

12.2 Transform View (Frequency Domain)

Need to look at:

- Z- Transform
- DTFT

M-Fold Decimation – Frequency-Domain

Notation: $\{Zx_{(\downarrow M)}\}(z) = X_{(\downarrow M)}^z(z) = \{X^z(z)\}_{(\downarrow M)}$

- Similar for DTFT
- Similar for Expansion

Q: What is $X_{(\downarrow M)}(z)$ in terms of $X(z)$???

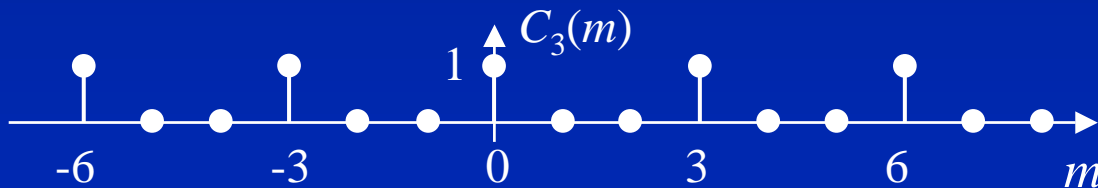
What do we expect????!!!!

Lower F_s causes Spectral Replicas to Move to Lower Frequencies
Should look exactly like sampling at a lower F_s

Thus... increased aliasing is possible!!!

To answer this we need to define a useful function (“comb” function):

$$c_M[n] = \sum_{k=-\infty}^{\infty} \delta[n - kM] = \delta[n \bmod M] = \frac{1}{M} \sum_{m=0}^{M-1} W_M^{mn}$$



See Eq. (4.6)
Call this (★)

M-Fold Decimation – Frequency-Domain (cont.)

Now... use the comb function to write decimation:

$$\begin{aligned} x_{(\downarrow M)}[n] &= x[nM] \\ &= x[nM]c_M[nM] \end{aligned}$$

Doesn't Really Do
Anything Here...
But Later it Will!!

Now... take Z-Transform, using this form:

$$X_{(\downarrow M)}^z(z) = \sum_{n=-\infty}^{\infty} x[nM]c_M[nM]z^{-n} = \sum_{n=-\infty}^{\infty} x[n]c_M[n]z^{-n/M}$$

$\cdots + x[0]z^0 + x[M]z^{-1} + x[2M]z^{-2} + \cdots$

$\cdots + x[0]z^0 + \underbrace{0 + \cdots + 0}_{c_M[n]} + x[M]z^{-1} + 0 + \cdots$

Now... take Z-Transform, using this form:

$$X_{(\downarrow M)}^z(z) = \sum_{n=-\infty}^{\infty} x[n] \left[\frac{1}{M} \sum_{m=0}^{M-1} W_M^{mn} \right] z^{-n/M}$$

Action of $C_M[n]$

M-Fold Decimation – Frequency-Domain (cont.)

Now... just manipulate:

$$\begin{aligned} X_{(\downarrow M)}^z(z) &= \sum_{n=-\infty}^{\infty} x[n] \left[\frac{1}{M} \sum_{m=0}^{M-1} W_M^{mn} \right] z^{-n/M} \\ &= \frac{1}{M} \sum_{m=0}^{M-1} \left[\sum_{n=-\infty}^{\infty} x[n] \left(W_M^{-m} z^{1/M} \right)^{-n} \right] \\ &= \frac{1}{M} \sum_{m=0}^{M-1} X^z \left(W_M^{-m} z^{1/M} \right) \end{aligned}$$

ZT of Decimated Signal is...

$$X_{(\downarrow M)}^z(z) = \frac{1}{M} \sum_{m=0}^{M-1} X^z \left(W_M^{-m} z^{1/M} \right)$$

M-Fold Decimation – Frequency-Domain (cont.)

Now to see a little better what this says... convert ZT to DTFT.

