Multirate Digital Signal Processing

Proakis & Manolakis Ch. 11

Multirate DSP

Main Question: How to change the sampling rate of a discrete-time signal without reconstructing and then re-sampling???

Why?

1. Interconnect subsystems having different Fs (e.g., CD to DAT)
2. Improve Certain DSP Operations (e.g., very narrow filters)
3. Efficient Implementation of Certain DSP Tasks (e.g. Correlation)

Two Basic Operations

1. Decimation – Decrease Fs by an integer factor: $Fs_{new} = (Fs_{old})/M$
2. Expansion – Increase Fs by an integer factor: $Fs_{new} = (Fs_{old}) \times L$

Can Combine to Get a Change by a **Rational** Factor: $Fs_{new} = (Fs_{old}) \times L/M$

Our Approach to Study Decimation & Expansion

1. Specify Operations in Time Domain – Easy, but Not Enlightening
2. Determine Impact in Frequency Domain – Harder, but More Enlightening
3. Explore Implementation and Applications (e.g., Polyphase, Filterbanks)

FIR Filter Review

Much of the material in Multirate DSP uses FIR filtering and therefore it is important to have a good grasp before we proceed.

So… let's review some ways of viewing FIR filters.

Let the FIR filter be $h[0]$, $h[1]$, … $h[N]$ and the input be $x[n]$, then the output of the filter is given by convolution in either of <u>two equivalent forms</u>:

