# II-1: Bandpass Signals
# Equivalent Lowpass Signals
# I&Q Signals

(PM-6.4.3)

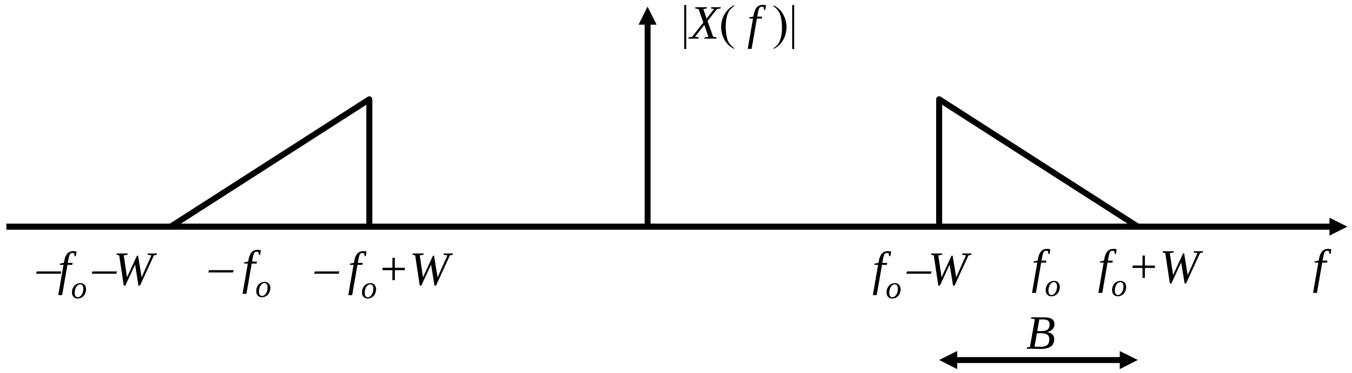
Also called:
Equiv. Baseband Signals

# Definition: Bandpass Signal

A Bandpass Signal is a signal $x(t)$ whose Fourier transform $X(f)$ is nonzero only in some small band around some "central" frequency $f_o$.

**For example**:

$$X(f) = 0 \quad \text{for} \quad |f - f_o| > W \quad \text{where} \quad W < f_o.$$

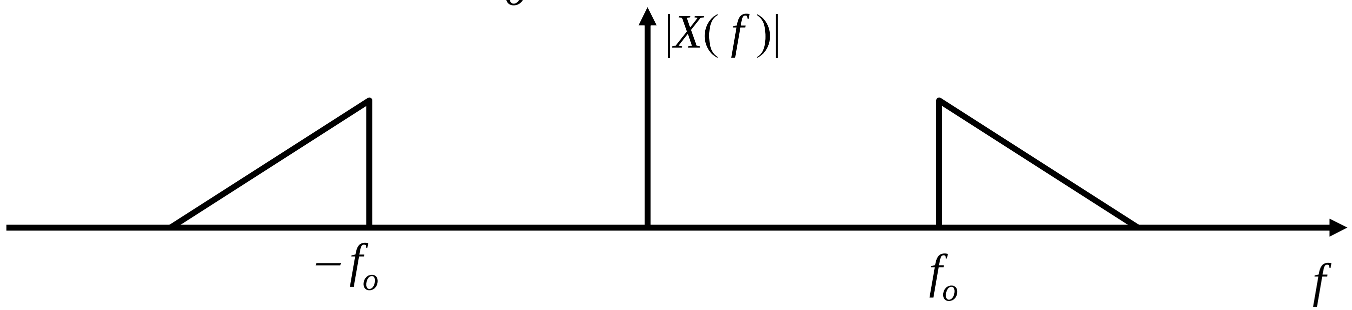


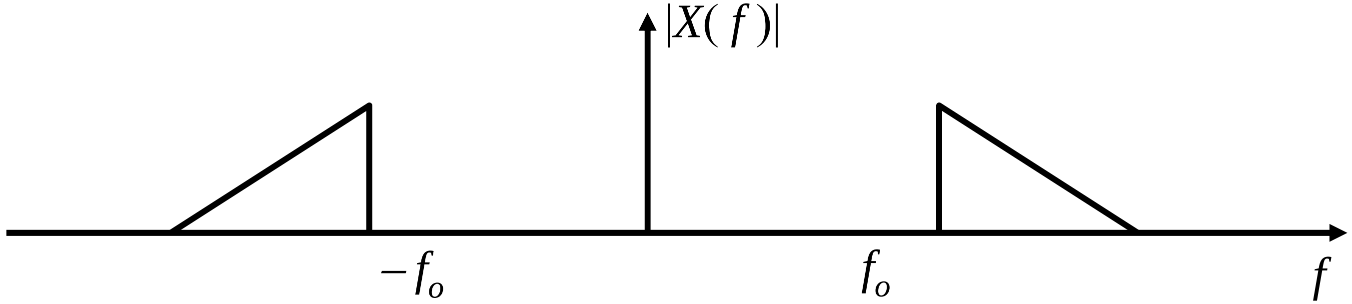

The **bandwidth** $B$ of the bandpass signal = the width of the positive-frequency interval on which the signal is nonzero.

