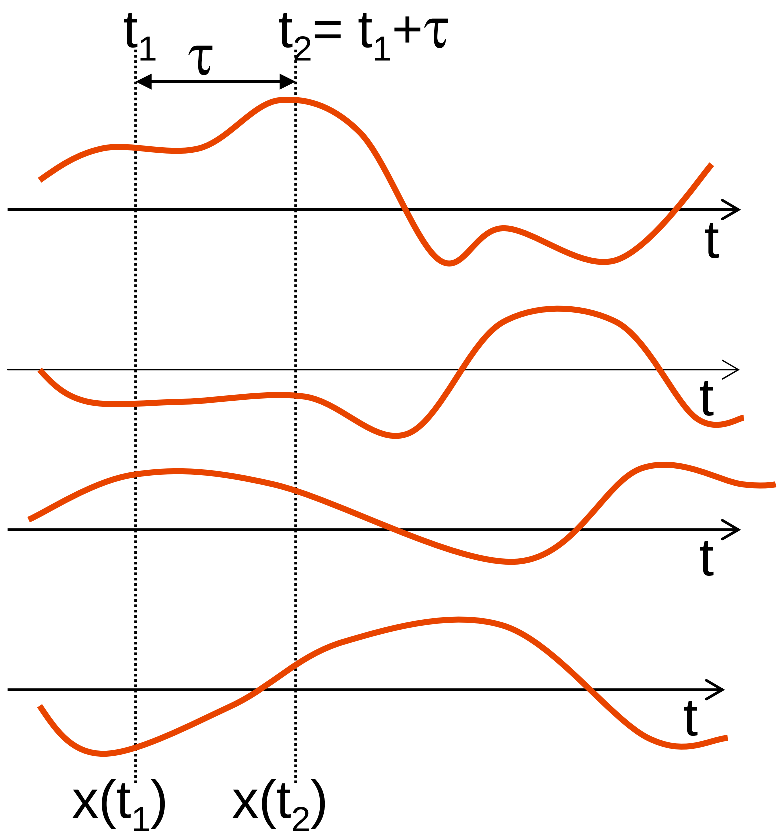
$R_X(t_1, t_2) =$

- **Positive** value if: $x(t_1)$& $x(t_2)$ are highly likely to have the same sign
- **Negative** value if: $x(t_1)$& $x(t_2)$ are highly likely to have opposite signs
- **Near Zero** if: Product $x(t_1)x(t_2)$ is ≈ equally likely to be positive or neg.