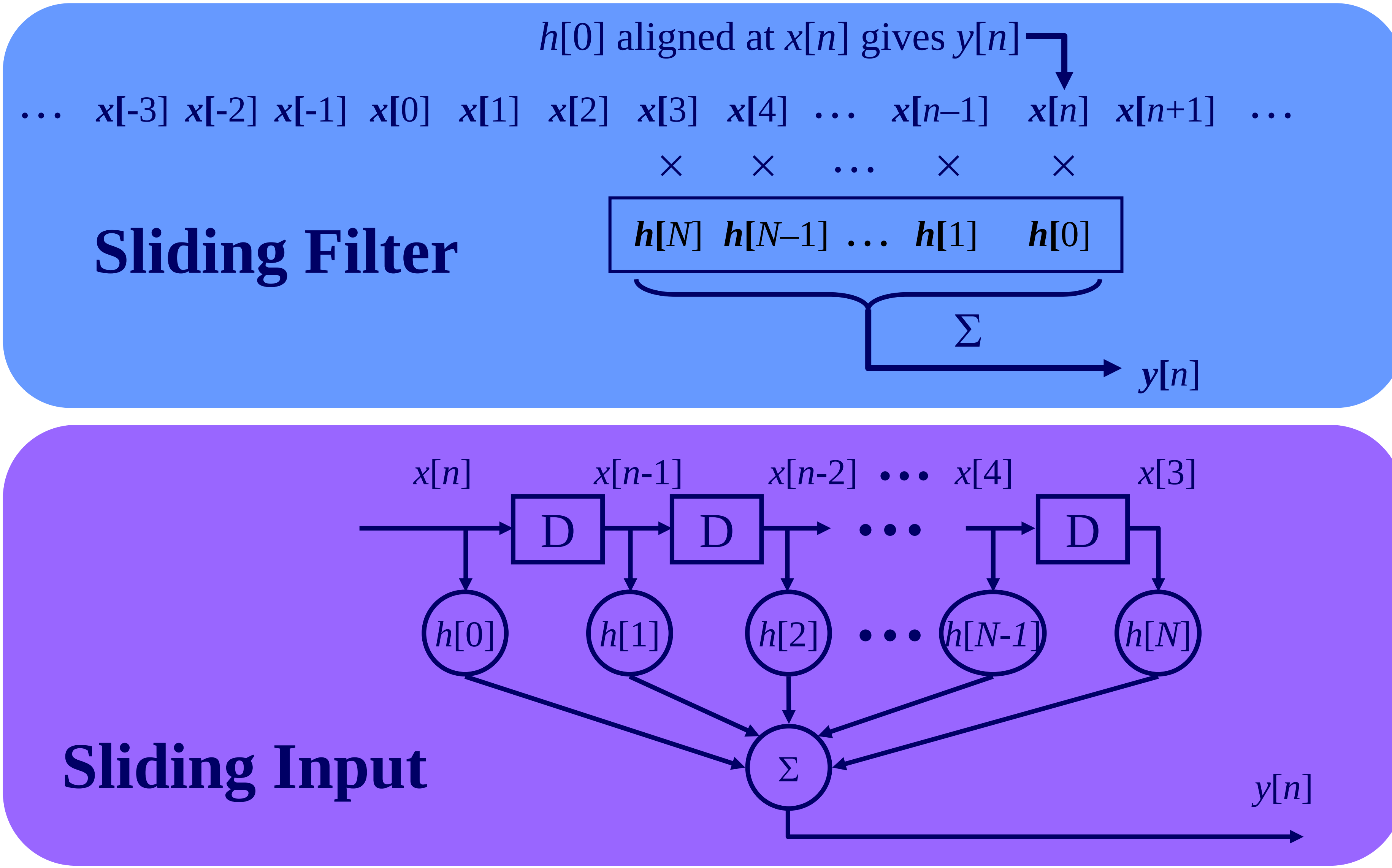
$$y[n] = \sum_{k=0}^{N} h[k]x[n-k]$$

"Input Slides Past Filter"

…or….

$$y[n] = \sum_{k}^{N} x[k]h[n-k]$$

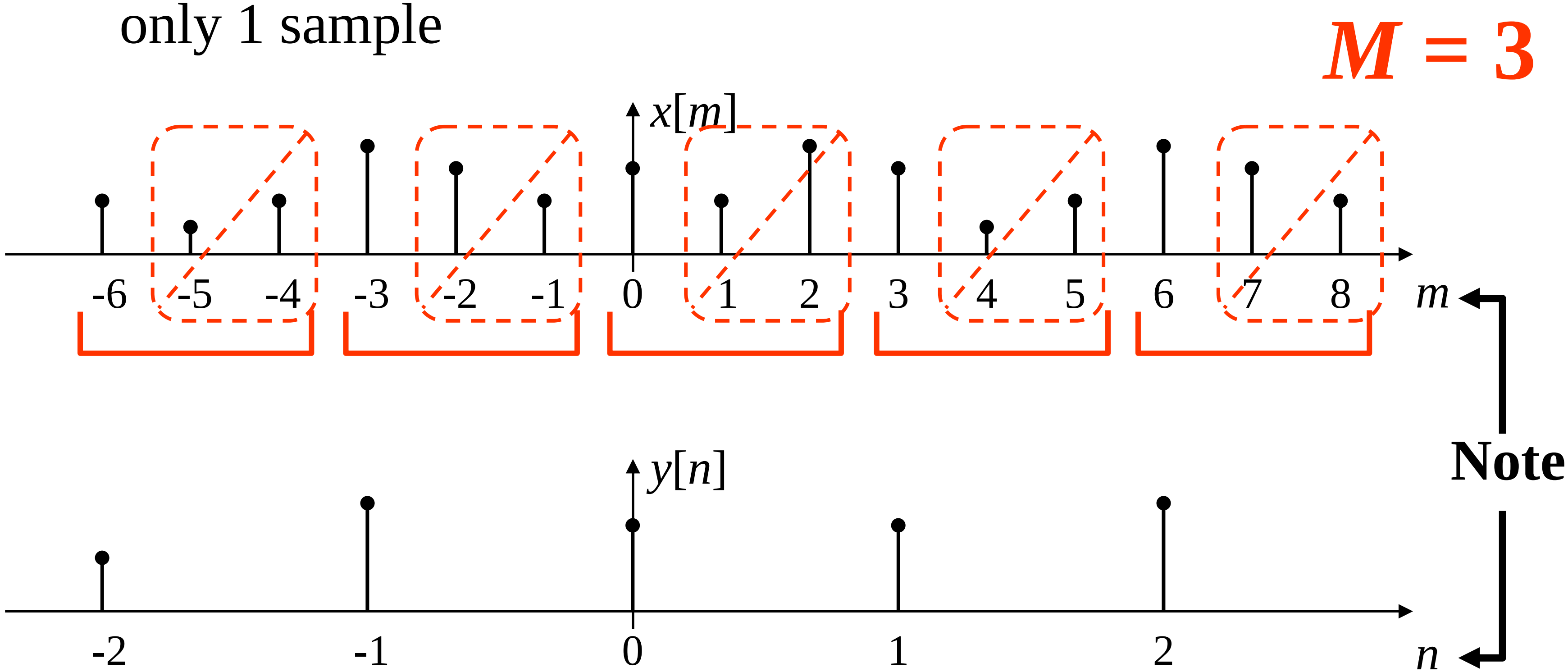
"Filter Slides Past Input"


h[0] aligned at x[n] gives y[n]
. . . x[-3] x[-2] x[-1] x[0] x[1] x[2] x[3] x[4] . . . x[n–1] x[n] x[n+1] . . .
× × . . . × ×
h[N] h[N–1] . . . h[1] h[0]
Σ
y[n]
Sliding Filter
x[n] x[n-1] x[n-2] • • • x[4] x[3]
D D D
h[0] h[1] h[2] • • • h[N-1] h[N]
Σ
y[n]
Sliding Input

