DFT-Based FIR Filtering

See Porat's Book: 4.7, 5.6

<u>Motivation: DTFT View of Filtering</u>

There are <u>two views of filtering</u>:

* Time Domain
* Frequency Domain

$$\boxed{\begin{matrix} x[n] \\ X^{\mathrm{f}}(\theta) \end{matrix}} \longrightarrow \boxed{\begin{matrix} h[n] \\ H^{\mathrm{f}}(\theta) \end{matrix}} \longrightarrow \boxed{\begin{matrix} y[n] = h[n] * x[n] \\ Y^{\mathrm{f}}(\theta) = H^{\mathrm{f}}(\theta)\, X^{\mathrm{f}}(\theta) \end{matrix}}$$

The FD viewpoint is indispensable for <u>analysis and design</u> of filters:

* Passband, Stopband, etc. of $|H^{\mathrm{f}}(\theta)|$
* Linearity of Phase $\angle H^{\mathrm{f}}(\theta)$, etc.

Q: What about using <u>DTFT for implementation</u>?

* Compute DTFT of input signal and filter
* Multiply the two and take inverse DTFT

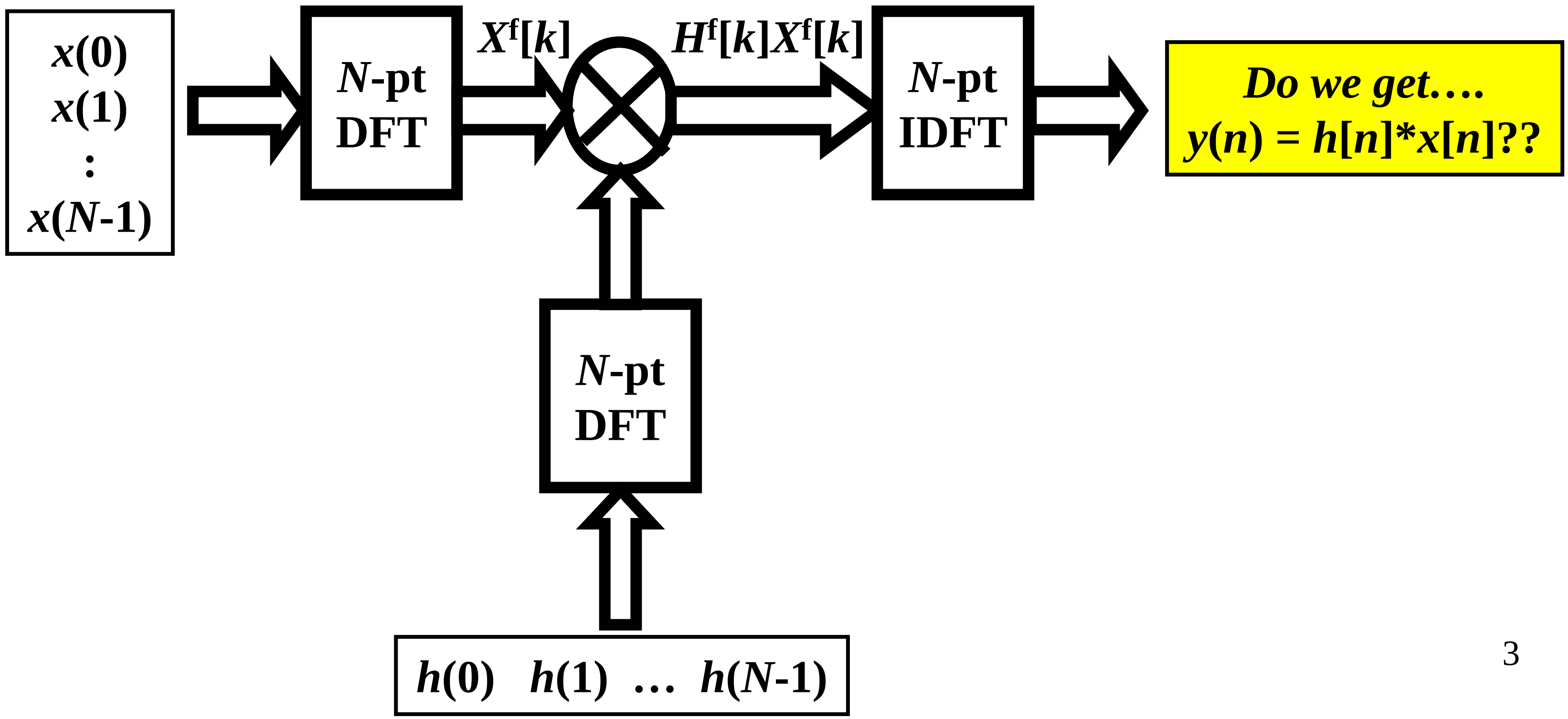
A: <u>NO!!!</u> Can’t compute DTFT – must compute at <u>infinite</u> many frequency values

Desired Intention: But Does It Work?

