Frequency Measurement

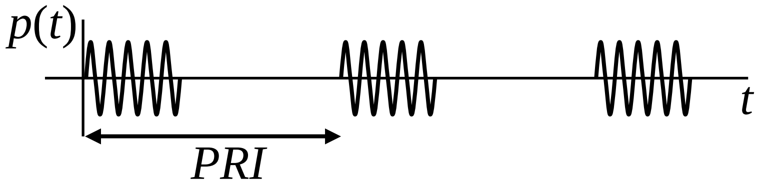
Porat Section 6.4

Frequency Measurement Problem

In many applications it is necessary to estimate the frequencies of narrow spikes in the DTFT of a signal. Often these spikes are due to sinusoids in the underlying infinite-duration signal.

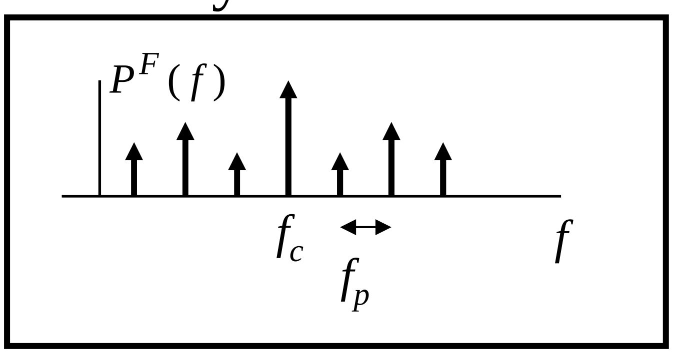
Examples:

1. Carrier Frequency of LPE Signal for Radio/Radar
2. Radar Pulse Train (Recall Electronic Warfare Example)

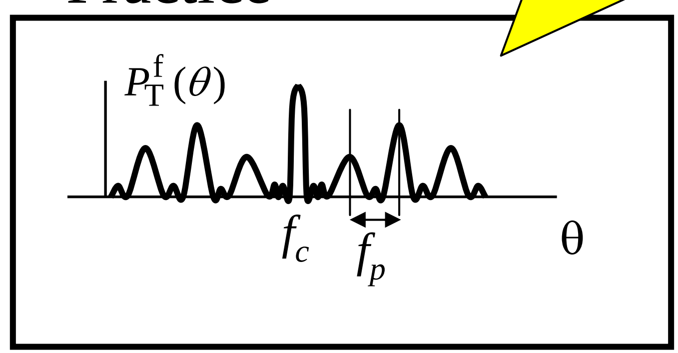


<u>Wish to Ensure Seeing</u>:
- Closely Spaced Peaks
- Weak Spikes Despite Strong Spikes