Decimation & Expansion (Time Domain View)

Decimation – Time-Domain Graphic View

***M*-Fold Decimation**: Out of every block of M input samples, keep only 1 sample



m is original sample index
n is new sample index

$$m = Mn \quad \text{for} \quad n \in Z$$

Maps Output Index to Input Index Needed

Notation: $x_{(\downarrow M)}[n] = y[n]$

Decimation – Time-Domain Math View

Recall: m is original sample index, n is new sample index $\Big\}$ $m = Mn \quad \text{for} \quad n \in Z$

M-Fold Decimation

$$x_{(\downarrow M)}[n] = x[Mn] \quad \text{for} \quad n \in Z$$

For M=3

$$x_{(\downarrow 3)}[0] = x[0]$$

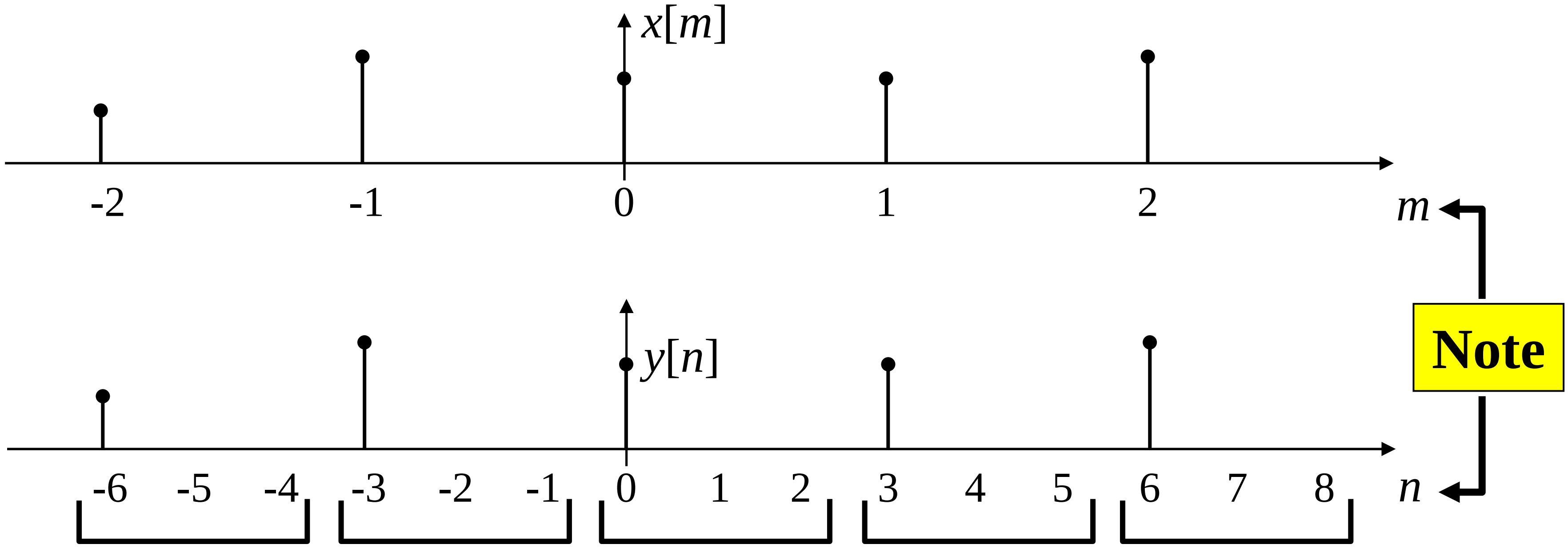
$$x_{(\downarrow 3)}[1] = x[3]$$

$$x_{(\downarrow 3)}[2] = x[6]$$

$$\vdots$$

Expansion – Time-Domain Graphic View

L-Fold Decimation: To each input sample, "tack on" L-1 zeros.



$$n = Lm \quad \text{for} \quad m \in Z \qquad\qquad m = \frac{n}{L} \quad \text{for} \quad n \in Z$$

Maps Output Index to Input Index Needed

Notation: $x_{(\uparrow L)}[n] = y[n]$

Expansion – Time-Domain Math View

Recall: $m = \dfrac{n}{L} \quad \text{for} \quad n \in Z$

Fractional Values for *m*???

