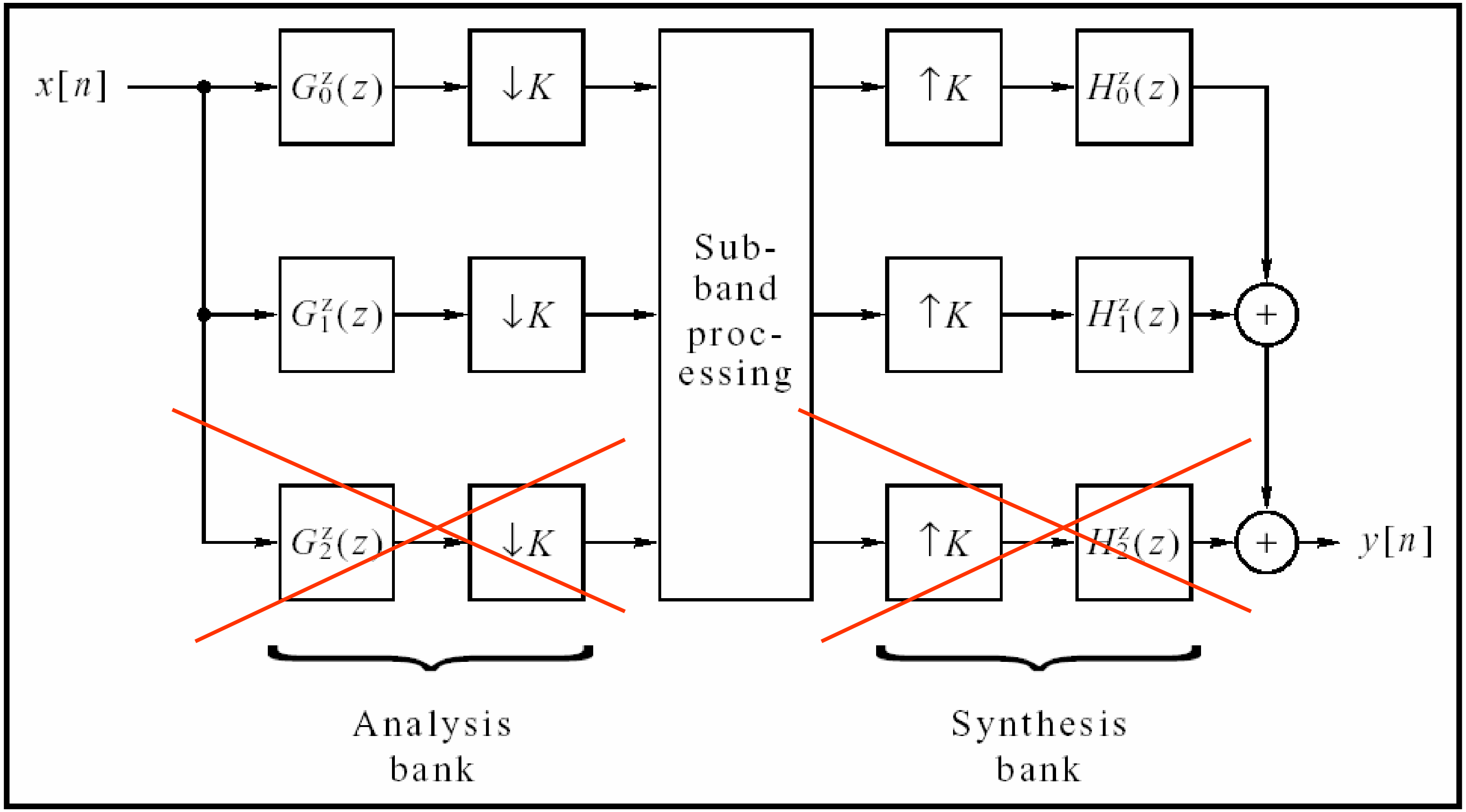
# 11.11 Two-Channel Filter Banks

# Two-Channel Filter Banks ($M$ = 2)

We want to look at methods that are not based on the DFT

In general we want to look at

Fig. 12.26 from Porat




… and figure out how to choose $G_i(z)$ & $H_i(z)$ to get Perfect Recon

# Why Two-Channel Filter Banks?

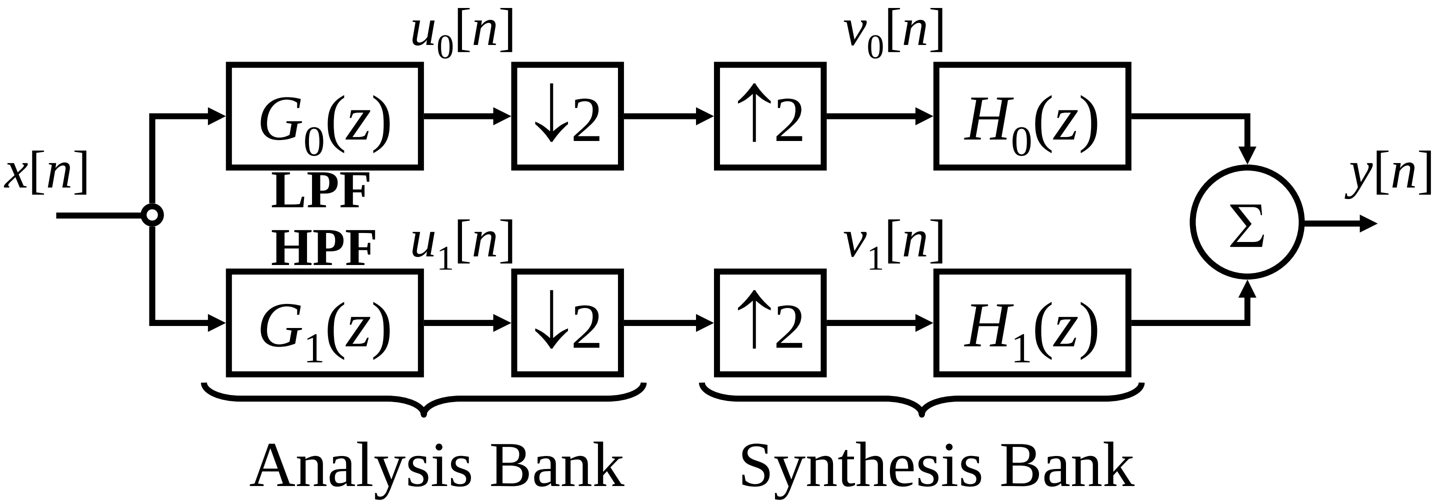
Why Consider Only Two Channels?

