Multistage Rate Change

Motivation for Multi-Stage Schemes

Consider Decimation:

When M is large (typically > 10 or so) it is usually inefficient to implement decimation in a single step (i.e., in a single stage).

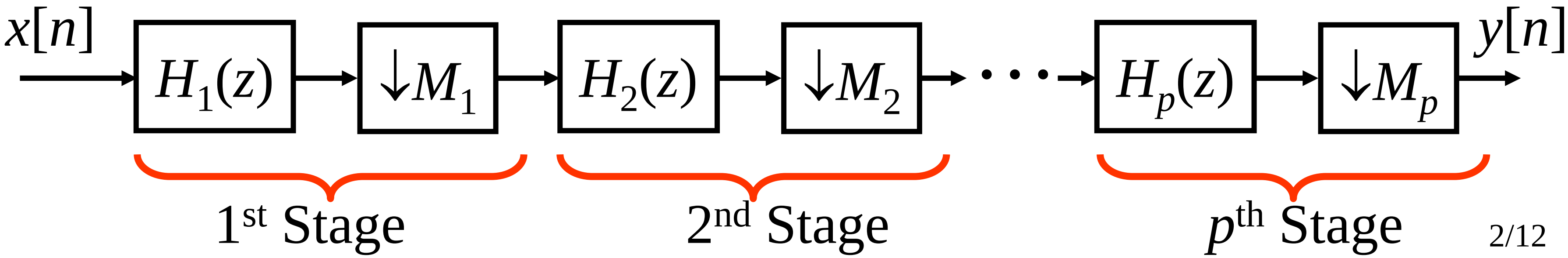
The Culprit: Large M requires the LPF to have a stopband edge of $\theta_s = \pi/M$, which is small for large M

➔ Need a LPF with a very narrow passband

➔ Requires a long FIR filter

➔ Inefficient since long filters require a large # of multiplies

Solution: If M can be factored into a product of integers ($M = M_1 M_2 M_3 \ldots M_p$). Then decimation by M can be done by:



Trick to Get Efficiency from Multi-Stage

The design of $H_1(z)$ (& other "front-end" stages) can be relaxed from what you would use for a single-stage design.

Certainly, you need $H_1(z)$ to have $\theta_s = \pi/M_1 > \pi/M$ so no aliasing occurs after $\downarrow M_1$….

But… it is even better than that.

Can let $\theta_s > \pi/M_1$ … which lets some aliasing occur

But… only so much aliasing – such that the aliasing that occurs gets suppressed by the next filter, $H_2(z)$

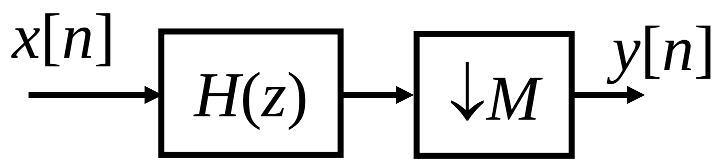
Higher Stopband Edge ➔ Shorter Filter ➔ More Efficient

Let's See Why for a 2-Stage Case

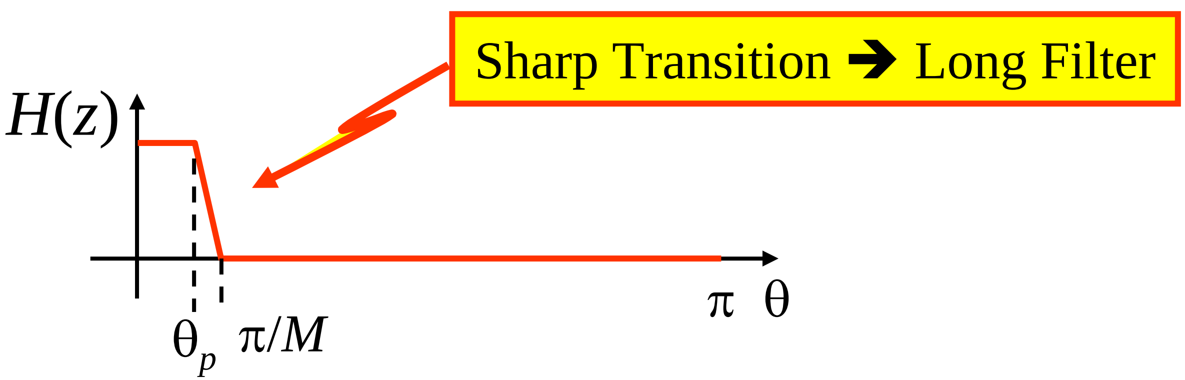
Say that the signal $x[n]$ has "spectral content of worth" only up to frequency $\theta = \theta_p < \pi/M$… with $M = M_1 M_2$.