L-Fold Expansion

$$x_{(\uparrow L)}[n] = \begin{cases} x[n/L], & \text{if} \quad n/L \in Z \\ 0, & \text{if} \quad n/L \notin Z \end{cases}$$

For L=3

$$x_{(\uparrow 3)}[0] = x[0]$$

$$x_{(\uparrow 3)}[1] = 0$$

$$x_{(\uparrow 3)}[2] = 0$$

$$x_{(\uparrow 3)}[3] = x[1]$$

$$x_{(\uparrow 3)}[4] = 0$$

$$x_{(\uparrow 3)}[5] = 0$$

$$x_{(\uparrow 3)}[6] = x[2]$$

$$\vdots$$

Properties of Rate Change Processing

1. Linear: $\{x_1 + x_2\}_{(\downarrow M)}[n] = x_{1(\downarrow M)}[n] + x_{2(\downarrow M)}[n]$ (similar for $\uparrow L$)
2. Time-Varying System (Not Time-Invariant!!!)

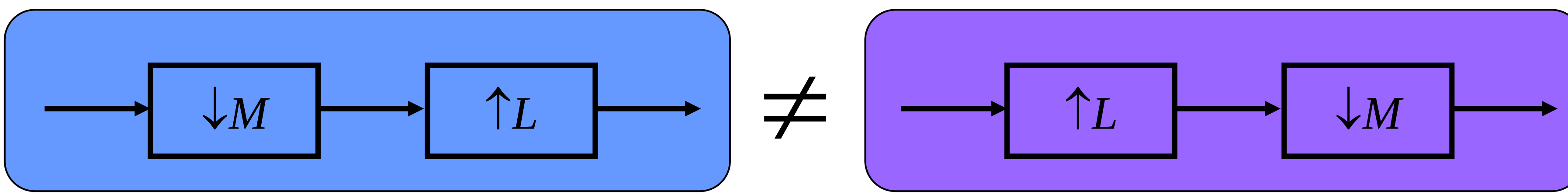
   If $x[n] \rightarrow y[n]$, where either $y[n] = x_{(\downarrow M)}[n]$ or $y[n] = x_{(\uparrow L)}[n]$

   Then in general….

   $x[n-k] \not\rightarrow y[n-k]$ for all k integer

   (To prove this doesn't hold all we need is one example – see p. 464)

   Exercise: For M-fold decimation, $x[n-k] \rightarrow y[n-k]$ holds for certain values of k… find them!!! How about for L-fold expansion???
3. Expansion & Decimation Don't Commute (In General)



$$\underbrace{\left\{x_{(\downarrow M)}\right\}_{(\uparrow L)} \neq \left\{x_{(\uparrow L)}\right\}_{(\downarrow M)}}_{\text{Don't Commute}}$$

Special Case: Commutation Works!!!

Theorem: If M & L are co-prime (also called "relatively prime"), then

$$(\bigstar) \quad \underbrace{\{x_{(\downarrow M)}\}_{(\uparrow L)}[n] = \{x_{(\uparrow L)}\}_{(\downarrow M)}[n]}_{\text{Commute}}$$

Two Integers are **Co-Prime** if they have no common factors.

$\underline{M = 9}$ & $\underline{L = 16}$
(1,3,9) (1,2,4,8)

Proof Approach Write down both sides of (★) using definitions; then see how to make them =

Proof: Write Down Left Side of (★)

First decimate:

$$x_{(\downarrow M)}[n] = x[nM]$$

Then expand it:

$$\{x_{(\downarrow M)}\}_{(\uparrow L)}[n] = \begin{cases} x[nM/L], & nM/L \text{ is integer} \\ 0, & \text{otherwise} \end{cases}$$

Note: nM/L is integer only when nM is divisible by L

Special Case: Commutation Works (cont.)

Write Down Right Side of (★)

First expand:

$$x_{(\uparrow L)}[n] = \begin{cases} x[n/L], & \text{if } n \text{ is divisible by } L \\ 0, & \text{otherwise} \end{cases}$$

Then decimate it:

$$\{x_{(\uparrow L)}\}_{(\downarrow M)}[n] = \begin{cases} x[nM/L], & \text{if } n \text{ is divisible by } L \\ 0, & \text{otherwise} \end{cases}$$

Now… what is needed to make Left = Right???

Left
$x[nM/L]$
nM div. by L

Need these to both be true for all the same values of n and want no values of n that cause only one to be true

Right
$x[nM/L]$
n div. by L

If M & L are **not** co-prime, then there *are* values of n for which $nM/L \in Z$ but $n/L \notin Z$

<End of Proof>