The Concept of Digital Frequency

EE521
Prof. Fowler

Department of Electrical Engineering
State University of New York at Binghamton

A continuous-time complex exponential can take on any frequency in the range $-\infty$ to $+\infty$, but consider what happens when we sample it at rate F_s:

$$e^{j2\pi f_o t} \rightarrow e^{j2\pi f_o nT_s} = e^{j2\pi\left[\frac{f_o}{F_s}\right]n} \triangleq e^{j\tilde{\omega}_o n}$$

Now imagine f_o varying from $-\infty$ to $+\infty$. Each time f_o is a multiple of F_s the argument of the exponential is a multiple of 2π (then $\tilde{\omega}_o$ is an integer multiple of 2π). Thus we can view the C-T frequency axis as being wrapped around a circle whose circumference is F_s Hz.

Thus, the **digital frequency** $\tilde{\omega}_o$ is cyclic with period 2π: see Fig. below.

The range for $\tilde{\omega}_o$ can be chosen to be any interval of length 2π. Common choices are:

$$\tilde{\omega}_o \text{ in } (-\pi, \pi] \text{ or } [-\pi, \pi)$$

$$\tilde{\omega}_o \text{ in } [0, 2\pi)$$

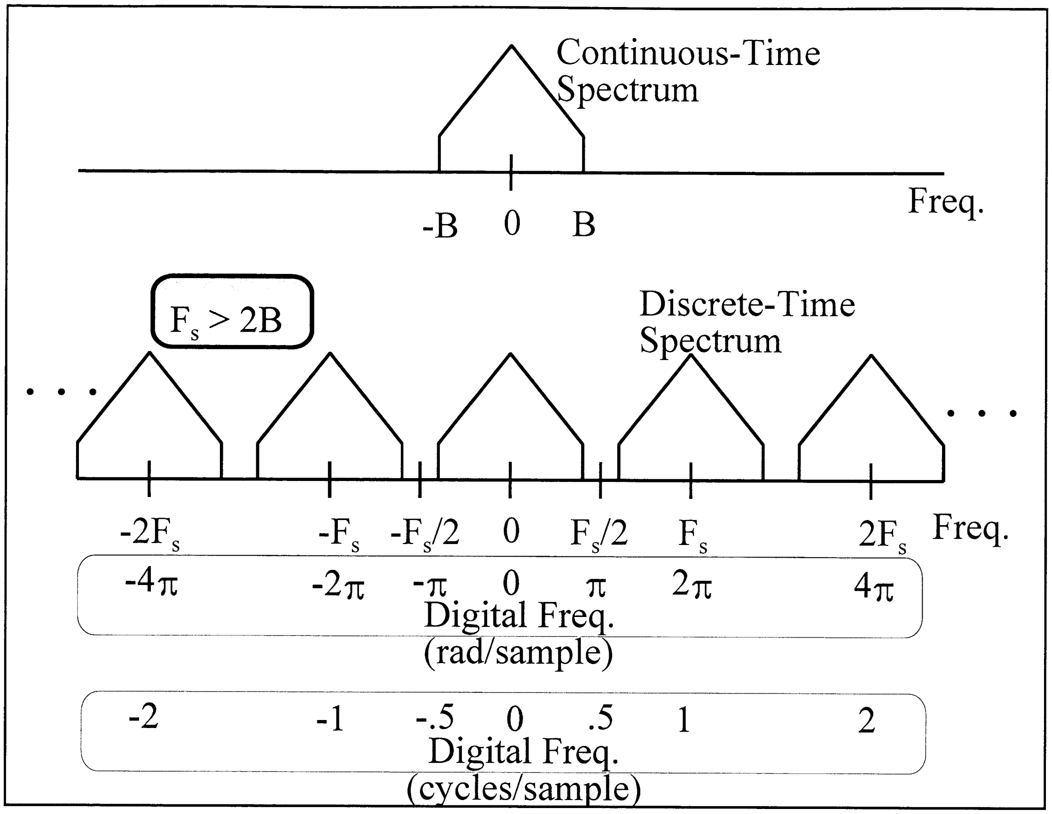
The units of $\tilde{\omega}_o$ are radians/sample: $\omega_o = \dfrac{2\pi f_o}{F_s} \rightarrow \dfrac{\text{(rad/cycle) (cycles/sec)}}{\text{samples/sec}}$