(Note: this is consistent with the bandwidth definition for lowpass signals).

# Definition: Bandpass Signal (cont.)

Note that the choice of $f_o$ is arbitrary:




Bandpass signals are encountered when receiving radio frequency (RF) signals such as communication and radar signals.

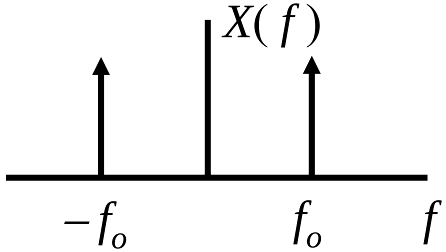
In the analysis and actual processing of BP signals it is convenient to work with a related, equivalent signal called the Equivalent Lowpass Signal. This is a natural generalization of the idea of phasor used in sophomore-level circuits.

# Recall: Phasor Idea Used in Circuits

Idea: Replace $A\cos(2\pi f_o t+\theta)$ by complex DC value $Ae^{j\theta}$

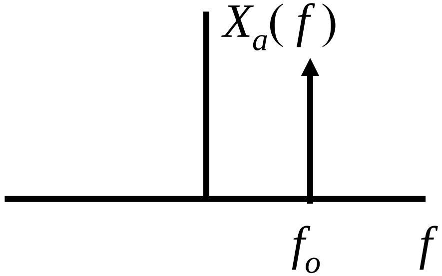
First, the **sinusoid**

$$\boxed{\begin{aligned} x(t) &= A\cos(2\pi f_o t+\theta) \\ &= \frac{A}{2}e^{j(2\pi f_o t+\theta)} + \frac{A}{2}e^{-j(2\pi f_o t+\theta)} \end{aligned}}$$



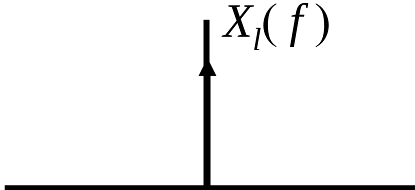

gets represented by a complex-valued signal called the **analytic signal**:

$$\begin{aligned} x_a(t) &= A\,exp\{\,j(2\pi f_o t+\theta)\} \\ &= A\cos(2\pi f_o t+\theta) + j\,A\sin(2\pi f_o t+\theta) \end{aligned}$$




Then to get the **phasor**, we frequency-shift the analytic signal down by $f_o$ to get:

$$\begin{aligned} x_l &= exp\{-j2\pi f_o t\}\; x_a(t) \\ &= Ae^{j\theta} \end{aligned}$$



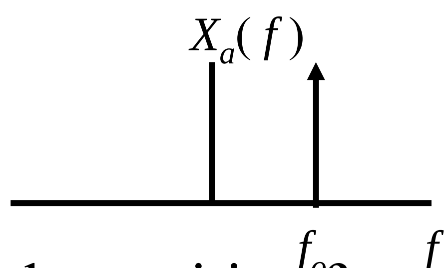
# Recall: Phasor Idea Used in Circuits (cont.)

<u>phasor</u> = equivalent *<u>l</u>owpass* signal representing the sinusoid (that's why we used the subscript *l* – for lowpass).

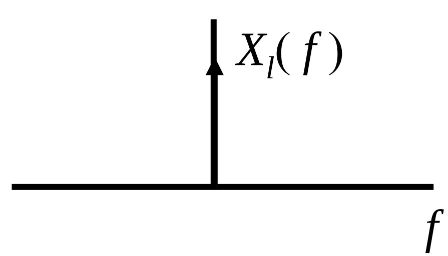
Note that this equivalent lowpass signal is complex valued, whereas the bandpass signal (the sinusoid) it represents is real valued.

