Frequency Measurement in Noise

Porat Section 6.5
Frequency Meas. in Noise Problem

Want to now look at the effect of noise on using the DFT to measure the frequency of a sinusoid.

Consider single complex sinusoid case:

\[ y[n] = Ae^{j\theta_0 n} + v[n], \quad 0 \leq n \leq N - 1 \]

Define: Input Signal-to-Noise Ratio (SNR):

\[
SNR_i = \frac{\text{signal power}}{\text{noise power}} = \frac{A^2}{\sigma_v^2} \quad \text{In dB: } 10\log_{10}\left(\frac{A^2}{\sigma_v^2}\right)
\]

Model for Windowed DTFT of Received Signal:

\[ Y_w^f(\theta) = AW^f(\theta - \theta_0) + V_w^f(\theta) \]
**Impact of Noise**

1. Makes it difficult to “see” the signal peak
   - Need **signal peak** well above the **noise floor**
   - If not…. Might not **detect** presence of signal
2. Noise perturbs the peak location
   - Degrades accuracy of the frequency estimate

So Processing Needs To….

- **First, Detect** the Signal
  - Look for peaks in the DFT
- **Then, Estimate** the Frequency (and amplitude/phase)
  - Same as before

Need to do analysis to determine the performance of these two† processing tasks. ➔ (Use DTFT in analysis rather than DFT)

† We’ll only consider Detection Performance (see Porat’s Book or EE522 for Estimation).
Signal Detection Analysis

Goal: Analyze relationships between peak level in DTFT due to signal and the noise floor height to answer:

Q: What parameters determine how high the signal’s peak is above the noise floor?

DTFT of Windowed Noisy Signal:

\[
Y_w^f(\theta) = DTFT \left\{ w[n](Ae^{j\theta_0n} + v[n]) \right\}
\]

\[
= A \sum_{n=0}^{N-1} w[n]e^{j(\theta_0-\theta)n} + \sum_{n=0}^{N-1} w[n]v[n]e^{-j\theta n}
\]

Signal Part + Noise Part
Signal Detection Analysis (pt. 2)

Signal part peaks at $\theta = \theta_0$, so look there:

$$Y_w^f(\theta_0) = A \sum_{n=0}^{N-1} w[n] + \sum_{n=0}^{N-1} w[n]v[n]e^{-j\theta_0 n}$$

Peak Height = $A$ “Boosted” by $\sum w[n]$

For Rect. Window this “Boost” is:

$$\sum_{n=0}^{N-1} w_R[n] = N$$

Q: What is the boost for other windows? Compare $\sum w[n]$ for other windows to that for the Rect window:

$$CG = \frac{\sum_{n=0}^{N-1} w[n]}{\sum_{n=0}^{N-1} w_R[n]} = \frac{\sum_{n=0}^{N-1} w[n]}{N}$$

Define Coherent Gain of Window

“Boost Lost” due to using a Non-Rect Window

- Note: $CG \leq 1$ (“=“ for Rect. Window)
- CG nearly independent of $N$
Signal Detection Analysis (pt. 3)

Re-write DTFT Peak Using CG:

\[ Y_w^f(\theta_0) = A(N \times CG) + \sum_{n=0}^{N-1} w[n]v[n]e^{-j\theta_0 n} \]

Impact of Signal on Peak Height

Impact of “Processing Length”

Impact of “Window Shape”

“More Rect” \( \Rightarrow \) \( \uparrow \) Peak

\[ \uparrow \text{Length} \Rightarrow \uparrow \text{Peak} \]

\[ \Rightarrow \text{Output Peak} = (\text{Input Amplitude}) \times (N \cdot CG) \]

However, the noise floor also increases…. So we need a way to measure “Improvement”…. “Output SNR”
Signal Detection Analysis (pt. 4)

Output SNR = $SNR_o = \frac{\text{Power of DTFT's Signal Peak}}{\text{DTFT Noise Power at Peak}}$

DTFT Power at Peak:

$$\left| Y_w^f(\theta_0) \right|^2 = Y_w^f(\theta_0) \times \overline{Y_w^f}(\theta_0)$$

$$= (NA \times CG)^2 + 2NA \times CG \times \text{Re} \left\{ \sum_{n=0}^{N-1} w[n]v[n]e^{-j\theta_0n} \right\}$$

$$+ \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} w[n]w[m]v[n]\overline{v}[m]e^{-j\theta_0(n-m)}$$

“FOIL”
The Terms

Signal Part
Noise Part
Signal Detection Analysis (pt. 5)

Now… need to look at the average output power:

Expected Value of 1st noise term is zero because \( E\{v[n]\} = 0 \)

\[
E\left\{ Y^f_w(\theta_0)^2 \right\} = (NA \times CG)^2 + \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} w[n]w[m]E\{v[n]\bar{v}[m]\} e^{-j\theta_0(n-m)} \sigma_v^2 \delta[n-m] \sigma_v^2 \sum_{n=0}^{N-1} w^2[n]
\]

Use Sifting Prop.

