Examples of Finding
Differential Equations
for Electrical Systems

For these simple circuits the trick is to first write a KVL or a KCL for the circuit.

Once you have this written identify the parts of it (i.e. voltage and current variables) that "aren't" the input and output signals. You need to find a way to replace these with the input and output signals…

You do this by using:
- "device rules" (like ohms law and the I-V rules for caps and inductors)
- other rules (e.g., if you wrote a KVL first, try writing a KCL and vice versa).

Then simplify it to put it in the standard form.
**Example:** Determine the Diff. Eq.

Write KVL around loop:

\[ x(t) = V_c(t) + V_L(t) + y(t) \]

\[ = \frac{1}{C} \int_{-\infty}^{t} i(t) dt + L \frac{d^2 i(t)}{dt^2} + R \frac{di(t)}{dt} \]

get rid of this by differentiating both sides

\[ \Rightarrow \frac{dx(t)}{dt} = \frac{1}{C} i(t) + L \frac{d^2 i(t)}{dt^2} + R \frac{di(t)}{dt} \]

These are "not" input or output variables so we can't have those in a DiffEq so we differentiate to get rid of it!!!

Now relate to output variable: \( y(t) = i(t) R \)

\[ \Rightarrow \text{replace } i(t) \text{ by } y(t)/R \]

\[ \Rightarrow \frac{L}{R} \frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{dx(t)}{dt} \]
Example #1

Determine the differential equation for this circuit...

(a) ... when $v_c(t)$ is taken as the output

(b) ... when $v_c(t)$ is taken as the output
(a) From KVL: \( V_R(t) + V_L(t) = V(t) \)

or \( V_R(t) = V(t) - V_L(t) \)

or \( i_R(t) = \frac{V(t) - V_L(t)}{R} \)

(b) From KCL:

\[
i_c(t) = i_R(t) + i_L(t)
\]

Now we use KVL and device rules to write as much as we can in terms of input & output.

\[
i_R(t) = \frac{V(t) - V_L(t)}{R}
\]

Device Rule

\[
i_L(t) = \frac{V(t) - V_L(t)}{R} + C \frac{dV(t)}{dt} - C \frac{dV_L(t)}{dt}
\]

in terms of ins & outs

Now we have something that is almost completely in terms of in's & out's so we are almost there!!

We need one more device rule...

Must relate to ins/outs

Device Rule

\[
\frac{dV(t)}{dt} = \frac{1}{L} V_L(t)
\]

So left, right side of (a)

We do this differentiation so we can plug this device rule in!! When we do that we get the equation below.

\[
\frac{1}{L} V_L(t) = \frac{1}{R} \frac{dV(t)}{dt} - \frac{1}{R} \frac{dV(t)}{dt} + C \frac{dV(t)}{dt} - C \frac{dV_L(t)}{dt}
\]
Re-arrange to get:

\[
\frac{d^2 V_c(t)}{dt^2} + \frac{1}{RC} \frac{dV_c(t)}{dt} + \frac{1}{LC} V_c(t) = \frac{d^2 x(t)}{dt^2} + \frac{1}{RC} \frac{dx(t)}{dt}
\]

(b) From KVL: \( V_c(t) + V_e(t) = x(t) \) \( \Rightarrow \) \( V_c(t) = x(t) - V_e(t) \)

From device rule for \( L \):

\[
V_c(t) = L \frac{di_c(t)}{dt}
\]

use KCL: \( i_c(t) = i_R(t) + i_e(t) \)

\[
= L \left[ \frac{di_R(t)}{dt} + \frac{di_e(t)}{dt} \right]
\]

\( \Rightarrow \) \( x(t) - V_c(t) = L \frac{di_R(t)}{dt} + L \frac{di_e(t)}{dt} \)

\( i_R(t) = V_c(t) \)

\( i_e(t) = \frac{C}{R} \frac{dV_c(t)}{dt} \)

\( \Rightarrow \)

\[
\frac{dV_c(t)}{dt} = \frac{x(t)}{R} \]

\( \downarrow \)

\( x(t) - V_c(t) = \frac{L}{R} \frac{dV_c(t)}{dt} + LC \frac{d^2 V_c(t)}{dt^2} \)