Recall: DTFT is the ZT evaluated on the unit circle:

$$z = e^{j\theta} \Rightarrow z^{1/M} = e^{j\theta/M}$$

Also, by definition: $W_M^{-m} = e^{-j2\pi m/M}$

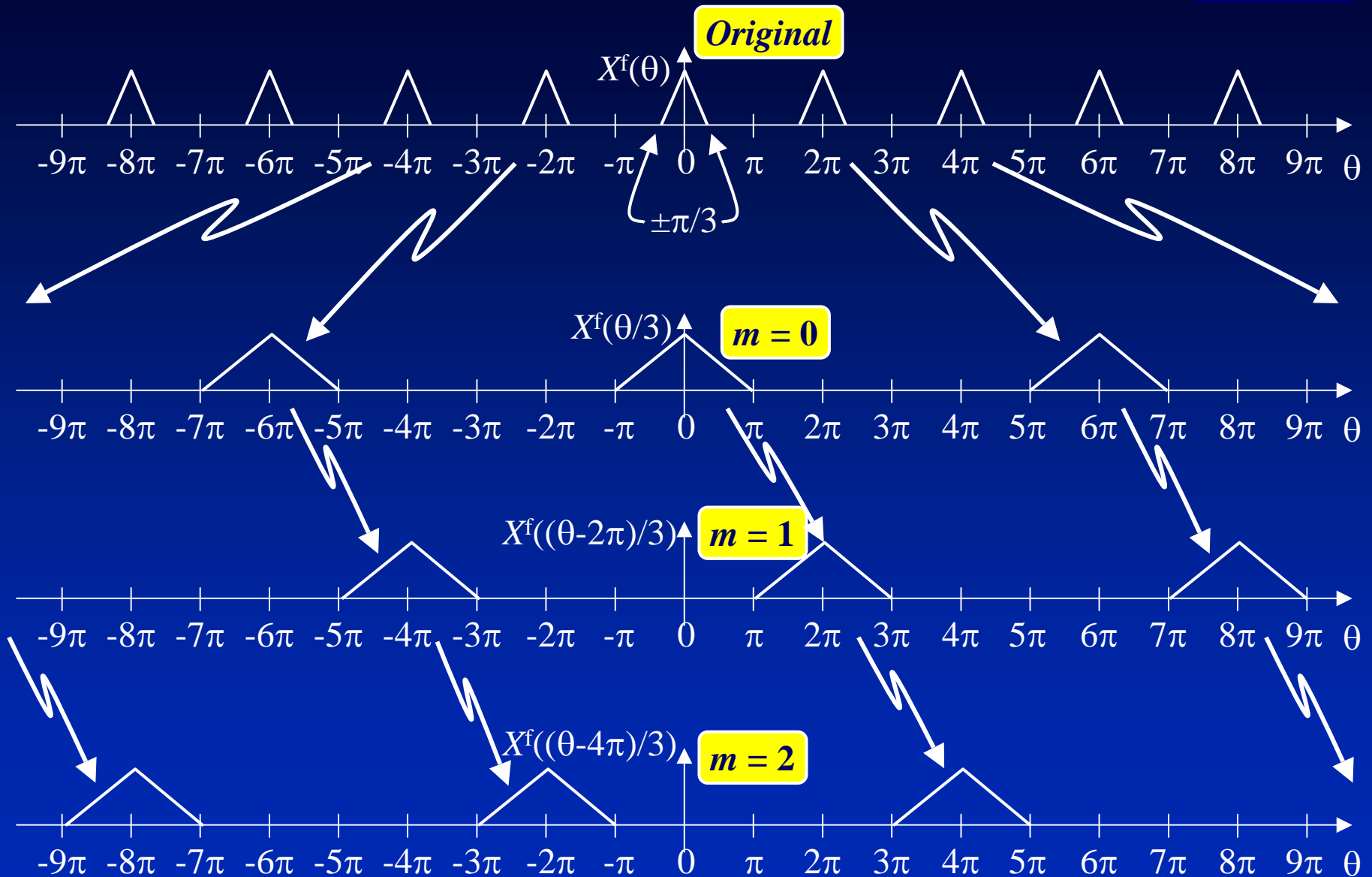
Then we get.... **DTFT of Decimated Signal is...**

$$X_{(\downarrow M)}^f(\theta) = \frac{1}{M} \sum_{m=0}^{M-1} X^f\left(\frac{\theta - 2\pi m}{M}\right)$$