But wait….

- If input signal is finite length, the DFT computes "samples of the DTFT"
- Likewise, if filter impulse response is finite length

Q: So… can we use this?



DFT Theory and Cyclic Convolution

A: Not Necessarily!!!!

DFT Theory (Theorem. 4.3 in Porat) tells us:

$$\boxed{\textbf{IDFT}\{H^{\mathrm{f}}[k]X^{\mathrm{f}}[k]\} = h[n] \circledast x[n]}$$

Circular (Cyclic) Convolution

Thus… this block diagram gives something called cyclic convolution, not the “ordinary” convolution we <u>want</u>!!!

Q: When <u>*does*</u> it work???

A: Only when we <u>“trick” the DFT Theory</u> into making

<u>*circular*</u> = <u>*linear*</u> convolution!!!!!!

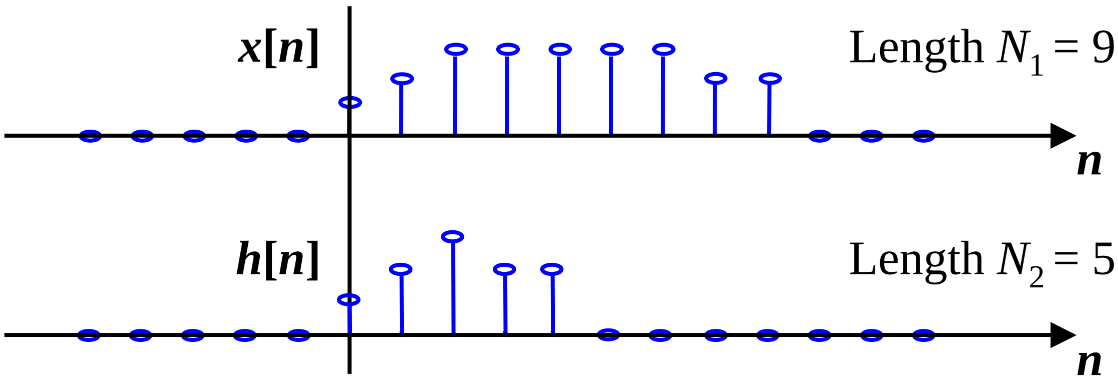
Q: So… when <u>does</u> Cyclic = Linear Convolution???

Easiest to see from an example!!!!!

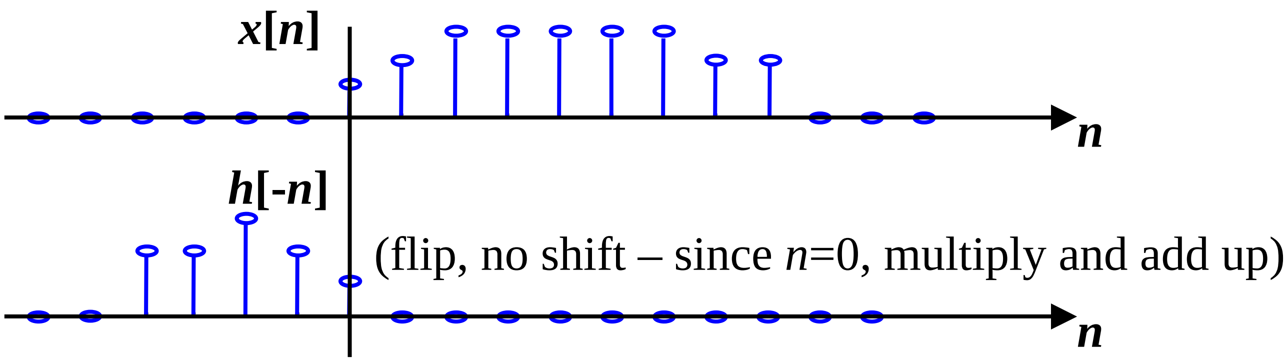
Linear Convolution for the Example

What does ***linear convolution*** give for 2 finite duration signals:

Original Signals:

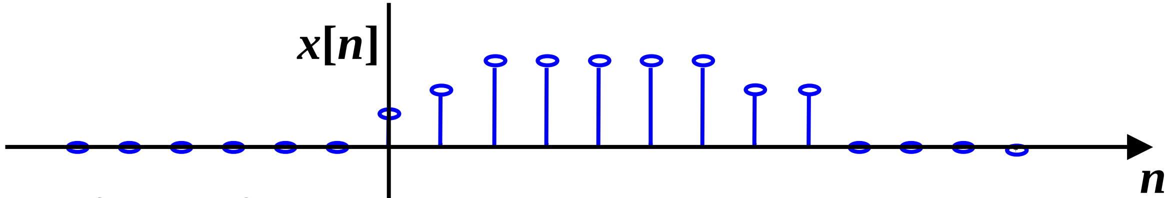


First Non-Zero Output is at n=0:

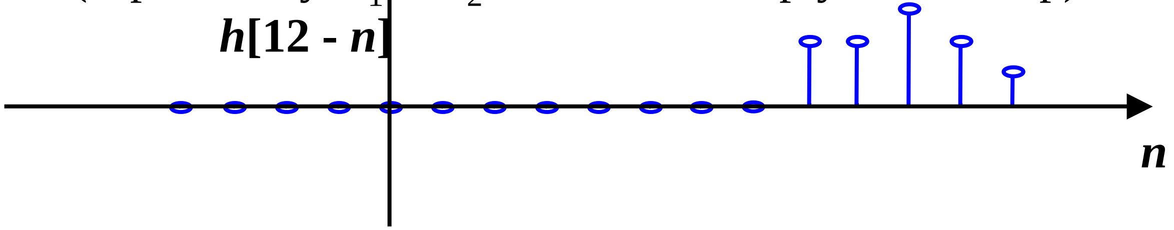


Linear Convolution for the Example (cont.)

Last Non-Zero Output is at $n = N_1 + N_2 - 2 = 12$:



(flip, shift by $N_1 + N_2 - 2 = 12$, multiply and add up)



The non-zero outputs are for $n = 0, 1, \ldots, 12$ ➔ 13 of them

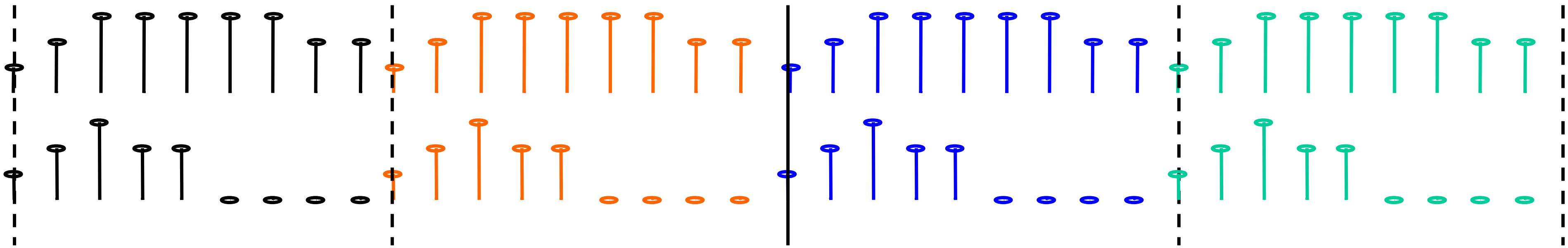
In General: Length of Output of Linear Convolution = $N_1 + N_2 - 1$

Cyclic Convolution for the Example

Now… What does ***cyclic* convolution** give for these 2 signals:

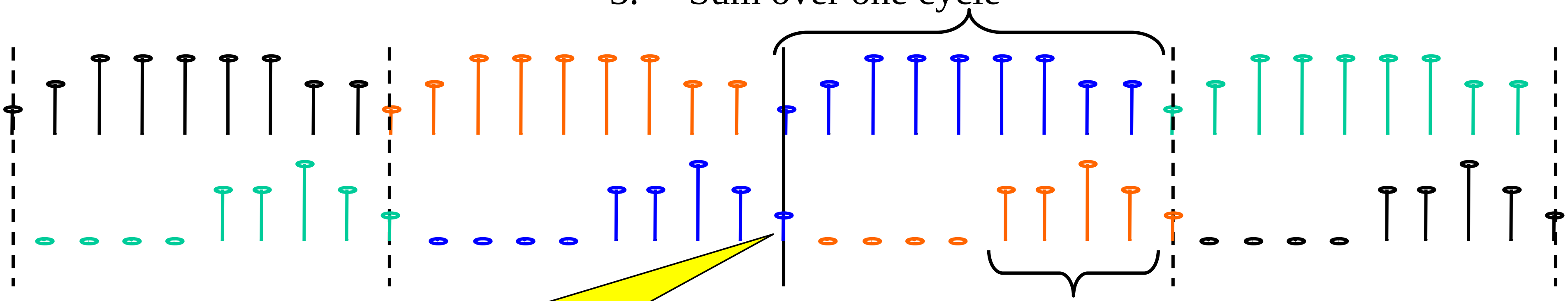
“Original” Signals:

1. Zero-Pad Shorter Signal to Length of Longer One
2. Then Periodize Each



“First” Output Sample:

1. Flip periodized version around this point
2. No shift needed to get n = 0 Output Value
3. Sum over one cycle



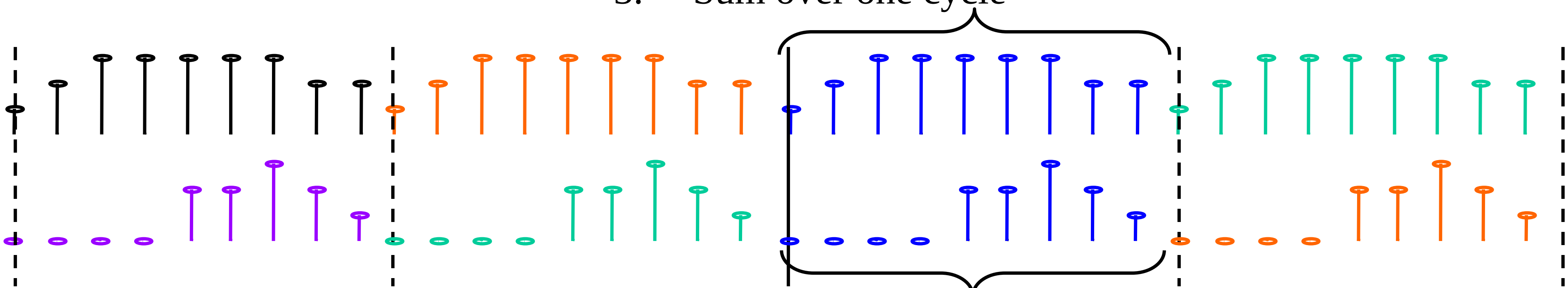
Same as in Linear Conv!!!!

Not Present in Linear Conv!!!!

Linear Convolution for the Example (cont.)

"Last" Output Sample (i.e., n = 8):

1. Flip periodized version around this point
2. Shift by 8 to get n = 8 Output Value
3. Sum over one cycle



Same as in Linear Conv!!!!

Note: If I try to compute the output for n = 9 ➔ Exactly the same case as for n = 0!!
Thus, the output is cyclic (i.e., periodic) with unique values for n = 0, 1, …. 8