**<u>Alternate View – Frequency Domain</u>:**

1. Suppress the negative frequency part of the sinusoid:




2. Frequency-shift the positive frequency part down to DC:

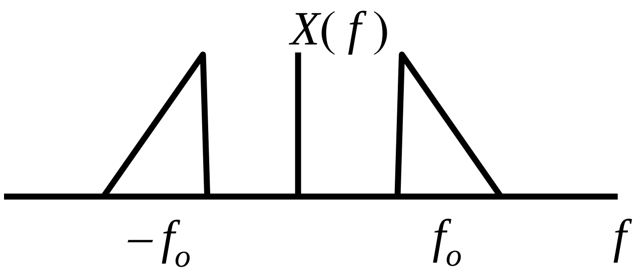


# Frequency-Domain View of Equiv. LP Signal

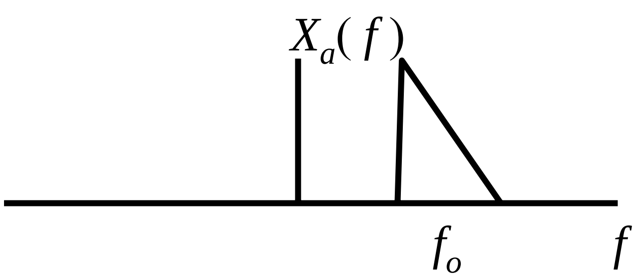
Now… use this FD view to do the same thing for a general bandpass signal that consists of more than one frequency.

….Then after that we interpret the results in the time domain.

Bandpass Signal's Fourier Transform:




Now to get the FT of the so-called Analytic Signal we suppress the negative frequencies:




Note: Since $|X_a(f)|$ is NOT even-symmetric, the TD signal $x_a(t)$ is complex-valued. (see Porat p. 12, #9)

# F-D View of Equiv. LP Signal (cont.)

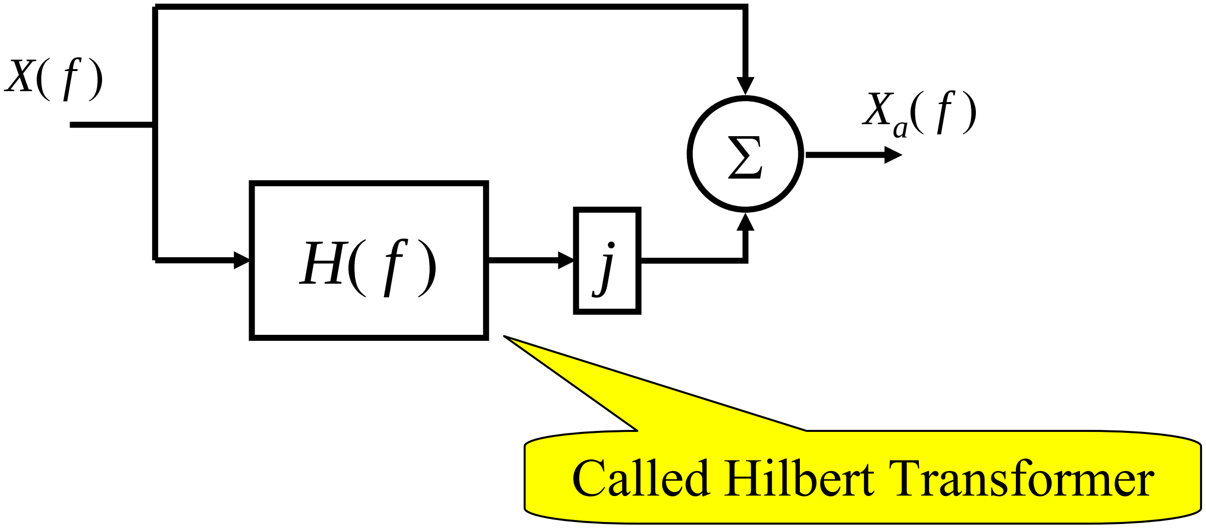
System View of Generating Analytic Signal: define a system frequency response $H(f)$ such that

$$H(f) = \begin{cases} -j & f > 0 \\ 0 & f = 0 \\ +j & f < 0 \end{cases}$$

then

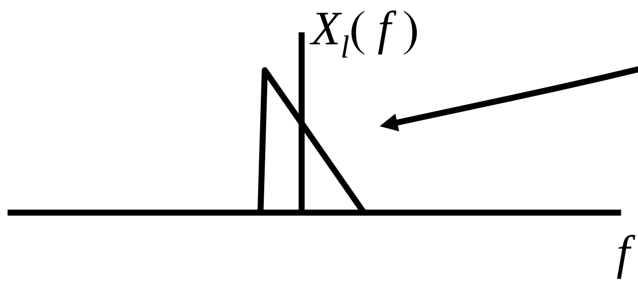
$$\boxed{\begin{aligned} X_a(f) &= X(f) + jH(f)X(f) \\ &= X(f) + j\hat{X}(f) \end{aligned}}$$

where $\hat{X}(f) = H(f)X(f)$



# F-D View of Equiv. LP Signal (cont.)

Then to get the **FT of the Equivalent Lowpass Signal**, frequency-shift the analytic signal down by $f_o$ to get:




Note that because $|X_l(f)|$ does not necessarily have even symmetry, the equivalent lowpass signal is complex valued,
….. whereas the bandpass signal it represents is real valued.