Single-Stage Method

Suppose we decimate using a single-stage scheme:

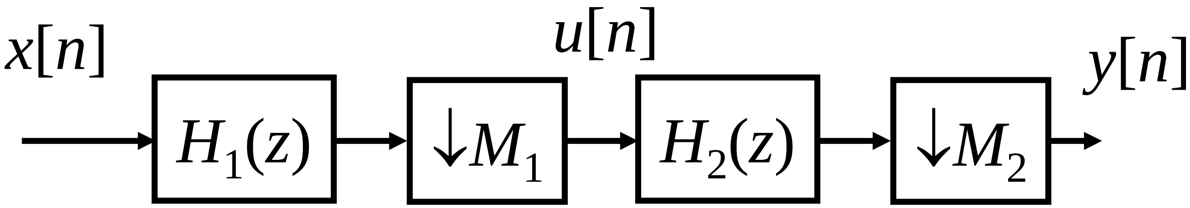


Then we need

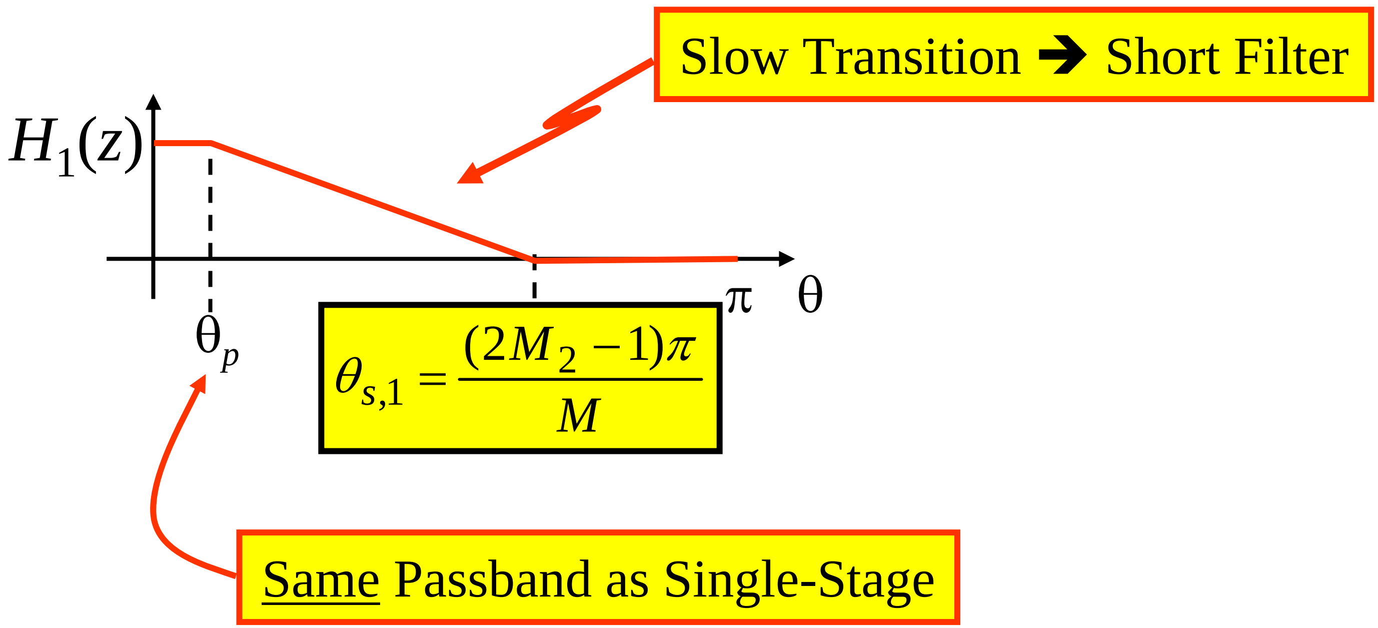


Let's See Why for a 2-Stage Case (cont.)