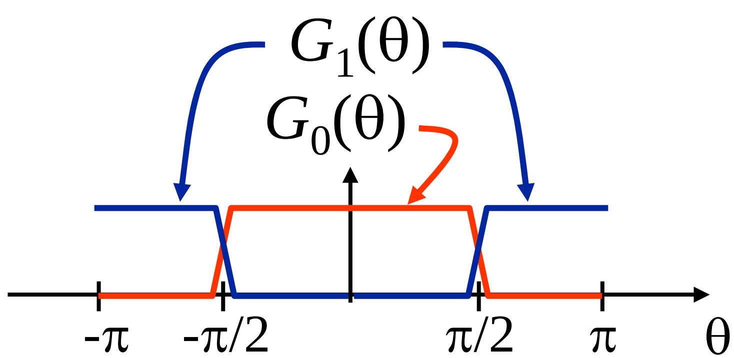
- Simple Place to Start
- Easier to Derive
  - But ideas extend to more channels
- Provide building blocks for tree-structure used to get more channels
  - This last reason is the most important!!

In our analysis we will assume:

- Subband Processing is Ignored
  - i.e., we focus on perfect recon w/o subband processing
- Maximal Decimation is to be used
  - i.e., we set decimation factor $K = 2$

# Two-Channel Structure to Consider




Q: How to choose $G_0(z)$, $G_1(z)$, $H_0(z)$, $H_1(z)$ ?

Q: Can we get Perfect Reconstruction (PR)?

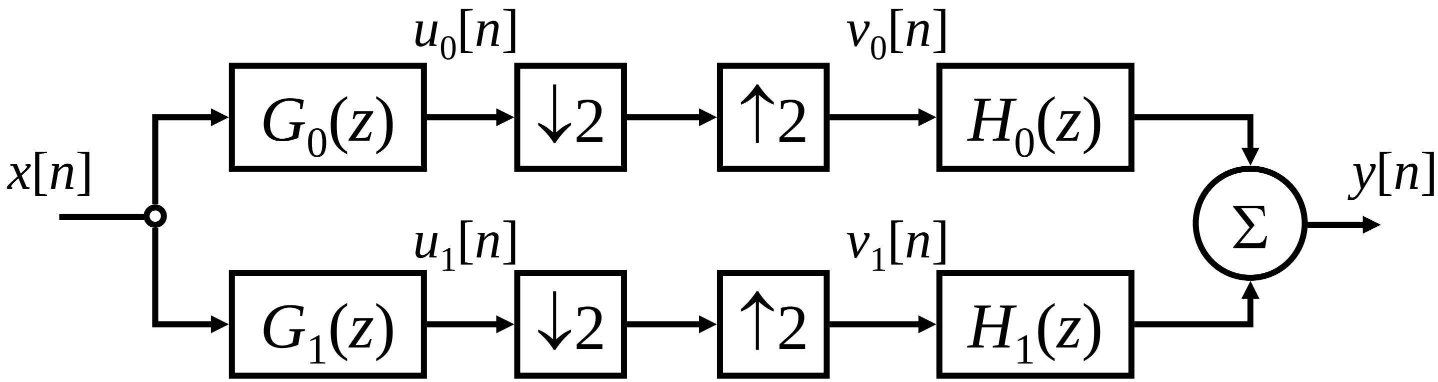
If so… HOW?!!!

# Two-Channel Properties

Write output $Y(z)$ in terms of input $X(z)$

… see what we get & what it tells us

Start at input and work towards output:




$$U_i^z(z) = G_i^z(z) X^z(z), \quad i = 0, 1 \qquad (\star)$$

Can easily show that for $\downarrow 2$ followed by $\uparrow 2$ we get:

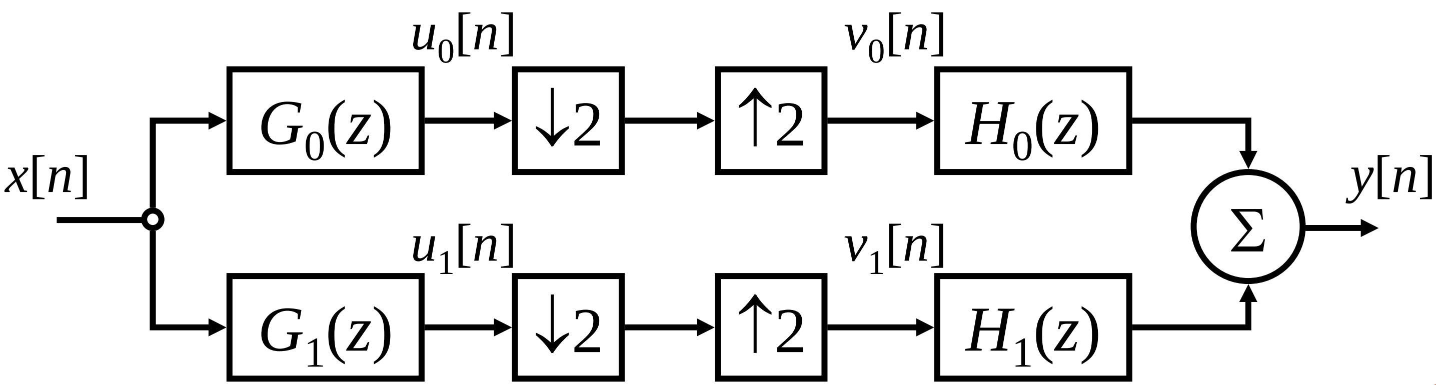
$$V_i^z(z) = \frac{U_i^z(z) + U_i^z(-z)}{2}, \quad i = 0, 1$$