Now… how do we describe the ELP signal in the Time-Domain?

# T-D View of ELP Signal

Consider the IFT of $H(f)X(f)$: $\hat{x}(t) = \mathfrak{F}^{-1}\{\hat{X}(f)\} = \mathfrak{F}^{-1}\{H(f)X(f)\}$

$$x_a(t) = x(t) + j\hat{x}(t) \qquad (\blacktriangle)$$

Let $x_l(t)$ be the time-domain signal that corresponds to $X_l(f)$. Because it is the frequency-shifted version of $x_a(t)$ …. using the frequency-shift property of FT gives:

$$x_l(t) = e^{-j2\pi f_o t} x_a(t) \qquad (\bigstar)$$

(Note: this is the same as an equation above for the phasor case!)

This, can then be written as:

$$x_l(t) = e^{-j2\pi f_o t}\left[x(t) + j\hat{x}(t)\right]$$

# <u>I&Q Form of ELP Signal</u>

An <u>extremely</u> useful viewpoint for the ELP signal is the I&Q form:

… since $x_l(t)$ is complex-valued (see comment above in frequency-domain discussion), we can write its real and imaginary parts, which we will denote as

$$x_l(t) = x_i(t) + jx_q(t) \qquad (\dagger)$$

where subscripts $i$ and $q$ are for In-phase (I) and Quadrature (Q).

We'd now like to find relationships between the bandpass signal $x(t)$ and the I-Q components of the lowpass equivalent signal.

# Relationship: I&Q Parts and BP Signal

Solving (★) for the analytic signal gives

$$x_a(t) = e^{j2\pi f_o t} x_l(t) \qquad (■)$$

(Makes sense… $x_a(t)$ is $x_l(t)$ shifted up.)

Using the I-Q form given in (✝) gives:

$$\begin{aligned}
x_a(t) &= e^{j2\pi f_o t}\left[x_i(t) + jx_q(t)\right] \\
&= \left[\cos(2\pi f_o t) + j\sin(2\pi f_o t)\right]\left[x_i(t) + jx_q(t)\right] \\
&= \boxed{\left[x_i(t)\cos(2\pi f_o t) - x_q(t)\sin(2\pi f_o t)\right]} \\
&\qquad + j\left[x_i(t)\sin(2\pi f_o t) + x_q(t)\cos(2\pi f_o t)\right] \\
&= \boxed{x(t)} + j\hat{x}(t)
\end{aligned}$$

By (▲)

# Relationship: I&Q Parts and BP Signal (cont.)

This shows how the I&Q components are related to the BP signal:

$$\boxed{x(t) = x_i(t)\cos(2\pi f_o t) - x_q(t)\sin(2\pi f_o t)} \qquad (\triangle)$$

Similarly – but less important – we have:

$$\boxed{\hat{x}(t) = x_i(t)\sin(2\pi f_o t) + x_q(t)\cos(2\pi f_o t)}$$

# Envelope/Phase Form of ELP Signal

This is an alternate form (but equally important to IQ form) of the ELP signal. Note in (✝) that the I&Q form is a “rectangular form” for the complex ELP signal.

So… converting to a “polar form” gives:

$$x_l(t) = A(t)e^{j\theta(t)} \qquad (❖)$$

where…

$$A(t) = \sqrt{x_i^2(t) + x_q^2(t)} \geq 0$$

$$\theta(t) = \arctan\left\{\frac{x_q(t)}{x_i(t)}\right\}$$

Note Similarity to Phasor!!

But… Time-<u>varying</u> Envelope & Magnitude

# Relationship: Env/Phase and I&Q

Often we need to convert between the two forms (rect & polar).

