Note Set #8

- D-T Convolution: The Tool for Finding the Zero-State Response
- Reading Assignment: Section 2.1-2.2 of Kamen and Heck
Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).
Convolution

Our Interest: Finding the output of LTI systems (D-T & C-T cases)

C-T

LTI System

Differential Equation

(solve)

“zero input” solution

“zero state” solution

Use char. poly. roots

HOW??

D-T

Difference Equation

(solve)

“zero input” solution

“zero state” solution

Use char. poly. roots

HOW??

Notice the parallel structure between C-T and D-T systems! We’ll see that they are solved using similar but slightly different tools!!

Our focus in this chapter will be on finding the zero-state solution… (we already know how to find the zero-input solution for C-T differential equations and later we’ll learn how to do that for D-T difference equations)
How do we find the Zero-State Response?

(Remember… that is the response (i.e., output) of the system to a specific input when the system has zero initial conditions)

Recall that in the examples for **differential** equations we always saw:

\[ y_{ZS}(t) = \int_{t_0}^{t} h(t - \lambda)x(\lambda)d\lambda \]

Where does this come from?

How do we deal with it?

C-T “convolution”

Recall that in the examples for **difference** equations we saw:

\[ y_{ZS}[n] = \sum_{i=1}^{n} h[n - i]x[i] \]

Where does this come from?

How do we deal with it?

D-T “convolution”

We’ll handle D-T systems first because they are easier to understand!
Convolution for LTI D-T systems

We are trying to find $y_{ZS}(t)$… so the ICs = 0  
\( i.e. \) no stored “energy”

We’ll drop the “zs” subscript to make the notation easier!

Before we can find the Z-S output… we need something first:

**Impulse Response** (Warning: book calls it “unit-pulse response”)

The impulse response $h[n]$ is what “comes out” when $\delta[n]$ “goes in” w/ ICs=0

Note: If system is causal, then $h[n] = 0$ for $n < 0$
The impulse response $h[n]$ uniquely describes the system... so we can identify the system by specifying its impulse response $h[n]$.

Thus, we often show the system using a block diagram with the system’s impulse response $h[n]$ inside the box representing the system:

![Block Diagram](image)

Because impulse response is only defined for LTI systems, if you see a box with the symbol $h[n]$ inside it you can assume that the system is an LTI system.

![Block Diagram](image)
How do we know/get the impulse response \( h[n] \)?

Many possible ways:

1. Given by the designer of D-T systems
2. Measured experimentally
   - Put in sequence \( \ldots 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ldots \)
   - See what comes out
3. Mathematically analyze the D-T system
   - Given difference equation
   - Derive \( h[n] \)

In what form will we know \( h[n] \)?

1. \( h[n] \) known analytically as a function
   
   Ex: \( h[n] = \left( \frac{1}{2} \right)^n u[n] \)

2. \( h[n] \) known numerically as a finite-duration sequence
   
   Ex:
   
<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h[n] )</td>
<td></td>
<td>0.5</td>
<td>1</td>
<td>2.1</td>
<td>1.3</td>
<td>.6</td>
</tr>
</tbody>
</table>

We assume that \( h[n] = 0 \) for \( n < 0 \)

There are many ways to do this, as we will see!
**Example of analytically finding $h[n]$**

Given a system described by a 1\textsuperscript{st} order difference equation:

$$y[n] = -ay[n-1] + bx[n]$$  \quad (a \text{ and } b \text{ are arbitrary numbers})

Recall that $h[n]$ is what comes out when $\delta[n]$ goes in (with zero ICs). So we can re-write the above difference equation as follows:

$$h[n] = -ah[n-1] + b\delta[n]$$

Here we solve for $h[n]$ recursively and then examine the results to deduce a closed-form solution (note: we can’t always use this “deductive” approach):

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\delta[n]$</th>
<th>$h[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
<td>$-a\times0 + b\times0 = 0$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>$-a\times0 + b\times1 = b$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$-ab + b\times0 = -ab$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$-a\times(-ab) = (-a)^2b$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$-a\times(-a)^2b = (-a)^3b$</td>
</tr>
</tbody>
</table>

By examining these results we see…

$$= b(-a)^n u[n]$$

So… we now have the impulse response for this system!!! Next we’ll learn how to \underline{use} it to \underline{solve} for the zero-state response!!!
Q: How do we use $h[n]$ to find the Zero-State Response?

A: “Convolution”  We’ll go through three analysis steps that will derive “The General Answer” that convolution is what we need to do to find the zero-state response

After that… we won’t need to re-do these steps… we’ll just “Do Convolution”

**Step 1: Using time-invariance we know:**

\[ \delta[n-i] \rightarrow h[n] \rightarrow h[n-i] \]  
(w/ ICs = 0)

Shifted input gives shifted output

**Step 2: Use “homogeneity” part of linearity:**

\[ x[i] \delta[n-i] \rightarrow h[n] \rightarrow x[i]h[n-i] \]  
(w/ ICs = 0)

The input is a function of $n$ so we view $x[i]$ as a fixed number for a given $i$

So… we scale the output by the same fixed number
Let’s see step 2… for a specific input:

\[ x[i] \delta[n-i] \quad h[n] \quad x[i]h[n-i] \]

(w/ ICs = 0)

\[ x[i] \delta[n-i] \quad n \]

1. \[ x[0] \delta[n] \quad h[n] \]
   ICs = 0
   1 \( h[n] \)

2. \[ x[1] \delta[n-1] \quad h[n] \]
   ICs = 0
   2 \( h[n-1] \)

3. \[ x[2] \delta[n-2] \quad h[n] \]
   ICs = 0
   1 \( h[n-2] \)

4. \[ x[3] \delta[n-3] \quad h[n] \]
   ICs = 0
   2.5 \( h[n-3] \)

This In

This Out

\( w/ ICs = 0 \)
Step 3: Use “additivity” part of linearity

In Step 2 we looked at inputs like this:

\[ x[i] \delta[n-i] \rightarrow h[n] \rightarrow x[i]h[n-i] \]

ICs = 0

For each \( i \), a different input \( \Rightarrow \) For each \( i \), a different response

Now we use the additivity part of linearity:

Put the **Sum of Those Inputs** In \( \Rightarrow \) Get the **Sum of Their Responses** Out

Input: \( \sum_{i=-\infty}^{\infty} x[i] \delta[n-i] \) \( \Rightarrow \) Output: \( \sum_{i=-\infty}^{\infty} x[i]h[n-i] \)

But… what is this??
On the next slide we show that it is the desired input signal \( x[n] \)!
Let’s see step 3 for a specific input:

\[
\sum_{i=-\infty}^{\infty} x[i] \delta[n - i]
\]

Note: The Sum of these “x-weighted” impulses gives \(x[n]!!\)
So… what we’ve seen is this:

\[
\sum_{i=-\infty}^{\infty} x[i] \delta[n-i] = x[n]
\]

Or in other words… we’ve derived an expression that tells what comes out of a D-T LTI system with input \( x[n] \):

\[
y[n] = \sum_{i=-\infty}^{\infty} x[i] h[n-i]
\]

So… now that we have derived this result we don’t have to do these three steps… we “just” use this equation to find the zero-state output:

\[
y[n] = \sum_{i=-\infty}^{\infty} x[i] h[n-i]
\]
For a LTI D-T system in **zero state** we no longer need the difference equation model…

- Instead we need the impulse response $h[n]$ & convolution

**New alternative model!**

**Equivalent Models (for zero state)**

Difference Equation

Convolution & Impulse resp.