2-Stage Method

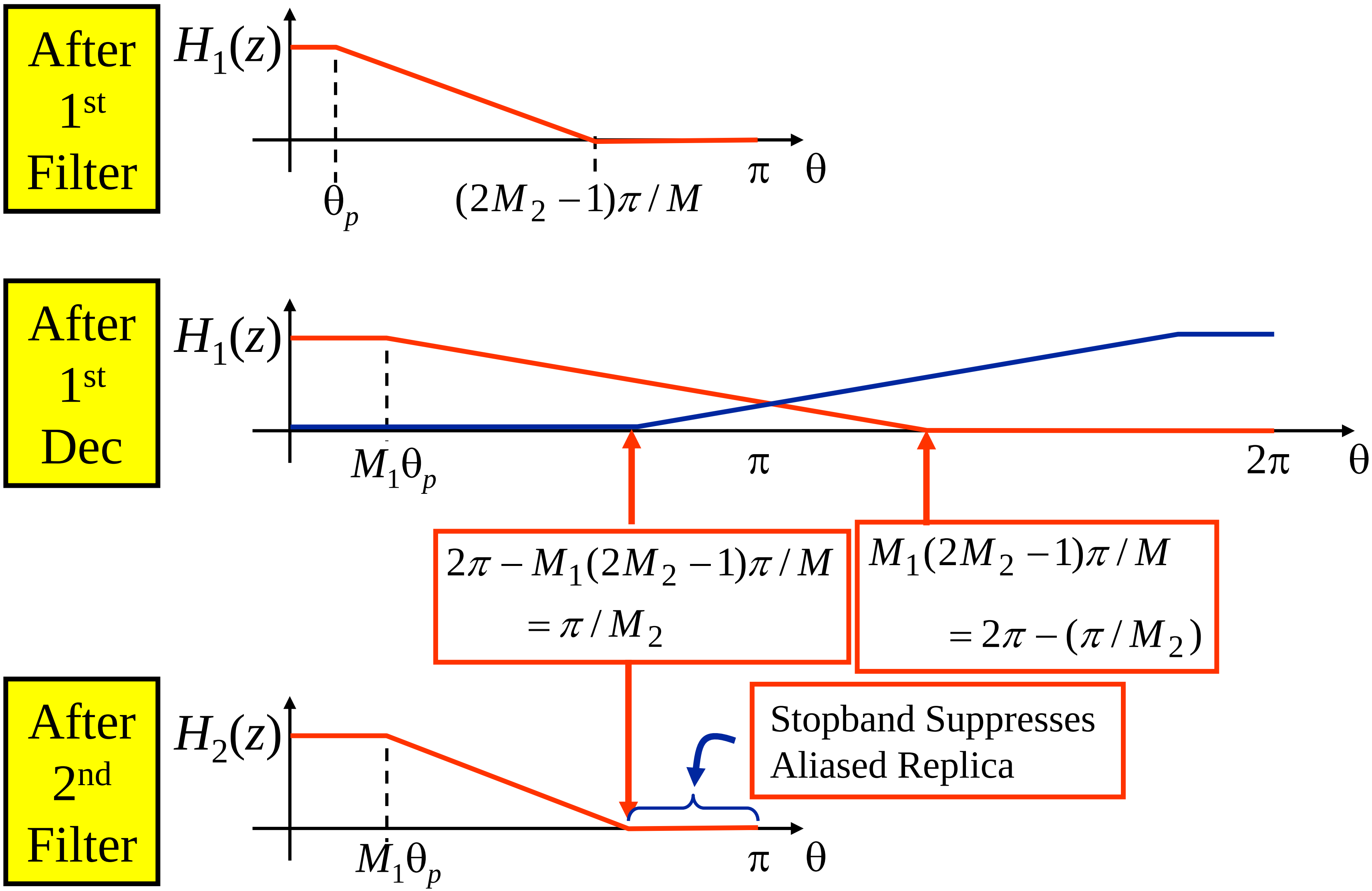


After $H_1(z)$ but before $\downarrow M_1$ we need:



Let’s See Why for a 2-Stage Case (cont.)

Let’s see the impact of this slower transition on aliasing:



Design Requirements

So… say you want to design a 2-stage multirate scheme instead of a 1-stage multirate scheme:

If single stage, say the specs need to be:

- Passband Cutoff = θ_p Passband Ripple = δ_p
- Stopband Cutoff = θ_s Stopband Level = δ_s

For a 2-stage scheme, our above results say we need:

- 1st Stage
  - $\theta_{p,1} = \theta_p$ $\delta_{p,1} = \delta_p/2$
  - $\theta_{s,1} = (2M_2\text{-}1)\pi/M > \theta_s$ $\delta_{s,1} = \delta_s$
- 2nd Stage
  - $\theta_{p,2} = M_1\theta_p$ $\delta_{p,2} = \delta_p/2$
  - $\theta_{s,2} = \pi/M_2$ $\delta_{s,2} = \delta_s$