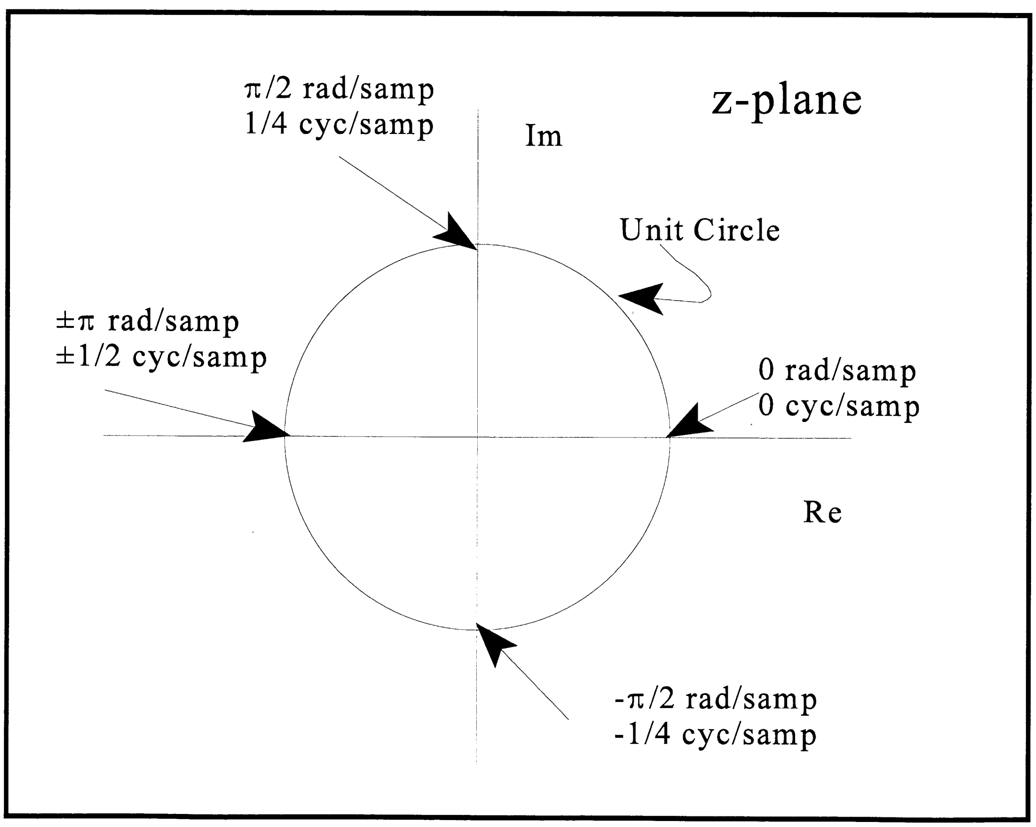
An alternative to $\tilde{\omega}_o$ is to define:

$$\tilde{f}_o \triangleq \frac{f_o}{F_s} = \frac{\omega_o}{2\pi}$$

with range (-0.5, 0.5) and units cycles/sample.

The diagrams below show the above ideas graphically. The first diagram shows how a C-T signal w/ triangular spectrum gets replicated in the discrete-time frequency domain; it also shows three different ways of viewing the frequency axis in the discrete-time frequency domain: (1) labeled using the corresponding C-T frequencies - this is useful when you are trying to keep in mind that the signal samples you are working with came from some sampled continuous-time signal (it helps keep your perspetive tied to the realworld of signals), note that this view requires knowledge of the particular sampling frequency or could be done in terms of an arbitray sampling frequency; (2) asa digital frequency between -π and π rad/sample; (3) asa digital frequency between -0.5 and 0.5 cycles/sample (remember 1 cycle = 2π, which is what is used to convert rad/sample into cycles/sample). These second two ways of viewing digital frequency are VERY useful when you are working on developing digital signal processing (DSP) techniques in general and not thing about a specific apllication tied to a specific sampling rate - that is, it is useful when talking about DSP theory in general. For this reason we will often talk about D-T signals as having frequencies that lie between -π and π rad/sample,or equivalently between -0.5 and 0.5 cycles/sample. In those cases we don't really care WHAT the sampling rate is. However, we will also sometimes talk about a D-T signal as having frequencies between $-F_s/2$ and $F_s/2$ Hz. The important thing to remember is that once the sampling is done (or once we are given a set of samples - however they were created) we only need to "live" in the region between -π and π rad/sample (or equivalently -0.5 and 0.5 cycles/sample (or equivalently between $-F_s/2$ and $F_s/2$ Hz). This view comes from the first figure below by seeing that for a D-T signal the spectrum just repeats and there is no NEW information at the higher frequencies. This viewpoint is captured in the second figure below by viewing the D-T frequency axis as the unit circle in the z-plane. This idea comes from the fact that if we were to "walk" along the linear axis in the bottom of the first figure (starting say at 0) it would appear to us that we were really just "walking in circles" because we keep seeing the same spectrum over and over again - as if we kept ending up back where we started (Fs Hz looks like 0 Hz, or π rad/sample looks just like 0 rad/sample, or 1 cycle/sample looks just like 0 cycles/sample). Thus we can view this axis as being the unit circle on the z-plane where the replication that we see on the linear axis is made to occur because of the cyclic nature of this new, circular axis. Pretty Neat!, Huh?


Continuous-Time
Spectrum
-B 0 B
Freq.
$F_s > 2B$
Discrete-Time
Spectrum
$-2F_s$ $-F_s$ $-F_s/2$ 0 $F_s/2$ F_s $2F_s$ Freq.
-4π -2π $-\pi$ 0 π 2π 4π
Digital Freq.
(rad/sample)
-2 -1 -.5 0 .5 1 2
Digital Freq.
(cycles/sample)


z-plane
$\pi/2$ rad/samp
1/4 cyc/samp
Im
Unit Circle
$\pm\pi$ rad/samp
$\pm 1/2$ cyc/samp
0 rad/samp
0 cyc/samp
Re
$-\pi/2$ rad/samp
-1/4 cyc/samp