Now putting $(\star)$ into this gives:

# Two-Channel Properties (cont.)

$$V_i^z(z) = \frac{G_i^z(z)X^z(z) + G_i^z(-z)X^z(-z)}{2}, \quad i = 0, 1 \qquad (\star\star)$$

Now look at how the output is formed from the $V_i$'s:




$$Y^z(z) = H_0^z(z)\overbrace{V_0^z(z)} + H_1^z(z)\overbrace{V_1^z(z)}$$

Using $(\star\star)$ in this gives:

# Two-Channel Properties (cont.)

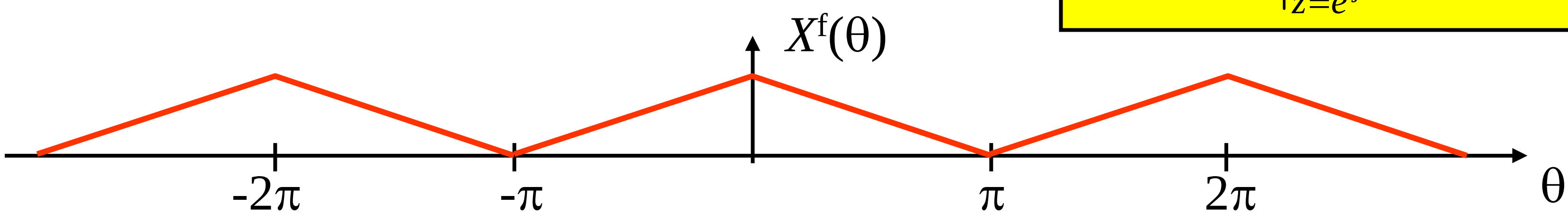
$$Y^z(z) = \left[\frac{G_0^z(z)H_0^z(z) + G_1^z(z)H_1^z(z)}{2}\right]X^z(z) + \left[\frac{G_0^z(-z)H_0^z(z) + G_1^z(-z)H_1^z(z)}{2}\right]X^z(-z)$$

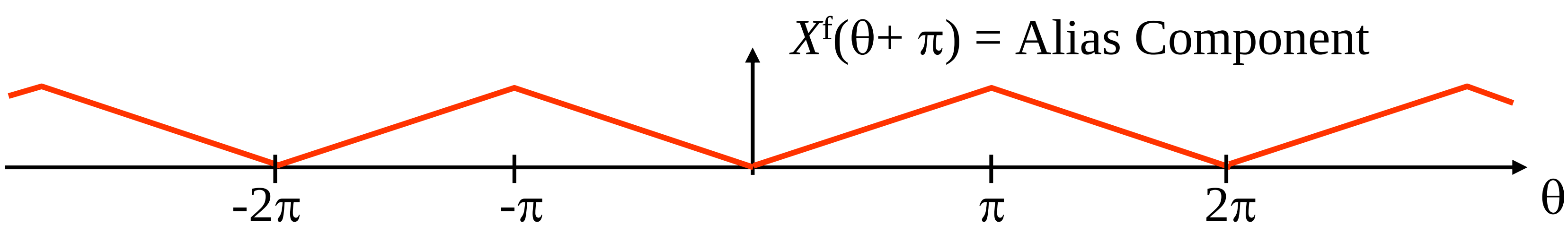
Two Terms:

- Response to Input Signal $X(z)$
- Response to Alias Component $X(-z)$

$$X^z(-z) = X^z(\underbrace{e^{j\pi}}_{-1} z)$$

$$\Rightarrow X^z(-z)\Big|_{z=e^{j\theta}} = X^f(\theta+\pi)$$



# Two-Channel Properties (cont.)

Don't want $X(-z)$ in output… to eliminate:

… necessary & sufficient to have:

$$\boxed{G_0^z(-z)H_0^z(z) + G_1^z(-z)H_1^z(z) = 0}$$

We'll call this the "Aliasing Cancellation Condition" (ACC)

One way to satisfy the ACC is to choose the synthesis filters as:

$$H_0^z(z) = 2G_1^z(-z) \leftarrow \textbf{Lowpass}$$

$$H_1^z(z) = -2G_0^z(-z) \leftarrow \textbf{Highpass}$$

(★★★)

Note: This does not eliminate aliasing in each channel!!!

The filters are chosen so:

Aliasing in the channels cancel each other in the summation!

# Two-Channel Properties (cont.)

Now that we have the aliasing cancelled….

What does the output signal look like???