Signal Peak’s Power: \( (NA \times CG)^2 \)

Noise Power @ Peak: \( \sigma_v^2 \sum_{n=0}^{N-1} w^2[n] \)
Signal Detection Analysis (pt. 6)

Now... Can write expression for “Output” SNR:

\[
SNR_o = \frac{(NA \times CG)^2}{\sigma_v^2 \sum_{n=0}^{N-1} w^2[n]} = \frac{A^2 (N \times CG)^2}{\sigma_v^2 \sum_{n=0}^{N-1} w^2[n]} = N \times SNR_i \sum_{n=0}^{N-1} \frac{(CG)^2}{w^2[n]} = PG
\]

Now... To “simplify” define “Processing Gain” PG:

\[
PG = \frac{N(CG)^2}{\sum_{n=0}^{N-1} w^2[n]} = \frac{N \left( \frac{1}{N} \sum_{n=0}^{N-1} w[n] \right)^2}{\sum_{n=0}^{N-1} w^2[n]}
\]

\[
SNR_o = N \times PG \times SNR_i
\]

Measures Effect of Signal Environment
Measures Effect of Window Type (i.e., Shape)
Measures Effect of Processing Length (Don’t Count Zero-Pads!!!)
Signal Detection Analysis (pt. 7)

Comments

• Generally Need $SNR_o \geq 14 \text{ dB}$ to ensure reliable detection!

• $PG \leq 1$ (with “=” for Rect Window)

• Coherent Gain (CG) vs. Processing Gain (PG)
  – CG relates Peak Level to Signal Amp: $\text{Peak Level} = N \times CG \times A$
  – PG relates Peak’s SNR to Signal SNR: $SNR_o = N \times PG \times SNR_i$

• CG and PG are usually Specified in dB
  – CG in dB: $10 \log_{10}(CG)^2$
  – PG in dB: $10 \log_{10}PG$

\[\text{Squared!} \quad *\text{Because CG is an Amplitude Gain}*\]

\[\text{Not Squared!} \quad *\text{Because PG is a Power Gain}*\]
Signal Detection Analysis (pt. 8)

Another View of Output \( SNR \)

Recall an earlier equation for output SNR:

\[
SNR_o = \frac{(NA \times CG)^2}{\sigma_v^2 \sum_{n=0}^{N-1} w^2[n]}
\]

Consider (for ease) the Rect Window \( (CG = 1 \text{ and } \sum w^2[n] = N) \) so…

\[
SNR_o = \frac{N^2 A^2}{N \sigma_v^2} = \frac{N^2 \times (\text{Input Signal Power})}{N \times (\text{Input Noise Power})}
\]

Since the Signal is Boosted More Than the Noise, we get a Boost in SNR:

\[
SNR_o = N \times SNR_i \quad \text{(recall : PG = 1 for Rect)}
\]
Signal Detection Analysis (pt. 9)

Yet Another View of Output SNR

Recall this form for the DTFT at the peak:

\[ Y_w^f(\theta)|_{\theta=\theta_0} = \left[ A \sum_{n=0}^{N-1} w[n]e^{j(\theta_0-\theta)n} \right]_{\theta=\theta_0} + \left[ \sum_{n=0}^{N-1} w[n]v[n]e^{-j\theta n} \right]_{\theta=\theta_0} \]

\[ = A \sum_{n=0}^{N-1} w[n]e^{j(0)n} + \sum_{n=0}^{N-1} w[n]v[n]e^{-j\theta_0 n} \]

Signal Terms Add “Coherently” ... Sum Grows Fast

Signal Terms Add “Incoherently” ... Sum Doesn’t Grow As Fast
Signal Detection Analysis (pt. 9)

Impact of Actually Using DFT rather than DTFT
Although we did our analysis using the DTFT, the actual processing is done using the DFT.

Q: What Impact Does This Have?
Recall: DFT is DTFT computed on a grid
→ DTFT Peak May Not Fall On the Grid
Worst Case: Peak Halfway Between Grid Points

Don’t Get the Full CG Here on the Grid!!!
Signal Detection Analysis (pt. 10)

Impact of Actually Using DFT rather than DTFT (cont.)

Leads to Defining “Worst-Case” Gains:

\[
CG = \frac{1}{N} \left| \sum_{n=0}^{N-1} w[n] e^{j0.5(\Delta \theta)n} \right|
\]

\[
PG = \frac{\left| \sum_{n=0}^{N-1} w[n] e^{j0.5(\Delta \theta)n} \right|^2}{N \left| \sum_{n=0}^{N-1} w^2[n] \right|}
\]

Don’t Need to Adjust Denom b/c it accounts for the Noise Effect (which is Flat, not Peaked)

Use Worst-Case Gains: when you need to be conservative in predicting detection performance!!