In General: Length of Output of Cyclic Convolution = max{N_1 , N_2}

Making Cyclic = Linear Convolution

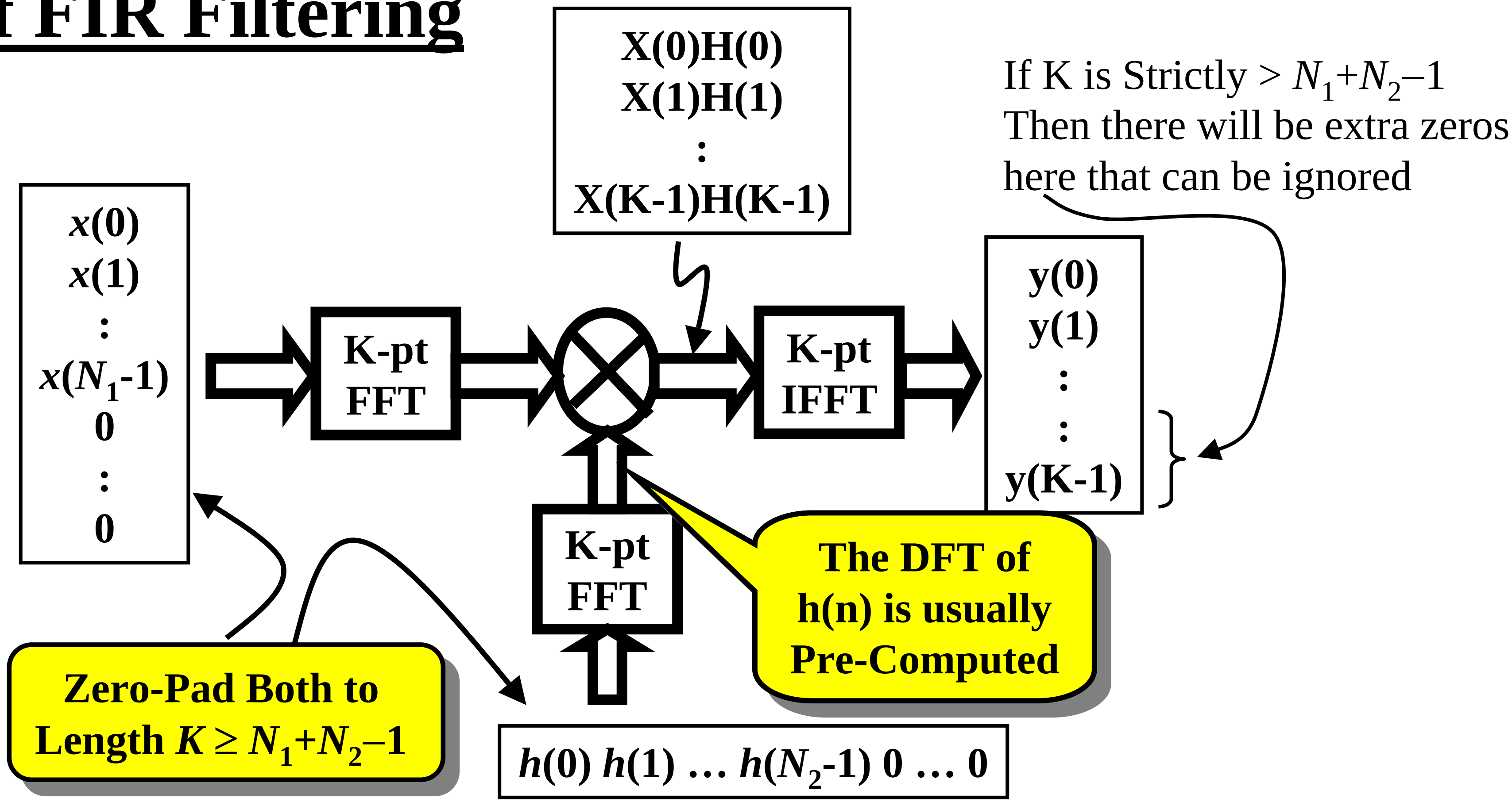
So…. Some of the output values of cyclic conv are <u>different</u> from linear conv!!!
Some of the output values of cyclic conv are <u>same as</u> linear conv
And….
The length of cyclic conv differs from the length of linear conv!!!

From the example above we can verify:
If we choose $K \geq N_1 + N_2 - 1$
And Zero-Pad Each Signal to Length L
Then <u>Cyclic = Linear Convolution</u>

N_1 **= Length of** $x[n]$
N_2 **= Length of** $h[n]$

<u>Exercise</u>: Verify this for the above example!!!!

Simple Frequency-Domain Implementation of FIR Filtering



<u>Why Do This</u>

The FFT's Efficiency Makes This Faster Than Time-Domain Implementation (In Many Cases)

Problems with the *Simple* FD Implementation

Q: What if $N_1 >> N_2$?
A: Then, need **Really Big FFT** ➔ Not Good!!!
(Input signal much longer than filter length)
Also… can't get any output samples until after whole signal is available and FFT processing is done. **Long Delay**.

Example: Filter 0.2 sec of a radar signal sampled at Fs = 50 MHz
$N_1 = (0.2 \text{ sec}) \times (50 \times 10^6 \text{ samples/sec}) = 10^7$ samples
FFT Size > 10^7 ➔ Really Big FFT!!!!