1. Axis-Scale $X^f(\theta)$ to get $X^f(\theta/M)$ – a stretch
 2. Then shift by $2\pi m$ to get new replicas
- Decimation Adds Spectral Replicas of Scaled DTFT

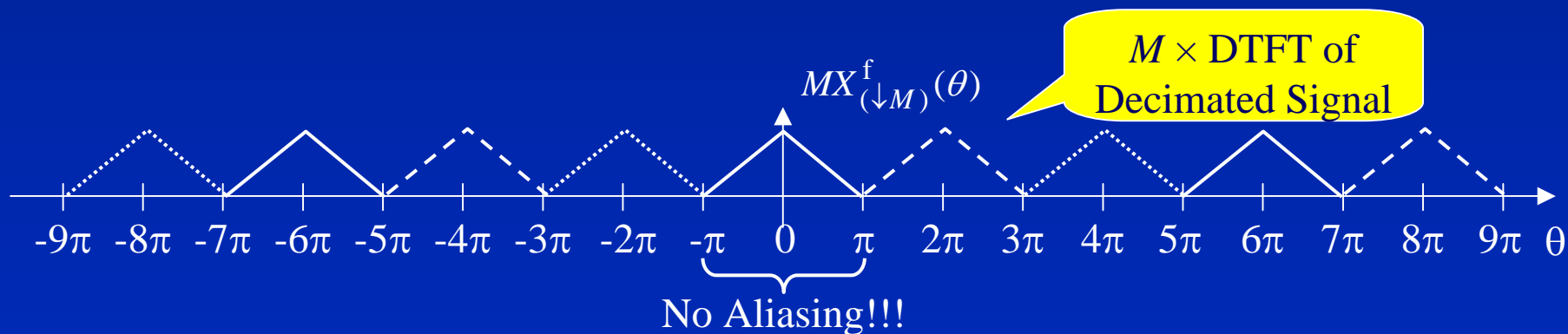
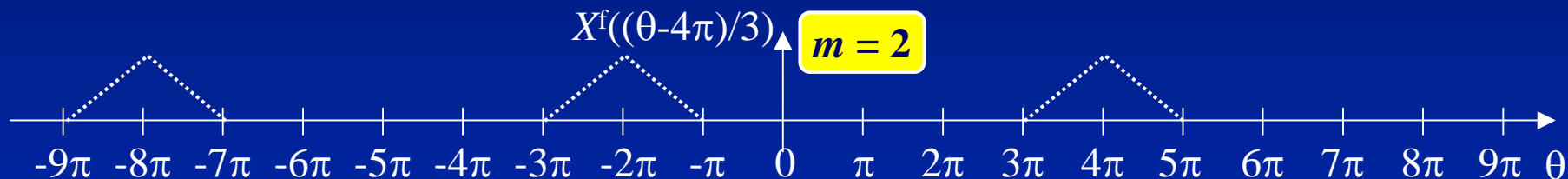
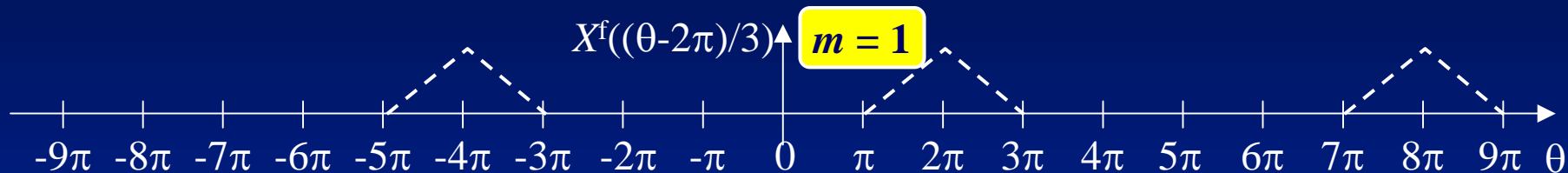
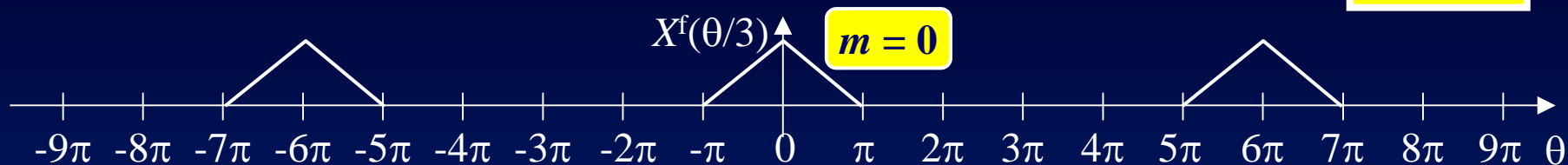
Stretches
Spectrum