If in (❖) we expand the complex exponential:

$$
\begin{aligned}
x_l(t) &= A(t)e^{j\theta(t)} \\
&= \underbrace{A(t)\cos[\theta(t)]}_{x_i(t)} + j\underbrace{A(t)\sin[\theta(t)]}_{x_q(t)}
\end{aligned}
$$

By (✝)

Thus….

$$
\boxed{
\begin{aligned}
x_i(t) &= A(t)\cos[\theta(t)] \\
x_q(t) &= A(t)\sin[\theta(t)]
\end{aligned}
}
$$

# Envelope/Phase Form of BP Signal

We already saw Env/Phase form for the ELP signal…

Do we get something similar for the original BP signal ??

Using (■) and (⌘) we can write

$$
\begin{aligned}
x_a(t) &= e^{j2\pi f_o t}\left[A(t)e^{j\theta(t)}\right] \\
&= A(t)e^{j[2\pi f_o t+\theta(t)]} \\
&= \boxed{A(t)\cos[2\pi f_o t+\theta(t)]} + jA(t)\sin[2\pi f_o t+\theta(t)] \\
&= \boxed{x(t)} + j\hat{x}(t)
\end{aligned}
$$

By (▲)

$$
\boxed{x(t) = A(t)\cos[2\pi f_o t+\theta(t)]} \qquad (♣)
$$

# <u>Envelope/Phase Form of BP Signal (cont.)</u>

So what we have just shown is:

**<u>Any</u> BP signal can be expressed as:**

$$\boldsymbol{x(t) = A(t)\cos[2\pi f_o t + \theta(t)]}$$

**where $A(t) \geq 0$.**

**Note: $A(t)$ and $\theta(t)$ vary slowly compared to $\cos(2\pi f_o t)$.**

The LPE signal has the same envelope and phase as the BP signal … compare (♣) and (⌘).

$$x_l(t) = A(t)e^{j\theta(t)}$$

# Analog Generation of I&Q Components

As stated earlier… processing for radar & communication is actually implemented using the ELP signal.

- Thus we need some way to get the ELP signal from a received BP signal…
- The I&Q form is the most commonly used

So… given the BP signal

$$x(t) = A(t)\cos[2\pi f_o t + \theta(t)]$$

we need to be able to extract through processing the I&Q signals:

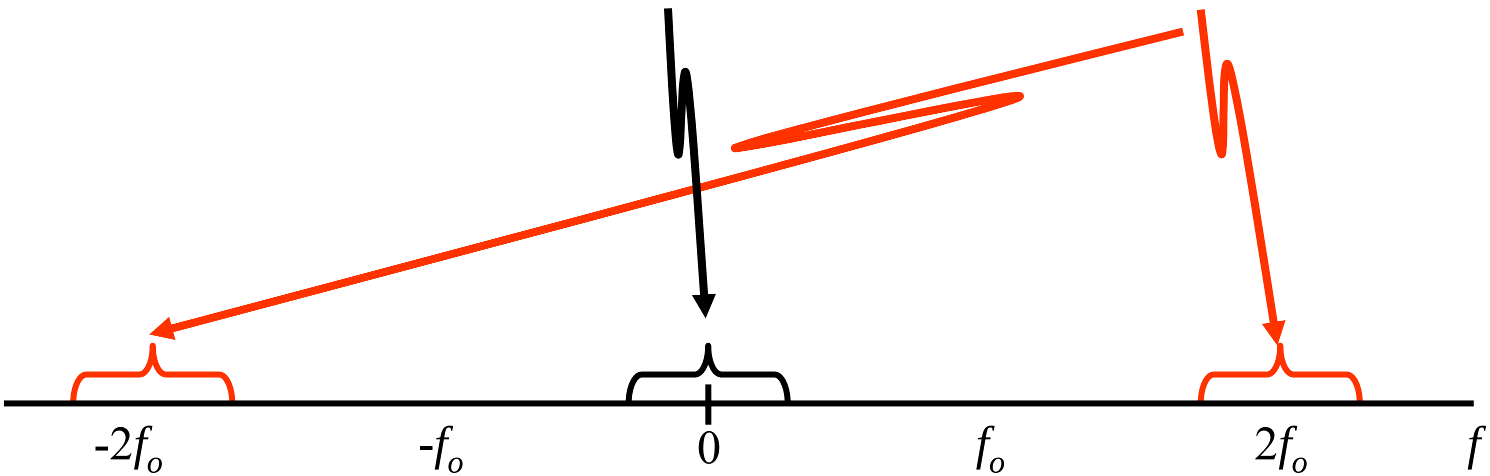
$$x_i(t) = A(t)\cos[\theta(t)]$$
$$x_q(t) = A(t)\sin[\theta(t)]$$