Using our previous results:

$$H_0^z(z) = 2G_1^z(-z)$$

$$H_1^z(z) = -2G_0^z(-z)$$

$$Y^z(z) = \left[\frac{G_0^z(z)H_0^z(z) + G_1^z(z)H_1^z(z)}{2}\right]X^z(z)$$

$$\cancel{+\left[\frac{G_0^z(-z)H_0^z(z) + G_1^z(-z)H_1^z(z)}{2}\right]X^z(-z)}$$

**Output When Aliasing Has Been Cancelled:**

$$Y^z(z) = \underbrace{\left[G_0^z(z)G_1^z(-z) - G_0^z(-z)G_1^z(z)\right]}_{\text{Define } F^z(z)}X^z(z)$$

Analysis-Synthesis Transfer Function

# Two-Channel Properties (cont.)

How do we get Perfect Reconstruction???

For PR we need: $y[n] = c\, x[n - l]$

$\Rightarrow F(z) = cz^{-l}$

$\Rightarrow F(\theta) = ce^{-jl\theta}$

Linear Phase Response

Flat Magnitude Response

If we don't have Perfect Reconstruction:

$$F^{\mathrm{f}}(\theta) = A(\theta)e^{j\phi(\theta)}$$

If $A(\theta) \neq$ constant, then we have "<u>Amplitude Distortion</u>"

If $\phi(\theta) \neq -l\theta$, then we have "<u>Phase Distortion</u>"

Choosing the filters to get PR drew much research effort

# Quadrature Mirror Filters (QMF)

QMF was one of the early attacks on filter bank design

But… it fell short of giving PR.

Note that the ACC puts no constraints on the individual filters:

– it says…. "if the analysis filters are <u>this</u> then the synthesis filters must be <u>that</u> "

So… we are free to choose the analysis filters to try to meet other needs:

- Good Stopband Rejection
- Perfect Reconstruction
- Linear Phase
- Etc.

QMF Banks were developed in 1977 by Esteban & Galand as an attempt to design good 2-channel FB's…

….. they are useful but not perfect!!!!!

# Quadrature Mirror Filters (cont.)

Definition of QMF: A pair of analysis filters are QMF if

$$-z\Big|_{z=e^{j\theta}} = e^{-j\pi}e^{j\theta} = e^{j\theta-\pi}$$

$$\boxed{G_1^z(z) = G_0^z(-z)}$$

$$\Updownarrow$$

$$\boxed{G_1^f(\theta) = G_0^f(\theta-\pi)}$$

QMF Condition

QMF Condition is equivalent to:

$$\boxed{G_1^f(0.5\pi+\theta) = \overline{G}_0^f(0.5\pi-\theta)}$$

Proof: Shift by 0.5π both sides of DTFT QMF Condition

$$G_1^f(\theta+0.5\pi) = G_0^f(\theta-0.5\pi)$$

Now use symmetry of DTFT ( $X^f(-\theta) = \overline{X}^f(\theta)$ ) on Rt Hand Side

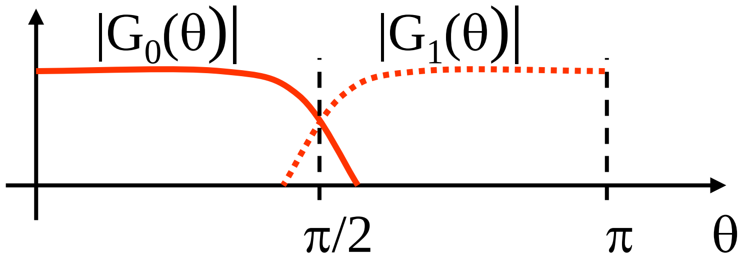
$$G_1^f(0.5\pi+\theta) = \overline{G}_0^f(0.5\pi-\theta)$$

# Quadrature Mirror Filters (cont.)

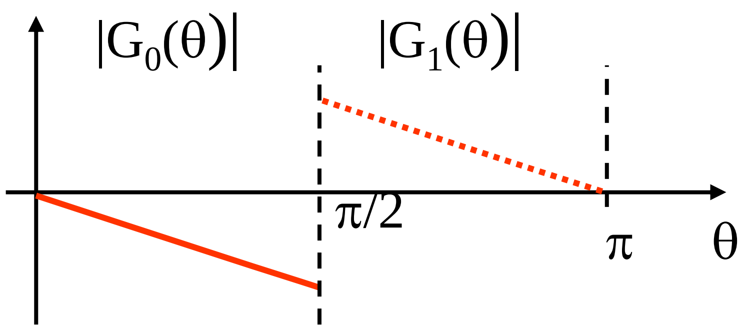
This is a "mirror image" criteria $\boxed{G_1^f(0.5\pi+\theta)=\overline{G_0^f}(0.5\pi-\theta)}$

➔ $G_1(\theta)$ is the "mirror image" of $G_0(\theta)$ around the point $\theta = \pi/2$

- Magnitudes are "even" images $\left|G_1^f(0.5\pi+\theta)\right| = \left|G_0^f(0.5\pi-\theta)\right|$




- Phases are "odd" images $\angle G_1^f(0.5\pi+\theta) = -\angle G_0^f(0.5\pi-\theta)$



# Quadrature Mirror Filters (cont.)

So… QMF design requires:

- Designing $G_0(z)$ as a lowpass filter
  - Design may strive to meet various spec's (e.g., stopband levels, transition widths, etc.)
- Then you get $G_1(z)$ from $G_0(z)$ using $G_1(z) = G_0(-z)$ for QMF
  - This ensures that $G_1(z)$ is a HPF, as desired.

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Now… recall that with ACC satisfied the analysis-synthesis transfer function is:

$$F^z(z) = G_0^z(z)G_1^z(-z) - G_0^z(-z)G_1^z(z)$$

Using the QMF Condition ( $G_1(z) = G_0(-z)$ ) in this gives:

$$\boxed{F^z(z) = [G_0^z(z)]^2 - [G_0^z(-z)]^2}$$

QMF
Analysis-Synthesis
Transfer Function

Note: This T-F is completely specified in terms of $G_0(z)$

This makes QMF design fairly easy compared to other FB design schemes

# Quadrature Mirror Filters (cont.)

Q: Can we get Perfect Reconstruction from QMF Filter Banks????