Re-arrange:

\[
\frac{d^2 V_c(t)}{dt^2} + \frac{1}{RC} \frac{dV_c(t)}{dt} + \frac{1}{LC} V_c(t) = \frac{1}{LC} x(t)
\]
Example #2

Write the Diff. Eq. for the following "2-stage" RC circuit:

\[ R_1 \quad \text{[Node 1]} \quad C_1 \quad \text{[Node 2]} \quad C_2 \quad \text{[Ground]} \]

Node Eq. at \( N_1 \): (Node Equations are a form of KCL)

\[
\dot{i}_{R_1}(t) = \dot{i}_{C_1}(t) + \dot{i}_{C_2}(t)
\]

\[
\frac{V(t) - V_{C_1}(t)}{R_1} = \frac{C_1}{R_1} \frac{dV_{C_1}(t)}{dt} + \frac{C_2}{R_1} \frac{dV_{C_2}(t)}{dt}
\]

Almost have what we want... need to get \( V_{C_1}(t) \) out of the equation!

KVL around 2nd Mesh

\[
V_{C_1}(t) = V_{R_2}(t) + V(t)
\]

\[
= R_{C_2}(t) = R_2 \left[ C_2 \frac{dV_{C_2}(t)}{dt} \right]
\]
Put (18) into (14) to get:

First re-write (14):

\[
C_2 \frac{dy(t)}{dt} + C_1 \frac{dV_c(t)}{dt} + \frac{1}{R_1} V_c(t) = \frac{1}{R_1} x(t)
\]

From (18) we get:

\[
V_c(t) = R_2 C_2 \frac{dy(t)}{dt} + y(t)
\]

and

\[
\frac{dV_c(t)}{dt} = R_2 C_2 \frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt}
\]

We get:

\[
C_1 R_2 C_2 \frac{d^2 y(t)}{dt^2} + \left( C_1 + \frac{R_2 C_2}{R_1} \right) \frac{dy(t)}{dt} + \frac{1}{R_1} y(t) = \frac{1}{R_1} x(t)
\]

which could then be normalized by multiplying by \( \frac{1}{C_1 R_2 C_2} \).
DC Motor Example ("Armature Control")

Assume fixed magnetic field

\[ v_a = K_e w(t) \]

output of system

\[ w = \text{speed rad/sec} \]

\[ T(t) = K_e i(t) \]

\[ \chi(t) = R i(t) + L \frac{di(t)}{dt} + K_e w(t) \]  \hspace{1cm} (A)

Now write KVL around the armature circuit:

- DC Motor consists of lots of wire wound on an "armature"  \hspace{1cm} \Rightarrow \text{motor has "parasitic" R & L}

- As motor armature turns, the coils cut magnetic flux lines \Rightarrow \text{induces voltage proportion to speed}

Now write KVL around the armature circuit:

\[ \chi(t) = R i(t) + L \frac{di(t)}{dt} + K_e w(t) \]  \hspace{1cm} (A)

- The current \( i(t) \) in the windings interacts w/ the magnetic field to create a force which provides the motor torque \( T(t) \)

\[ T(t) = K_e i(t) \]  \hspace{1cm} (A)
Finally, rotational mechanics provides:

\[ T(t) = I \frac{d\omega(t)}{dt} + b \omega(t) \]  

\( T(t) \) Torque due to Mass I \( b \) Torque due to friction

Putting (A) into (G) and dividing by \( K_e \) gives:

\[ \dot{\omega}(t) = \frac{I}{K_e} \frac{d\omega(t)}{dt} + \frac{b}{K_e} \omega(t) \]

Now put this in (A) to get:

\[ \frac{LI}{K_e} \frac{d^2\omega(t)}{dt^2} + \left( \frac{RI + Lb}{K_e} \right) \frac{d\omega(t)}{dt} + \left( K_e + \frac{Rb}{K_e} \right) \omega(t) = X(t) \]

\[ \Rightarrow \text{Second-Order Diff. Eq.} \]

\[ \Rightarrow \text{Could be Oscillatory!} \]

\[ \omega(t) \]

\[ \theta(t) \]