# Analog Generation of I&Q Components (cont.)

These give the clue as to how to extract the I-Q signals by using analog techniques.  Using trigonometric identities:

$$2x(t)\cos(2\pi f_o t) = 2\left[A(t)\cos(2\pi f_o t + \theta(t))\right]\cos(2\pi f_o t)$$

$$= \underbrace{A(t)\cos[\theta(t)]}_{x_i(t)} + \underbrace{A(t)\cos[2\pi(2f_o)t + \theta(t)]}_{x(t)\text{.... but centered at } 2f_o}$$



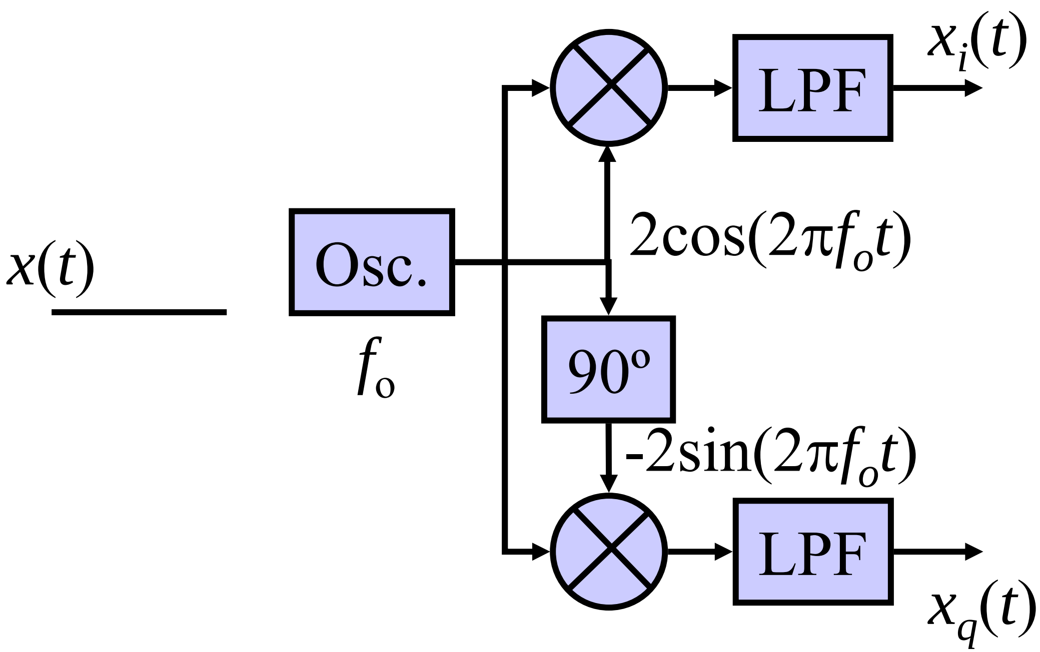
# Analog Generation of I&Q Components (cont.)

Similarly, we also get….

$$2x(t)\sin(2\pi f_o t) = 2\left[A(t)\cos(2\pi f_o t + \theta(t))\right]\sin(2\pi f_o t)$$

$$= \underbrace{A(t)\sin[\theta(t)]}_{x_q(t)} + \underbrace{A(t)\sin[2\pi(2f_o)t + \theta(t)]}_{\hat{x}(t) \text{.... but centered at } 2f_o}$$

## Analog Circuitry to Generate I&Q



# Digital Generation of I&Q Components

ANALOG DEVICES

## 67 MSPS Digital Receive Signal Processor

## AD6620

**FEATURES**

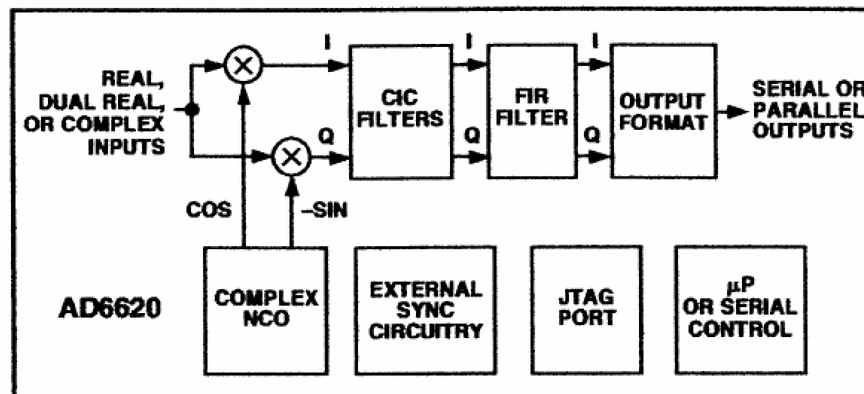
High Input Sample Rate
- 67 MSPS Single Channel Real
- 33.5 MSPS Diversity Channel Real
- 33.5 MSPS Single Channel Complex