**Fact**: FIR QMF's can only give PR when the filter has no more than 2 coefficients.

**Proof**: Recall that for the QMF case the FB transfer function is:

$$F^z(z) = [G_0^z(z)]^2 - [G_0^z(-z)]^2$$

But this can be re-written as:

$$F^z(z) = [G_0^z(z) + G_0^z(-z)][G_0^z(z) - G_0^z(-z)]$$

The requirement for PR is $F(z) = cz^{-l}$ which implies that we need:

$$G_0^z(z) + G_0^z(-z) = c_1 z^{-l_1}$$

$$G_0^z(z) - G_0^z(-z) = c_2 z^{-l_2}$$

for some $c_1$, $c_2$, $l_1$, $l_2$

Bit of big jump here… see other notes for better proof

Solving this leads to:

$$G_0^z(z) = 0.5c_1 z^{-l_1} + 0.5c_2 z^{-l_2}$$

$$G_0^z(-z) = 0.5c_1 z^{-l_1} - 0.5c_2 z^{-l_2}$$

These are 2-tap FIR Filters

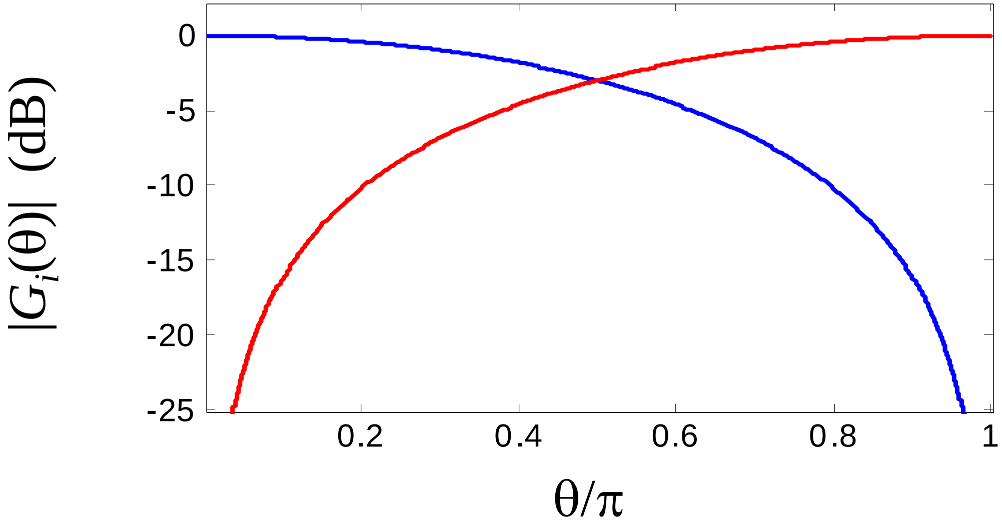
# Quadrature Mirror Filters (cont.)

So, only way to get PR from QMF is to use 2-tap filters:

➔ Serious Limitation… But Non-PR QMF's are still used.

Note: IIR QMF's don't have this limitation but have their own drawbacks (e.g., nonlinear phase)

Two-Tap FIR filters have poor passband, stopband, & transition band performance:



# Quadrature Mirror Filters (cont.)

**Result**: If $G_0(z)$ has linear phase, then the resulting QMF analysis-synthesis FB is free of phase distortion (i.e., the FB also has linear phase).

Note: Must use FIR to get linear phase

**Proof**: Let $G_0(z)$ be an $N^{\text{th}}$ order FIR filter with linear phase, then

$$G_0^{\mathrm{f}}(\theta) = A(\theta)e^{-j0.5\theta N}$$

Linear Phase

Amplitude Function
(real valued, but can go negative)

Then we have:

$$G_0^{\mathrm{f}}(\theta-\pi) = A(\theta-\pi)e^{-j0.5(\theta-\pi)N}$$
$$= e^{j\pi N/2}A(\theta-\pi)e^{-j0.5\theta N}$$

and…

$$F^{\mathrm{f}}(\theta) = [A^2(\theta) - (-1)^N A^2(\theta-\pi)]e^{-j\theta N}$$

# Quadrature Mirror Filters (cont.)

**Aside**: $N$ can't be chosen to be even valued!!! Here's why:

From standard FT properties (even magnitude): $A^2(\theta) = A^2(-\theta)$

Thus…

$$|F^{f}(\pi/2)| = A^2(\pi/2)[1 - (-1)^N] = 0 \quad \text{if } N \text{ is even!!}$$

Thus, even $N$ causes complete attenuation at $\pi/2$…. BAD!!!

Back to the proof… Since $N$ is odd we get:

$$F^{\mathrm{f}}(\theta) = \underbrace{[A^2(\theta) + A^2(\theta - \pi)]}_{\text{FB's Magnitude Response}} e^{-j\theta N}$$

FB's Phase Response

➔ The FB has linear phase…. End of Proof

# Quadrature Mirror Filters (cont.)

**Result**: A linear-phase QMF FB will have amplitude distortion (except for the 2-tap case). The amplitude distortion is given by

$$D_{amp}(\theta) = \left|F^{\mathrm{f}}(\theta)\right| - 1$$

$$= \left|G_0^{\mathrm{f}}(\theta)\right|^2 + \left|G_0^{\mathrm{f}}(\theta - \pi)\right|^2 - 1$$

**QMF Design Method**: Use numerical optimization techniques to find a linear-phase $G_0(\theta)$ that minimizes $D_{\mathrm{amp}}(\theta)$.