Example: DTFT for M -Fold Decimation $M = 3$



Example: Continued

$M = 3$



Example: Insights

1. The M -decimated signal will have no aliasing... only if the signal being decimated has: $X^f(\theta) = 0$ for $|\theta| > \pi/M$

This makes complete sense from an “ordinary” sampling theorem view point!!!

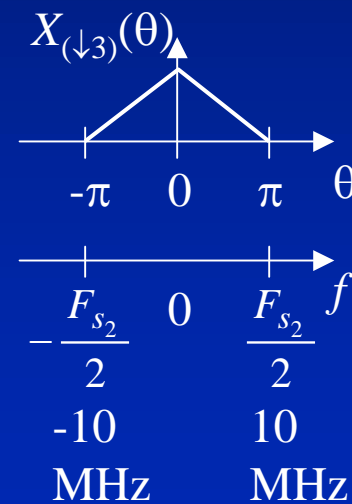
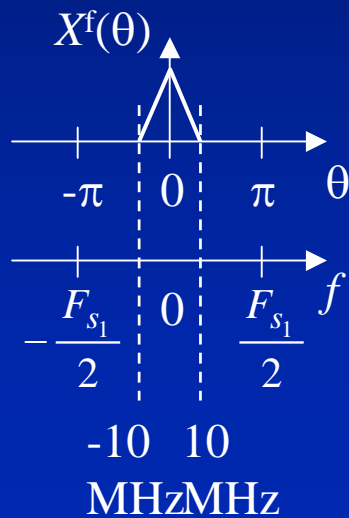
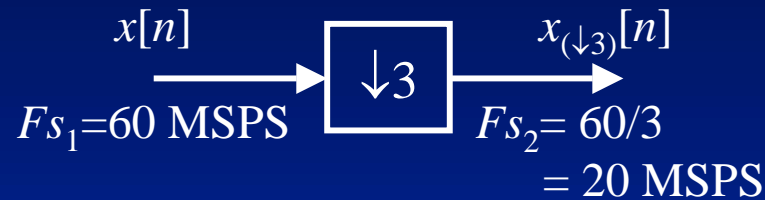


Such a signal is called an “ M^{th} -Band Signal”

2. After M -decimating an M^{th} -band signal, the spectrum of the decimated signal will fill the $[-\pi, \pi]$ band.

Effect on ‘Physical’ Frequency

Although decimation changes the digital frequency of the signal, the corresponding ‘physical’ frequency is not changed... as the following example shows:



Note
Expansion
Also Has
No Effect
on Physical
Frequency

Signal Still Occupies Same Physical Frequency

L-Fold Expansion – Frequency-Domain

Q: What is $X_{(\uparrow L)}(z)$ in terms of $X(z)$???

What do we expect????!!!!

Certainly **NOT** the same as *really* sampling at a higher rate because of the inserted zeros!!!