NCO Frequency Translation
- Worst Spur Better than –100 dBc
- Tuning Resolution Better than 0.02 Hz

2nd Order Cascaded Integrator Comb FIR Filter
- Linear Phase, Fixed Coefficients
- Programmable Decimation Rates: 2, 3 . . . 16

5th Order Cascaded Integrator Comb FIR Filter
- Linear Phase, Fixed Coefficients

**FUNCTIONAL BLOCK DIAGRAM**

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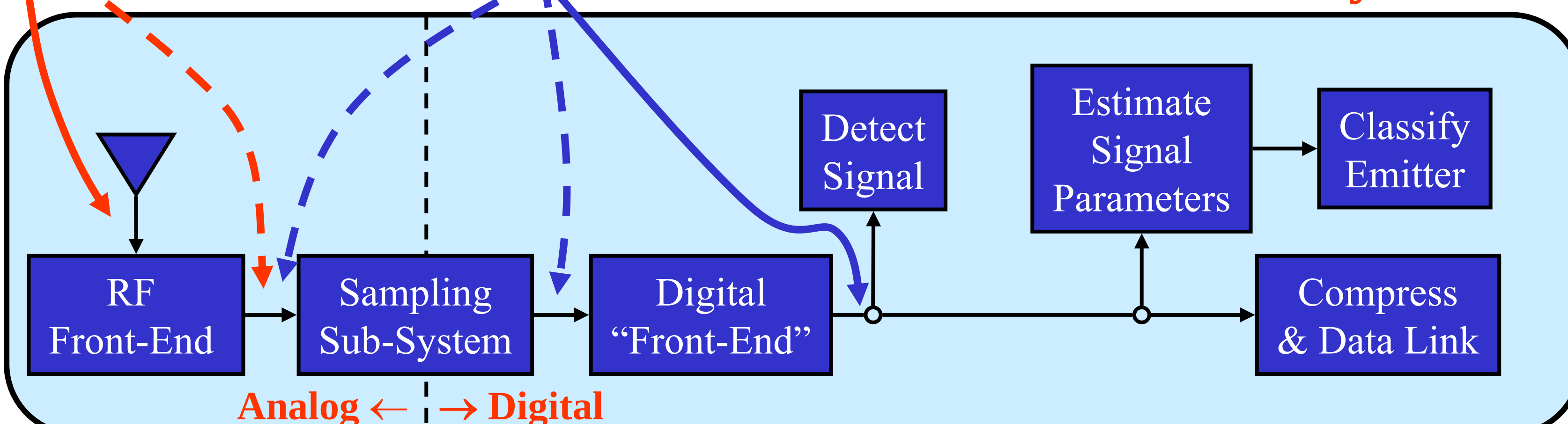
# Uses of These Ideas

- **Bandpass Signal Model**
  - usually used to *model* RF signals in radar and communications
  - also often used to *model* acoustic signals in sonar
  - not generally used for audio/speech signals
- **Lowpass Equivalent Signal**
  - used as a conceptual tool to aid analysis/design
  - used as the actual representation in real processing

**Receiver From Our Case Study**



# Uses of These Ideas (cont.)

- **Analytic Signal**
  - generally used as a conceptual tool to prove results
  - usually applied directly to the continuous-time RF bandpass signal
  - There are occasions where we actually compute the analytic signal of a real-valued digital signal,
    - but usually applying it some real-valued lowpass signal.
  - ***See MATLAB Warning Below***
- **Hilbert Transform of a Signal**
  - generally used as a conceptual tool to prove results
  - There are occasions where we actually compute the Hilbert transform of a real-valued digital signal,
    - but usually applying it some real-valued lowpass signal.
  - ***See MATLAB Warning Below***

# MATLAB Warning

MATLAB has a command that is called “hilbert”. The Help entry on MATLAB for this command is:

> “HILBERT(X) is the Hilbert transform of the real part of vector X. The real part of the result is the original real data; the imaginary part is the actual Hilbert transform.”

Thus, executing hilbert(x) does **NOT** return the Hilbert transform of x;

- It gives the analytic signal – see (▲).