<See Books Entirely on Multirate & Filterbanks>

# Perfect Reconstruction Filter Banks

With careful design you can get QMF's to give nearly perfect recon

To get perfect reconstruction… we must replace the QMF condition by a different condition.

Recall QMF: $G_1^z(z) = G_0^z(-z)$

New Condition: $\boxed{G_1^z(z) = (-z)^{-N} G_0^z(-z^{-1})}$

$N$ is FIR Order

Filters satisfying this: "Conjugate Quadrature Filters (CQF)"

The CQF condition is equivalent to:

$$\boxed{g_1[n] = (-1)^n g_0[N-n]}$$

# PR Filter Banks (cont.)

Proof of this equivalence:

Let $W_1(z) = G_0(1/z)$

$\Rightarrow w_1[n] = g_0[-n]$

See (3.2.12) in P&M

Let $W_2(z) = z^{-N} W_1(z)$

$\Rightarrow w_2[n] = w_1[n]\big|_{n=n-N}$

$= g_0[-(n-N)] = g_0[N-n]$

See (3.2.5) in P&M

Let $W_3(z) = W_2(-z)$

$\Rightarrow w_3[n] = (-1)^{-n} w_2[n]$

$= (-1)^{-n} g_0[N-n]$

See (3.2.9) in P&M (with $a$ = -1)

Finally note that:

$$W_3(z) = W_2(-z) = (-z)^{-N} W_1(-z) = (-z)^{-N} G_0(-1/z) = G_1(z)$$

$$\Rightarrow g_1[n] = (-1)^{-n} g_0[N-n]$$

**<End of Proof>**

# PR Filter Banks (cont.)

Restricting $N$ to be Odd (as we did for QMF) gives:

$$\boxed{F^z(z) = z^{-N}\underbrace{[G_0^z(z)G_0^z(z^{-1}) - G_0^z(-z)G_0^z(-z^{-1})]}}$$

If "this" = 1, then we get PR!

Note this result:

$$W^z(z^{-1}) \quad \rightarrow \quad w[-n]$$

$$\Rightarrow \mathcal{F}\{w[-n]\} = \sum_{n=-\infty}^{\infty} w[-n]e^{-j\theta n} = \sum_{m=-\infty}^{\infty} w[m]e^{j\theta m}$$

$$= \left[\sum_{m=-\infty}^{\infty} w[m]e^{-j\theta m}\right]^* = \overline{W}(\theta)$$

And….

$$W^z(-z) \quad \rightarrow \quad W^f(\theta - \pi)$$

# PR Filter Banks (cont.)

Since for PR we want

$$\boxed{G_0^z(z)G_0^z(z^{-1}) - G_0^z(-z)G_0^z(-z^{-1}) = 1}$$

Using these two results gives the equivalent equation:

$$\boxed{\left|G_0^{\mathrm{f}}(\theta)\right|^2 + \left|G_0^{\mathrm{f}}(\theta-\pi)\right|^2 = 1}$$

A filter satisfying this is called "<u>Power Symmetric</u>"

# PR Filter Banks (cont.)

Design Procedure for CQF-PR Filter Bank

1. Find an odd-order, power symmetric LPF with passband of [0,π/2] and having desired stopband attenuation
   Gives $G_0(z)$
2. Use CQF Condition to get $G_1(z)$
3. Use Aliasing Cancellation Criteria to get:

$$H_0^z(z) = 2G_1^z(-z)$$

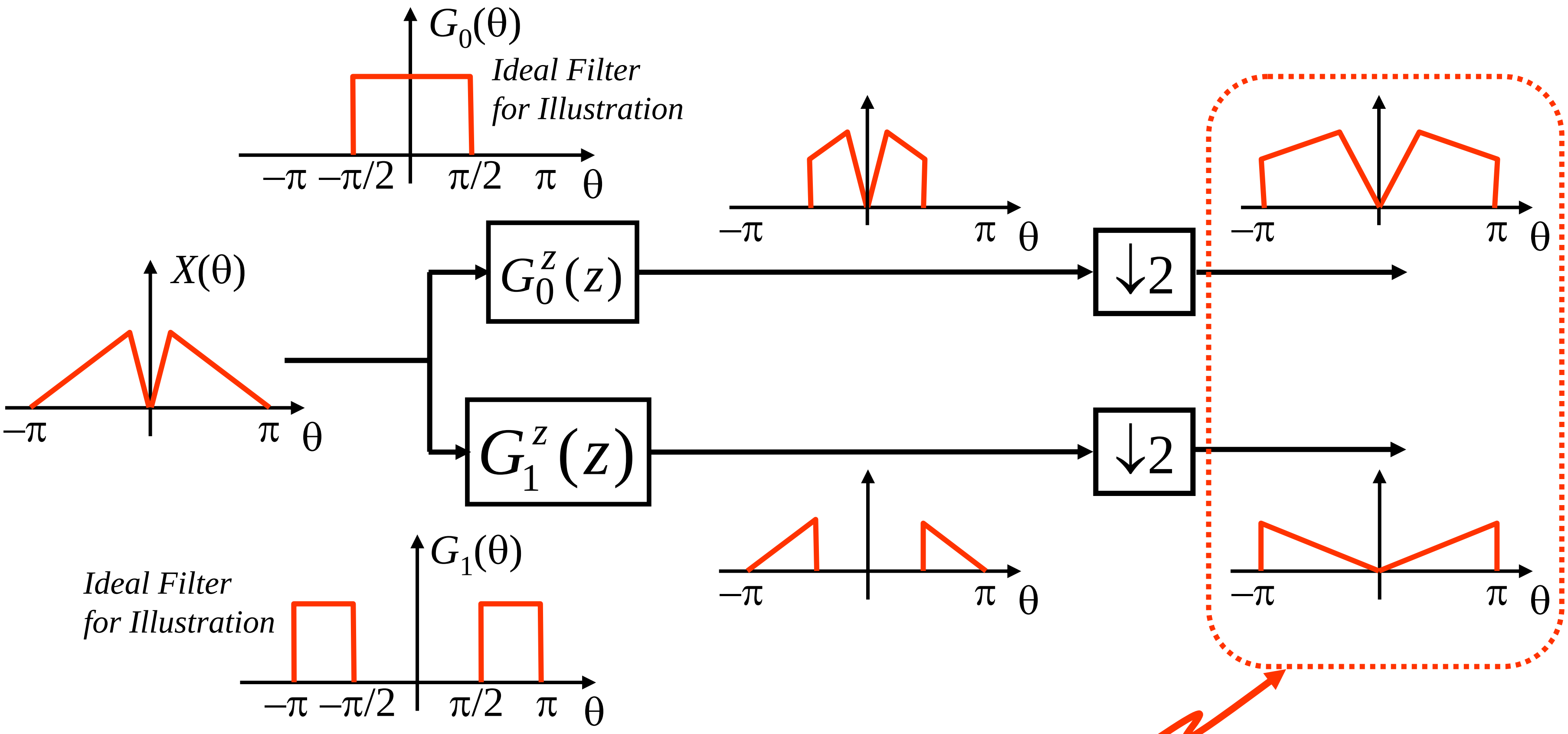
$$H_1^z(z) = -2G_0^z(-z)$$

Step #1 is Challenging!!!!

See books entirely on Multirate & Filterbanks

# Tree-Structured Filter Banks

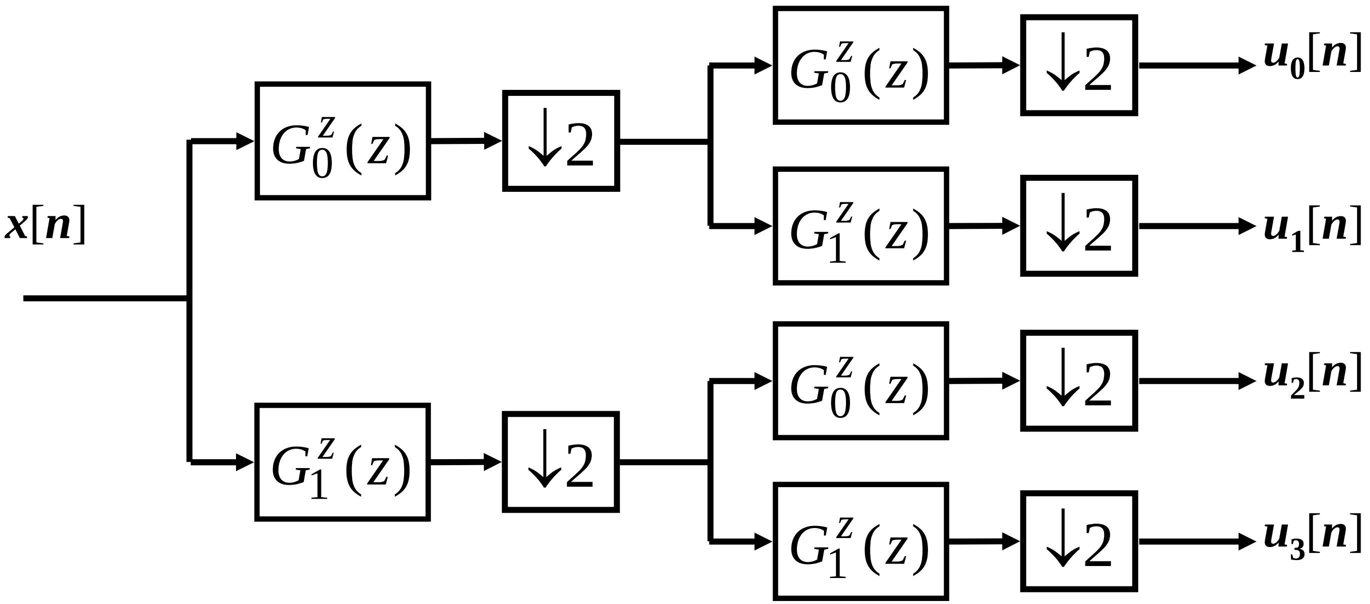
**Q: What do we have after a two-channel split & decimation?**