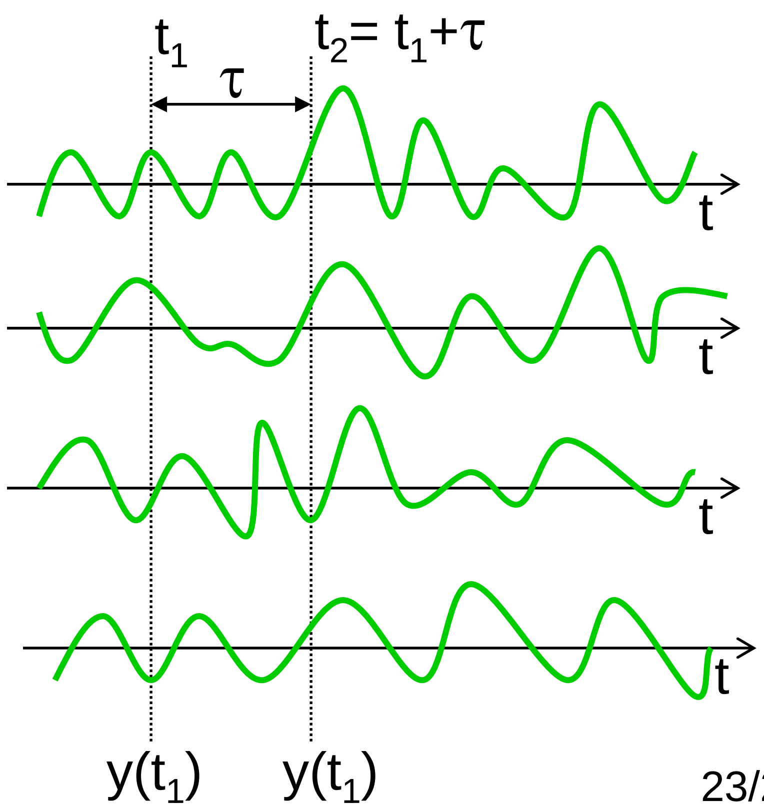
# Comparing ACF's of 2 RP's

**Note** : Both x(t) & y(t) have the same 1st Order PDF,
….yet they appear to be very different!

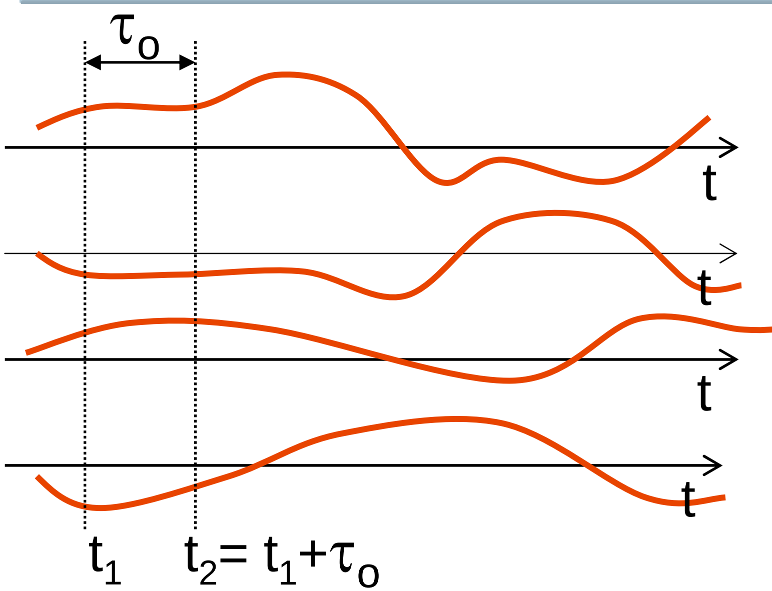
**Four Realizations of x(t)**



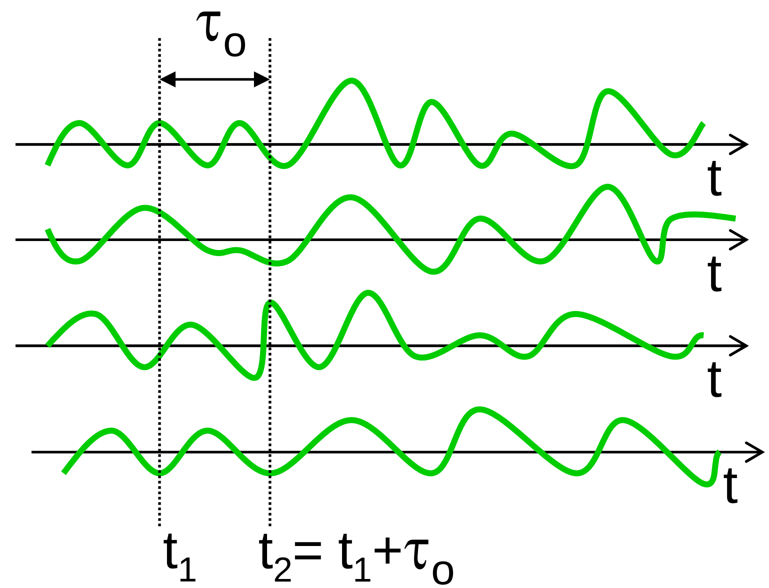

**Four Realizations of y(t)**

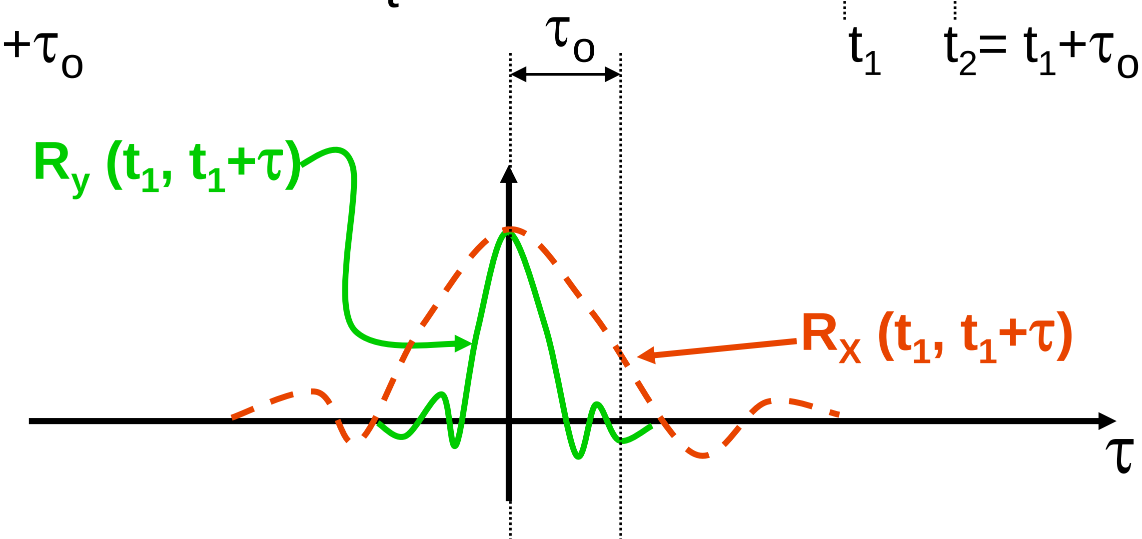


## Four Realizations of x(t)




## Four Realizations of y(t)




For $t_2 = t_1 + \tau$, $R_x(t_1, t_2) = \underbrace{R_x(t_1, t_1 + \tau)}$

View as a function of $\tau$ for each fixed $t_1$