Decimation & Expansion
(Frequency Domain View)

Need to look at:
  – Z- Transform
  – DTFT
**M-Fold Decimation – Frequency-Domain**

Notation: \( \{Z_{(\downarrow M)}\}(z) = X^{Z}_{(\downarrow M)}(z) = \{X^{Z}(z)\}_{(\downarrow M)} \)

- Similar for DTFT
- Similar for Expansion

Q: What is \( X_{(\downarrow M)}(z) \) in terms of \( X(z) \)???

What do we expect????!!!!

Lower \( F_s \) causes Spectral Replicas to Move to Lower Frequencies
Should look exactly like sampling at a lower \( F_s \)
Thus… increased aliasing is possible!!!

To answer this we need to define a useful function ("comb" function):

\[
c_M[n] = \sum_{k=-\infty}^{\infty} \delta[n - kM] = \delta[n \text{ mod } M] = \frac{1}{M} \sum_{m=0}^{M-1} W_M \delta[n - m]
\]

\( W_M \triangleq e^{j 2 \pi / M} \)

Call this \( \star \)…
This is like a DT Fourier Series and is easily verified!
**M-Fold Decimation – Frequency-Domain (cont.)**

Now… use the comb function to write decimation:

\[ x_{(\downarrow M)}[n] = x[nM] = x[nM]c_M[nM] \]

Now… take Z-Transform, using this form:

\[
X^z_{(\downarrow M)}(z) = \sum_{n=-\infty}^{\infty} x[nM]c_M[nM]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]c_M[n]z^{-n/M} \\
\cdots + x[0]z^0 + x[M]z^{-1} + x[2M]z^{2} + \cdots \cdots + x[0]z^0 + \cdots + 0 + x[M]z^{-1} + 0 + \cdots
\]

Now… take Z-Transform, using this form:

\[
X^z_{(\downarrow M)}(z) = \sum_{n=-\infty}^{\infty} x[n]\left[ \frac{1}{M} \sum_{m=0}^{M-1} W_M^{mn} \right]z^{-n/M}
\]
\( M \)-Fold Decimation – Frequency-Domain (cont.)

Now… just manipulate:

\[
X_{\downarrow M}^z(z) = \sum_{n=-\infty}^{\infty} x[n] \left[ \frac{1}{M} \sum_{m=0}^{M-1} W_M^{mn} \right] z^{-n/M}
\]

\[
= \frac{1}{M} \sum_{m=0}^{M-1} \left[ \sum_{n=-\infty}^{\infty} x[n] \left( W_M^{-m} z^{1/M} \right)^n \right]
\]

\[
= \frac{1}{M} \sum_{m=0}^{M-1} X^z (W_M^{-m} z^{1/M})
\]

ZT of Decimated Signal is…

\[
X_{\downarrow M}^z(z) = \frac{1}{M} \sum_{m=0}^{M-1} X^z (W_M^{-m} z^{1/M})
\]
**M-Fold Decimation – Frequency-Domain (cont.)**

Now to see a little better what this says… convert ZT to DTFT. 

**Recall:** DTFT is the ZT evaluated on the unit circle:

$$z = e^{j\theta} \Rightarrow z^{1/M} = e^{j\theta/M}$$

Also, by definition: $$W_M^{-m} = e^{-j2\pi m / M}$$

Then we get….

**DTFT of Decimated Signal is…**

$$X^f(\downarrow M)(\theta) = \frac{1}{M} \sum_{m=0}^{M-1} X^f\left(\frac{\theta - 2\pi m}{M}\right)$$

1. Axis-Scale $$X^f(\theta)$$ to get $$X^f(\theta/M)$$ – a stretch
2. Then shift by $$2\pi m$$ to get new replicas
   ➔ Decimation Adds Spectral Replicas of Scaled DTFT

Stretches Spectrum
Example: DTFT for $M$-Fold Decimation

$M = 3$

$X^f(\theta)$

$\pm \pi/3$

$X^f(\theta/3)$

$m = 0$

$X^f((\theta-2\pi)/3)$

$m = 1$

$X^f((\theta-4\pi)/3)$

$m = 2$
Example: Continued

\[ X^f(\theta/3) \]

\[ m = 0 \]

\[ X^f((\theta - 2\pi)/3) \]

\[ m = 1 \]

\[ X^f((\theta - 4\pi)/3) \]

\[ m = 2 \]

\[ M \times \text{DTFT of Decimated Signal} \]

No Aliasing!!!
Example: Insights

1. The $M$-decimated signal will have no aliasing… only if the signal being decimated has: $X_f(\theta) = 0$ for $|\theta| > \pi / M$

   This makes complete sense from an “ordinary” sampling theorem view point!!

2. After $M$-decimating an $M^{th}$-band signal, the spectrum of the decimated signal will fill the $[-\pi, \pi]$ band.

Such a signal is called an “$M^{th}$-Band Signal”
Effect on “Physical” Frequency

Although decimation changes the digital frequency of the signal, the corresponding “physical” frequency is not changed… as the following example shows:

\[
x[n] \quad \downarrow 3 \quad x(\downarrow 3)[n]
\]

- \( F_{s_1} = 60 \text{ MSPS} \)
- \( F_{s_2} = 60/3 = 20 \text{ MSPS} \)

\[
X^r(\theta) \quad \theta
\]

\[
\frac{F_{s_1}}{2} \quad 0 \quad \frac{F_{s_1}}{2}
\]

\[-10 \quad 10 \quad \text{MHz} \quad \text{MHz}\]

\[
X(\downarrow 3)(\theta) \quad \theta
\]

\[
\frac{F_{s_2}}{2} \quad 0 \quad \frac{F_{s_2}}{2}
\]

\[-10 \quad 10 \quad \text{MHz} \quad \text{MHz}\]

Signal Still Occupies **Same** Physical Frequency

Note
Expansion Also Has No Effect on Physical Frequency
**L-Fold Expansion – Frequency-Domain**

Q: What is $X_{(\uparrow L)}(z)$ in terms of $X(z)$???

What do we expect????!!!!

Certainly **NOT** the same as *really* sampling at a higher rate because of the inserted zeros!!!

Frequency Domain analysis answers this!!!

$$X_{(\uparrow L)}(z) = \sum_{n=-\infty}^{\infty} x_{(\uparrow L)}[n]z^{-n}$$

$$= + \cdots + x[0]z^0 + \underbrace{0 + \cdots + 0}_{{L-1} \text{ zeros}} + x[1]z^{-L} + \underbrace{0 + \cdots + 0}_{{L-1} \text{ zeros}} + x[2]z^{-2L}$$

$$= \sum_{n=-\infty}^{\infty} x[n]z^{-L} = X^z(z^L)$$

**ZT of Expanded Signal is…**

$$X_{(\uparrow L)}^z(z) = X^z(z^L)$$
$L$-Fold Expansion – Frequency-Domain (cont.)

Now to see a little better what this says… convert ZT to DTFT. **Recall**: DTFT is the ZT evaluated on the unit circle:

$$z = e^{j\theta} \Rightarrow z^L = e^{jL\theta}$$

**DTFT of Decimated Signal is…**

$$X^f_{(\uparrow L)}(\theta) = X^f(L\theta)$$

Scrunches Spectrum
Example: DTFT for $L$-Fold Expansion

$L = 3$

Expansion Causes Images
to Appear in the $[-\pi, \pi]$ Range

Here’s what we’d have if we **REALLY** sampled 3 times as fast… **No Images**!!!