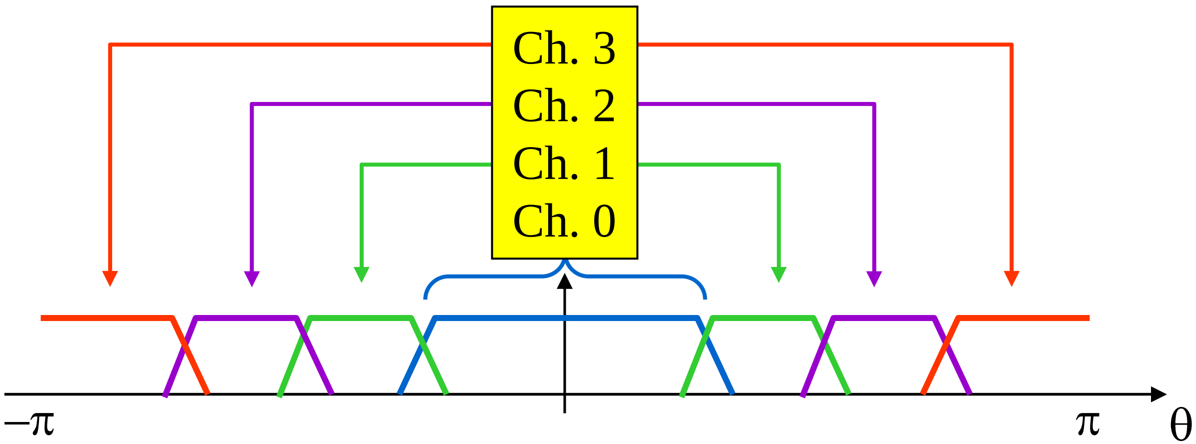



**You Get The Desired Half-Bands, But Spread Over [–π , π]!!!!**
**All Ready to Re-Apply the <u>SAME</u> 2-Band Split**

*Note: No Aliased terms – due to <u>Ideal</u> Filters*

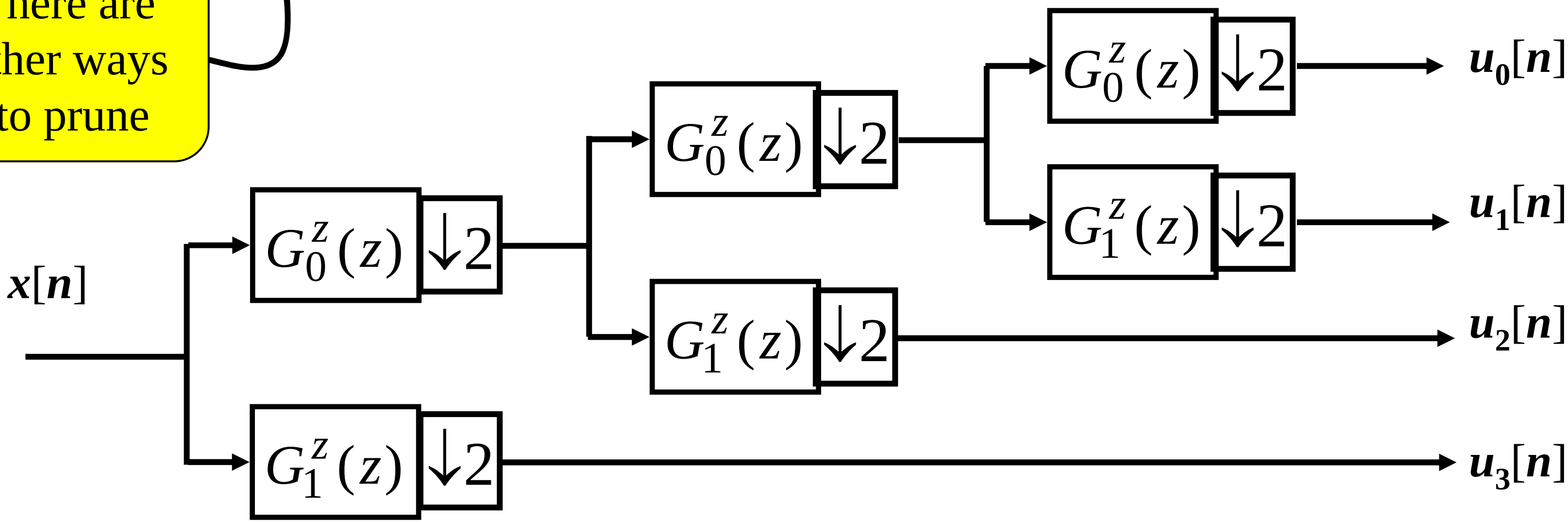
# A Full-Tree Cascade of Two-Band Filter Banks ➔ Uniformly Spaced Filter Bank

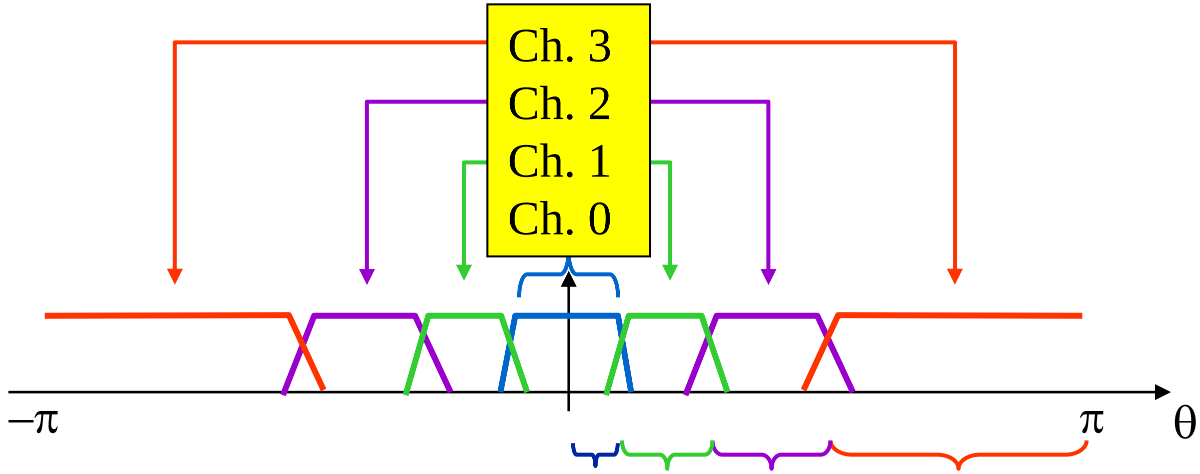


## HP-Pruned Tree Cascade of Two-Band Filter Banks
## ➔ Octave-Band Filter Bank
### (Used to Implement Wavelet Transform)

There are other ways to prune




**Each higher band is twice as wide**