# Summary of Relationships

**BP Signal & Its Hilbert Transform**

$$x(t) = A(t)\cos[2\pi f_o t + \theta(t)]$$
$$\hat{x}(t) = A(t)\sin[2\pi f_o t + \theta(t)]$$

------

$$x(t) = \mathrm{Re}\{x_a(t)\}$$
$$\hat{x}(t) = \mathrm{Im}\{x_a(t)\}$$

------

$$x(t) = x_i(t)\cos(2\pi f_o t) - x_q(t)\sin(2\pi f_o t)$$
$$\hat{x}(t) = x_i(t)\sin(2\pi f_o t) + x_q(t)\cos(2\pi f_o t)$$

*****************************

$$A(t) = \sqrt{x^2(t) + \hat{x}^2(t)}$$
$$\theta(t) = \arctan\left\{\frac{\hat{x}(t)}{x(t)}\right\} - 2\pi f_o t$$

**Analytic Signal**

$$x_a(t) = x(t) + j\hat{x}(t)$$
$$x_a(t) = x_l(t)e^{j2\pi f_o t}$$
$$x_a(t) = A(t)e^{j[2\pi f_o t + \theta(t)]}$$

***********

$$A(t) = |x_a(t)|$$
$$\theta(t) = [\angle x_a(t)] - 2\pi f_o t$$

**Equiv. Lowpass Model**

$$x_l(t) = x_a(t)e^{-j2\pi f_o t}$$
$$x_l(t) = A(t)e^{j\theta(t)}$$
$$x_l(t) = x_i(t) + jx_q(t)$$

*****************

$$x_i(t) = \mathrm{Re}\{x_l(t)\}$$
$$x_q(t) = \mathrm{Im}\{x_l(t)\}$$

-----

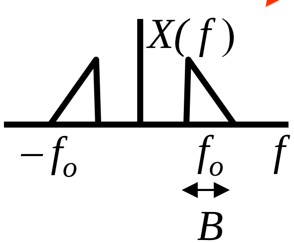
$$x_i(t) = A(t)\cos[\theta(t)]$$
$$x_q(t) = A(t)\sin[\theta(t)]$$

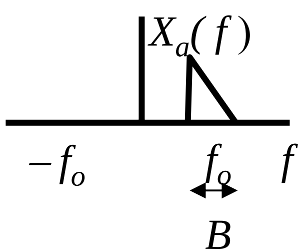
-----

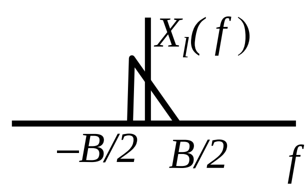
$$x_i(t) = x(t)\cos(2\pi f_o t) + \hat{x}(t)\sin(2\pi f_o t)$$
$$x_q(t) = \hat{x}(t)\cos(2\pi f_o t) - x(t)\sin(2\pi f_o t)$$

*****************

$$A(t) = |x_l(t)| = \sqrt{x_i^2(t) + x_q^2(t)}$$
$$\theta(t) = \angle x_l(t) = \arctan\left\{\frac{x_q(t)}{x_i(t)}\right\}$$



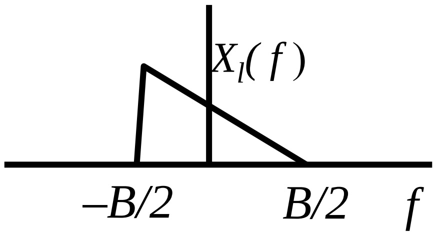



# Sampling Rate Needed for ELP Signal

Given a complex-valued equivalent lowpass signal, what is an appropriate sampling rate to use?

To answer this question… look at the ELP signal's Fourier transform:




Sampling this signal is no different than sampling some real-valued lowpass signal: choose $F_s > 2f_{\max}$

…. in this case gives $F_s > 2(B/2) = \mathrm{B}$.

Now does this make sense?

# <u>Sampling Rate Needed for ELP Signal (cont.)</u>

Now does this make sense?

<u>Bandpass Sampling</u> on the corresponding bandpass signal (BPS)

….would require $F_s > 2B$,

BUT… need only half that rate for the ELP signal!!!

Do we really need only half the amount of information to represent the ELPS as we need for the BPS?

Would that even make sense? Since ELPS ↔ BPS ?

It doesn't at first!!!! ***<u>BUT</u>*** … the ELPS is complex

it requires a real sample value *<u>and</u>*

an imaginary sample value for *each* signal sample

$$\underbrace{[(\mathrm{I+Q})\ @\ F_s = \mathrm{B}]}_{\text{\# of Real Values for ELPS}} \quad = \quad \underbrace{[\mathrm{BPS}\ @\ F_s = 2\mathrm{B}]}_{\text{\# of Real Values for BPS}}$$