Frequency Domain analysis answers this!!!

$$\begin{aligned} X_{(\uparrow L)}^z(z) &= \sum_{n=-\infty}^{\infty} x_{(\uparrow L)}[n]z^{-n} \\ &= +\cdots + x[0]z^0 + \underbrace{0 + \cdots + 0}_{L-1 \text{ zeros}} + x[1]z^{-L} + \underbrace{0 + \cdots + 0}_{L-1 \text{ zeros}} + x[2]z^{-2L} \\ &= \sum_{n=-\infty}^{\infty} x[n]z^{-Ln} = X^z(z^L) \end{aligned}$$

ZT of Expanded Signal is...

$$X_{(\uparrow L)}^z(z) = X^z(z^L)$$

L-Fold Expansion – Frequency-Domain (cont.)

Now to see a little better what this says... convert ZT to DTFT.

Recall: DTFT is the ZT evaluated on the unit circle:

$$z = e^{j\theta} \quad \Rightarrow \quad z^L = e^{jL\theta}$$

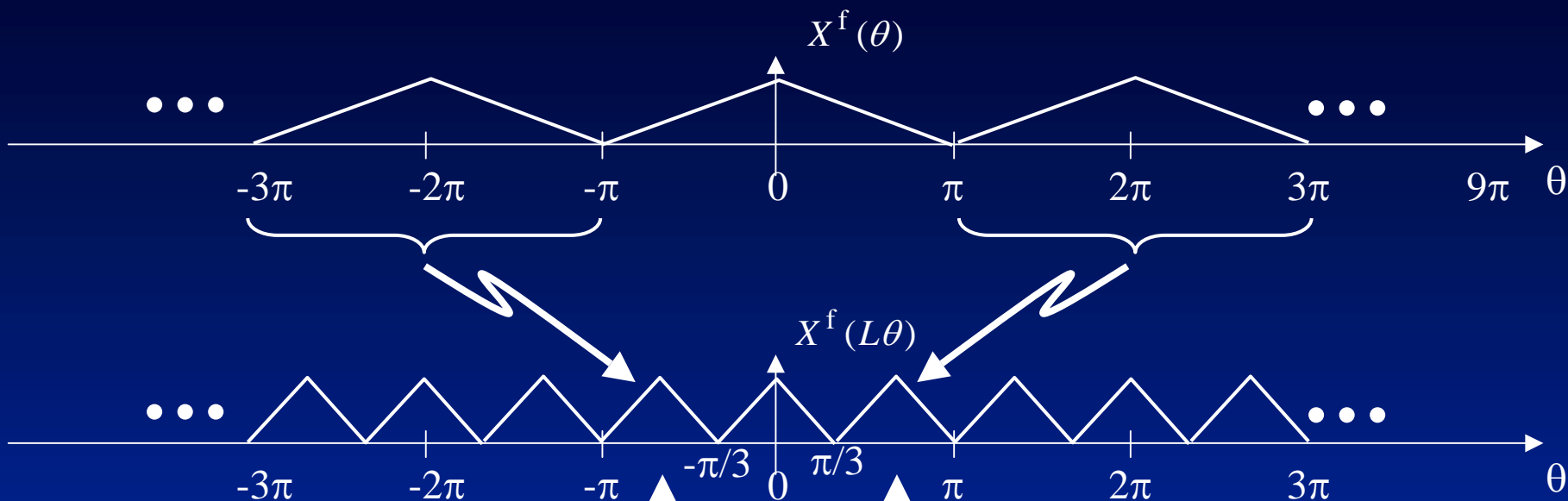
DTFT of Decimated Signal is...

$$X_{(\uparrow L)}^f(\theta) = X^f(L\theta)$$

Scrunches
Spectrum

Example: DTFT for L -Fold Expansion

$L=3$



Expansion Causes Images
to Appear in the $[-\pi, \pi]$ Range

Here's what we'd have if we REALLY sampled
3 times as fast... No Images!!!

