Frequency Measurement

Porat Section 6.4
Frequency Measurement Problem

In many applications it is necessary to estimate the frequencies of narrow spikes in the DTFT of a signal. Often these spikes are due to sinusoids in the underlying infinite-duration signal.

Examples:

1. Carrier Frequency of LPE Signal for Radio/Radar
2. Radar Pulse Train (Recall Electronic Warfare Example)

Wish to Ensure Seeing:
- Closely Spaced Peaks
- Weak Spikes Despite Strong Spikes
**Frequency Measurement Processing**

1. Collect \( N \) Samples of the Signal \( y[n] \) for \( 0 \leq n \leq N - 1 \)
2. Apply Window: \( y_w[n] = y[n]w[n] \) for \( 0 \leq n \leq N - 1 \)
   a. Usually use Non-Rectangular Window
3. Zero-Pad to \( M \gg N \)
   a. Done to get Needed Frequency Bin Spacing
   b. Also to get Power-of-Two to Enable FFT Use
   c. Rule of Thumb: \( M = 2^p \approx 4N \) (to get enough pts on a peak)
4. Compute \( M \)-pt. DFT using the FFT Algorithm
5. Find Peaks and Measure Frequency Location
   a. Optional (but a good idea): Quadratic Fit & Interpolate

   LS Gives Fit Coefficients: \( a_0, a_1, a_2 \) \( \Rightarrow \theta_{est} = -a_1/2a_2 \)

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\( \hat{X}^f (\theta) = a_2 \theta^2 + a_1 \theta + a_0 \)

To Find the Peak, Set:

\[
\frac{d}{d\theta} \hat{X}^f (\theta) = 0 \quad \Rightarrow \quad a_2 \theta + a_1 = 0
\]

\[
\Rightarrow \quad \theta_{est} = \frac{-a_1}{2a_2}
\]
Frequency Measurement Analysis

Use DTFT for analysis purposes

Consider the Two Complex Sinusoid Problem (Gives General Case Insight)

\[ y[n] = A_1 e^{j(\theta_1 n + \phi_1)} + A_2 e^{j(\theta_2 n + \phi_2)}, \quad \text{for } n \in \mathbb{Z} \]

DTFT of the infinite duration signal is:

\[ Y^f(\theta) = 2\pi A_1 \delta(\theta - \theta_1) e^{j\phi_1} + 2\pi A_2 \delta(\theta - \theta_2) e^{j\phi_2}, \quad \theta \in [-\pi, \pi] \] Repeats Elsewhere

But… what the finite-duration data will show is only the DTFT of the windowed signal \( y_w[n] \):

\[
\begin{align*}
Y_w^f(\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} Y^f(\xi) W^f(\theta - \xi) d\xi \\
&= \int_{-\pi}^{\pi} \left[ A_1 \delta(\theta - \theta_1) e^{j\phi_1} + A_2 \delta(\theta - \theta_2) e^{j\phi_2} \right] W^f(\theta - \xi) d\xi \\
&= A_1 e^{j\phi_1} W^f(\theta - \theta_1) + A_2 e^{j\phi_2} W^f(\theta - \theta_2)
\end{align*}
\]
Frequency Measurement Analysis (pt. 2)

\[ |Y^f(\theta)| \]

\[ A_1 \]

\[ A_2 \]

\[ |A_1 W^f(\theta - \theta_1)| \]

\[ A_1 P_w \]

\[ |A_2 W^f(\theta - \theta_2)| \]

\[ A_2 P_w \]

\[ P_w = W^f (0) = ??? \]

\[ P_w = W^f (\theta)|_{\theta=0} = \left[ \sum_{n=0}^{N-1} w[n] e^{j\theta n} \right]_{\theta=0} \]

\[ P_w = \sum_{n=0}^{N-1} w[n] \]

\[ = N \text{ for Rectangle} \]

\[ < N \text{ for others} \]

- Sidelobes Interfere with Peaks
- Mainlobes can also Interfere when \( \theta_1 & \theta_2 \) are closely spaced
Frequency Measurement Analysis (pt. 3)

Plotting Comments
Usually Plot the Spectra in dB & Sometimes Normalized to 0 dB Max

$$20 \log_{10} \left| Y_w^f (\theta) \right| = 10 \log_{10} \left| Y_w^f (\theta) \right|^2$$

plots in magnitude form

$$20 \log_{10} \left[ \frac{Y_w^f (\theta)}{P_y} \right]$$, where \( P_y = \max \left\{ Y_w^f (\theta) \right\} $$

plots in power form

Ignoring ML/SL Interference for Now
1. \( |Y_w(\theta)| \) has Peak Values of \( A_1 P_w \) & \( A_2 P_w \)
2. Ratio of peaks is \( A_1 / A_2 \), which in dB is

$$20 \log_{10} \left[ \frac{A_1}{A_2} \right] = 10 \log_{10} \left[ \frac{A_1^2}{A_2^2} \right]$$

(Recall: Power of \( Ae^{i\theta}n \) is \( A^2 \))

\( \Rightarrow \) Peak Ratio in dB = Power Ratio in dB
3. Location of Peaks are at \( \theta = \theta_1 \) & \( \theta = \theta_2 \)
4. Sinusoid Phases are computed as \( \phi_1 = \angle Y_w (\theta_1) \) & \( \phi_2 = \angle Y_w (\theta_2) \)

Computed Spectrum Provides Simple Way to Measure:
- Signal Powers and Power Ratios
- Signal Frequencies
- Signal Phases

IF … ML/SL Interference can be made negligible
Frequency Measurement Analysis (pt. 4)

Making ML/SL Interference Negligible

- You can’t always do this!!!
- Choose a Window with:
  - ML Width Narrower than expected $|\theta_2 - \theta_1|$  
  - SL level in dB relative to ML level exceeds maximum expected power separation of sinusoids

Impact of Non-Negligible ML/SL Interference

- Severe Case: Can’t see Weak Signal’s Peak
- Moderate Case: Perturbs Peak Location, Height, Phase
  Causes Inaccurate Measurements!!!