Theory



Practice

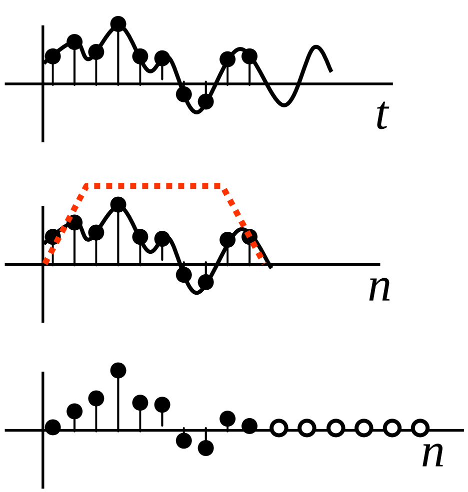


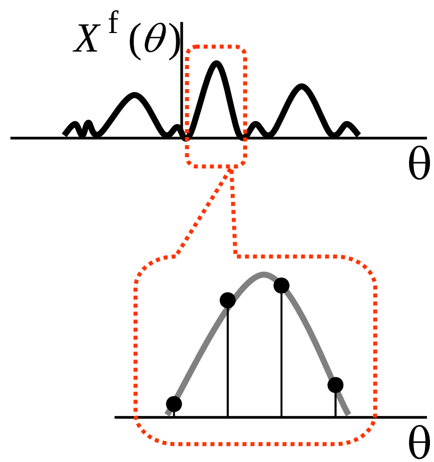
Frequency Measurement Processing

Note: Window First, Then Zero-Pad

1. Collect N Samples of the Signal $y[n]$ for $0 \le n \le N-1$
2. Apply Window: $y_w[n] = y[n]w[n]$ for $0 \le n \le N-1$
   a. Usually use Non-Rectangular Window
3. Zero-Pad to $M > N$
   a. Done to get Needed Frequency Bin Spacing
   b. Also to get Power-of-Two to Enable FFT Use
   c. Rule of Thumb: $M = 2^p \approx 4N$ (to get enough pts on a peak)
4. Compute M-pt. DFT using the FFT Algorithm
5. Find Peaks and Measure Frequency Location
   a. Optional (but a good idea): Quadratic Fit & Interpolate

LS Gives Fit Coefficients: a_0, a_1, a_2 ➔ $\theta_{est} = -a_1 / 2a_2$





Least-Squares Fit

$$\hat{X}^f(\theta) = a_2\theta^2 + a_1\theta + a_0$$

To Find the Peak, Set:

$$\frac{d}{d\theta}\hat{X}^f(\theta) = 0 \quad \Rightarrow \quad a_2\theta + a_1 = 0$$

$$\boxed{\Rightarrow \quad \theta_{est} = \frac{-a_1}{2a_2}}$$

Frequency Measurement Analysis

Use DTFT for analysis purposes

Consider the Two Complex Sinusoid Problem (Gives General Case Insight)

$$y[n] = A_1 e^{j(\theta_1 n + \phi_1)} + A_2 e^{j(\theta_2 n + \phi_2)}, \quad \text{for } n \in Z$$

DTFT of the infinite duration signal is:

$$Y^{\mathrm{f}}(\theta) = 2\pi A_1 \delta(\theta - \theta_1) e^{j\phi_1} + 2\pi A_2 \delta(\theta - \theta_2) e^{j\phi_2}, \quad \theta \in [-\pi, \pi] \text{ Repeats Elsewhere}$$

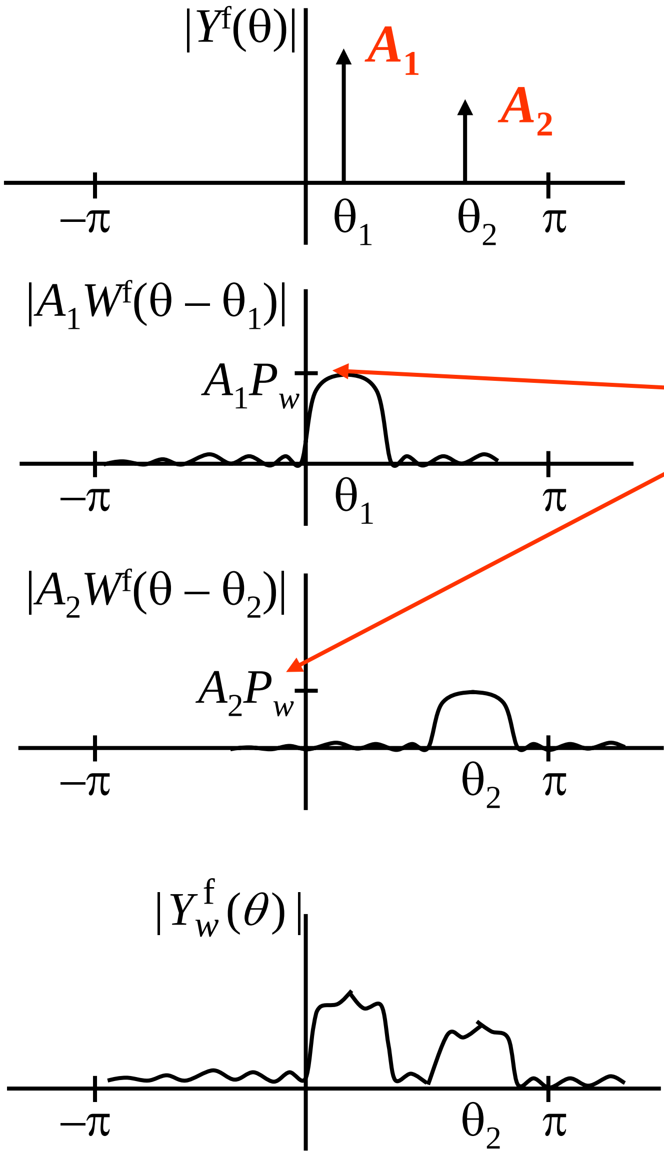
But… what the finite-duration data will show is only the DTFT of the windowed signal $y_w[n]$:

$$Y_w^{\mathrm{f}}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y^{\mathrm{f}}(\xi) W^{\mathrm{f}}(\theta - \xi) d\xi$$

$$= \int_{-\pi}^{\pi} \left[A_1 \delta(\theta - \theta_1) e^{j\phi_1} + A_2 \delta(\theta - \theta_2) e^{j\phi_2} \right] W^{\mathrm{f}}(\theta - \xi) d\xi$$

$$= A_1 e^{j\phi_1} W^{\mathrm{f}}(\theta - \theta_1) + A_2 e^{j\phi_2} W^{\mathrm{f}}(\theta - \theta_2)$$

Frequency Measurement Analysis (pt. 2)



$$P_w = W^{f}(0) = ????$$

$$P_w = W^{f}(\theta)\,|_{\theta=0} = \left[\sum_{n=0}^{N-1} w[n] e^{j\theta n}\right]_{\theta=0}$$

$$P_w = \sum_{n=0}^{N-1} w[n]$$

= N for Rectangle
< N for others

- Sidelobes Interfere with Peaks
- Mainlobes can also Interfere when θ_1 & θ_2 are closely spaced

Frequency Measurement Analysis (pt. 3)

Plotting Comments

Usually Plot the Spectra in dB & Sometimes Normalized to 0 dB Max

$$\underbrace{20\log_{10}\left|Y_w^{\mathrm{f}}(\theta)\right|}_{\text{magnitude form}} = \underbrace{10\log_{10}\left|Y_w^{\mathrm{f}}(\theta)\right|^2}_{\text{power form}}$$

$$20\log_{10}\left[\frac{\left|Y_w^{\mathrm{f}}(\theta)\right|}{P_y}\right], \quad \text{where } P_y = \max\left\{\left|Y_w^{\mathrm{f}}(\theta)\right|\right\}$$

Ignoring ML/SL Interference for Now

1. $|Y_w(\theta)|$ has Peak Values of $A_1 P_w$ & $A_2 P_w$
2. Ratio of peaks is A_1/A_2, which in dB is

$$20\log_{10}\left[A_1 / A_2\right] = 10\log_{10}\left[A_1^2 / A_2^2\right]$$

(Recall: Power of $Ae^{j\theta n}$ is A^2)

➔ Peak Ratio in dB = Power Ratio in dB

3. Location of Peaks are at $\theta = \theta_1$ & $\theta = \theta_2$
4. Sinusoid Phases are computed as $\phi_1 = \angle Y_w(\theta_1)$ & $\phi_2 = \angle Y_w(\theta_2)$

Computed Spectrum Provides Simple Way to Measure:

- **Signal Powers and Power Ratios**
- **Signal Frequencies**
- **Signal Phases**

IF … ML/SL Interference can be made negligible

Frequency Measurement Analysis (pt. 4)

Making ML/SL Interference Negligible

- You can't always do this!!!
- Choose a Window with:
  - ML Width Narrower than expected $|\theta_2 - \theta_1|$
  - SL level in dB relative to ML level exceeds maximum expected power separation of sinusoids

Impact of Non-Negligible ML/SL Interference

- Severe Case: Can't see Weak Signal's Peak
- Moderate Case: Perturbs Peak Location, Height, Phase
  Causes Inaccurate Measurements!!!