Q: What if N_1 is unknown in advance?
Example: Filtering a stream of audio
A: FFT size can't be set ahead of time – difficult programming

***Simple* FD Implementation Has Serious Limitations!!!**

Better **FD-Based FIR Filter Implementations**

Two Very Similar Methods Exist

- Overlap – Add (OLA)
- Overlap – Save (OLS)

Covered Here

<u>Both</u> methods exploit linearity of filter:

- Break input signal into a sum of blocks ➔ $x[n] = \sum_i x_i[n]$
- Output = sum of response to each block

$$\boxed{\begin{aligned} z[n] &= (x * h)[n] \\ &= \sum_i \underbrace{(x_i * h)[n]}_{=z_i[n]} \\ &= \sum_i z_i[n] \end{aligned}}$$

Use the *Simple* FD-Based Method to Compute Each Output Block

<u>The difference between OLA & OSA</u>

lies in how the $x_i[n]$ blocks are formed

OLA Method for FD-Based FIR

For OLA: Choose $x_i[n]$ to be non-overlapped blocks of length N_B

(blocks are contiguous)

N_B is a Design Choice

$$\boxed{x_i[n] = \begin{cases} x[n], & iN_B \le n < (i+1)N_B \\ 0, & \text{otherwise} \end{cases}}$$

<See Fig. 5.8 (a) in Porat>

Q: Now what happens when each of these length-N_B blocks gets convolved with the length-N_2 filter?

A: The output block has length $N_2 + N_B - 1 > N_B$

- Output Blocks are Bigger than Input Blocks
- But are separated by N_B points
- Thus… Output Blocks Overlap
- Total Output = "Sum of Overlapped Blocks"

"Overlap-Add"

<See Fig. 5.8 (b) – (d) in Porat>

OLA Method Steps

Assume: Filter $h[n]$ length N_2 is specified

Choose: Block Size N_B & FFT Size $N_{\text{FFT}} = 2^p \geq N_B + N_2 - 1$

Choose N_B such that:

$N_B + N_2 - 1 = 2^p$

It gives minimal complexity for method (see below)

Run:

Zero-Pad $h[n]$ & Compute N_{FFT}-pt FFT (can be pre-computed)

For each i value ("For Each Block")

- Compute $z_i[n]$ using Simple FD-Based Method
  - Zero-Pad $x_i[n]$ & Compute N_{FFT}-pt FFT
  - Multiply by FFT of $h[n]$
  - Compute IFFT to get $z_i[n]$
- Overlap the $z_i[n]$ with previously computed output blocks
- Add it to the output buffer

<See Fig. 5.8 (e) in Porat>

<u>OLA Method Complexity</u>

- The FFT of filter $h[n]$ can be pre-computed ➔ Don't Count it!
- We'll measure complexity using <u># Multiplies/Input Sample</u>
- Use $2N_{\mathrm{FFT}}\log_2 N_{\mathrm{FFT}}$ Multiplies as measure for FFT
- Assume input samples are Real Valued

  Can do 2 real-signal FFT's for price of $2N_{\mathrm{FFT}}\log_2 N_{\mathrm{FFT}}$ (See Porat 5.5)

- For Each Pair of Input Blocks
  - One FFT: $2N_{\mathrm{FFT}}\log_2 N_{\mathrm{FFT}}$ Multiplies
  - Multiply DFT × DFT: $4N_{\mathrm{FFT}}$ Multiplies
  - One IFFT: $2N_{\mathrm{FFT}}\log_2 N_{\mathrm{FFT}}$ Multiplies
  - <u>Total</u>: $4N_{\mathrm{FFT}}\,[1 + \log_2 N_{\mathrm{FFT}}]$ Multiplies

    $= \mathbf{4(}N_B + N_2 - \mathbf{1)\,[1 + \log_2(}N_B + N_2 - \mathbf{1)]}$
- The Number of Input Samples = 2 Blocks = $2N_B$

- **<u># Multiplies/Input Sample</u>** $= \mathbf{2(1 + (}N_2 - \mathbf{1)/}\,N_B\mathbf{)\,[1 + \log_2(}N_B + N_2 - \mathbf{1)]}$

Comparison to TD Method Complexity

Complexity of TD Method

- Filter $h[n]$ has length of N_2
- To get each output sample:
  - Multiply each filter coefficient by a signal sample: **N_2 Multiplies**
- **# Multiplies/Input Sample = N_2 Multiplies**

Condition Needed For OLA to Be More Efficient:

$$\boxed{2\left(1+\frac{N_2-1}{N_B}\right)\left[1+\log_2\left(N_2+N_B-1\right)\right] < N_2} \qquad (5.38)$$

In Porat

Thus… For a given N_2, Choose N_B to minimize the left-hand side

FD Complexity vs TD Complexity

Plot of: [Left-Hand Side]/[Right-Hand Side] of (5.38)