Passband Ripple is split between 2 filters

Example: How 2-Stage Reduces Computation

Goal: Decimation by $M = 12$

Filter Requirements

$\delta_p = 0.01$ ← to give some desired fidelity (application specific)
$\delta_s = 0.001$ ← to limit aliasing to desired level (application specific)

$\theta_p = \pi/16$ ← to pass desired band (application specific)
$\theta_s = \pi/12 = \pi/M$ ← to prevent aliasing for desired decimation rate

Single-Stage Method

Length of filter determines the # of computations

➔ Use (10.2.94) in P&M to estimate filter order needed:

$$N = \frac{-20\log_{10}\sqrt{\delta_p \delta_s} - 13}{2.32\underbrace{|\theta_s - \theta_p|}_{\text{transition width}}}$$

Example: Single-Stage Method (cont.)

Using this order estimate for the given filter requirements gives:

$N = 244 \;\Rightarrow\;$ Length: $L = N+1 = 245$

Note: #'s given below differ slightly from book because it (wrongly) uses N instead of L in its computation estimates

Our chosen complexity measure: # Multiplies/Input Sample

Each output sample (after decimation): **L Multiplies/Output Sample**

There are ***M* Input Samples/Output Sample** (due to decimation)

$$\Rightarrow \text{(\# Multiplies)/(Input Sample)} = \frac{L \text{ mult/output}}{M \text{ input/output}} = \frac{L}{M} = \frac{245}{12} \approx 20.4$$

Single-Stage Complexity = 20.4 multiplies/input

Ex. Cont.: Double-Stage Method ($M = M_1 M_2$: 12 = 3×4)

1st Stage

- $\theta_{p,1} = \theta_p = \pi/16$ $\delta_{p,1} = \delta_p/2 = 0.005$
- $\theta_{s,1} = (2M_2\text{-}1)\pi/M = 7\pi/12$ $\delta_{s,1} = \delta_s = 0.001$

Estimated filter order gives: ➔ $N_1 = 11$ ➔ $L_1 = 12$

So… Mult/Input = $L_1/M_1 = 12/3 = 4$

2nd Stage

- $\theta_{p,2} = M_1\theta_p = 3\pi/16$ $\delta_{p,2} = \delta_p/2 = 0.005$
- $\theta_{s,2} = \pi/M_2 = \pi/4$ $\delta_{s,2} = \delta_s = 0.001$

Estimated filter order gives: ➔ $N_2 = 88$ ➔ $L_2 = 89$

So… Mult/Input = $L_2/(M_1 M_2) = 89/12 = 7.4$

referenced back to input of whole system

Double-Stage Complexity = 4 + 7.4 = 11.4 multiplies/input

2-Stage has ≈ ½ Complexity of 1-Stage

Comments on Multistage Method

Q: What happens in this example when order of stages is switched?
i.e., $M_1 = 4$ and $M_2 = 3$ (Left as Exercise!!!)

These 2-Stage design ideas can be extended to p-stage designs:

$$M = M_1 M_2 M_3 \ldots M_p$$

The order of these multiple stages matters

Similar ideas can be used for multistage interpolation

Application: Multistage Rate Change

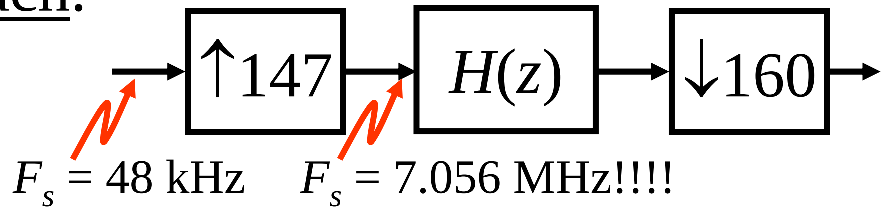
Convert Digital Audio Tape (DAT) format to Compact Disk (CD)

DAT uses $F_s = 48$ kHz
CD uses $F_s = 44.1$ kHz

$$\frac{44.1\times 10^3}{48\times 10^3} = \frac{441}{480} = \frac{3\times 147}{3\times 160} = \frac{147}{160}$$

$$\text{Rate Change Ratio} = \frac{L}{M} = \frac{147}{160}$$

Single-Stage Approach:



Multiple-Stage Approach:

$L = 147 = 7\times 7\times 3$

$M = 160 = 5\times 8\times 4$