BINGHAMTON UNIVERSITY
State University of New York

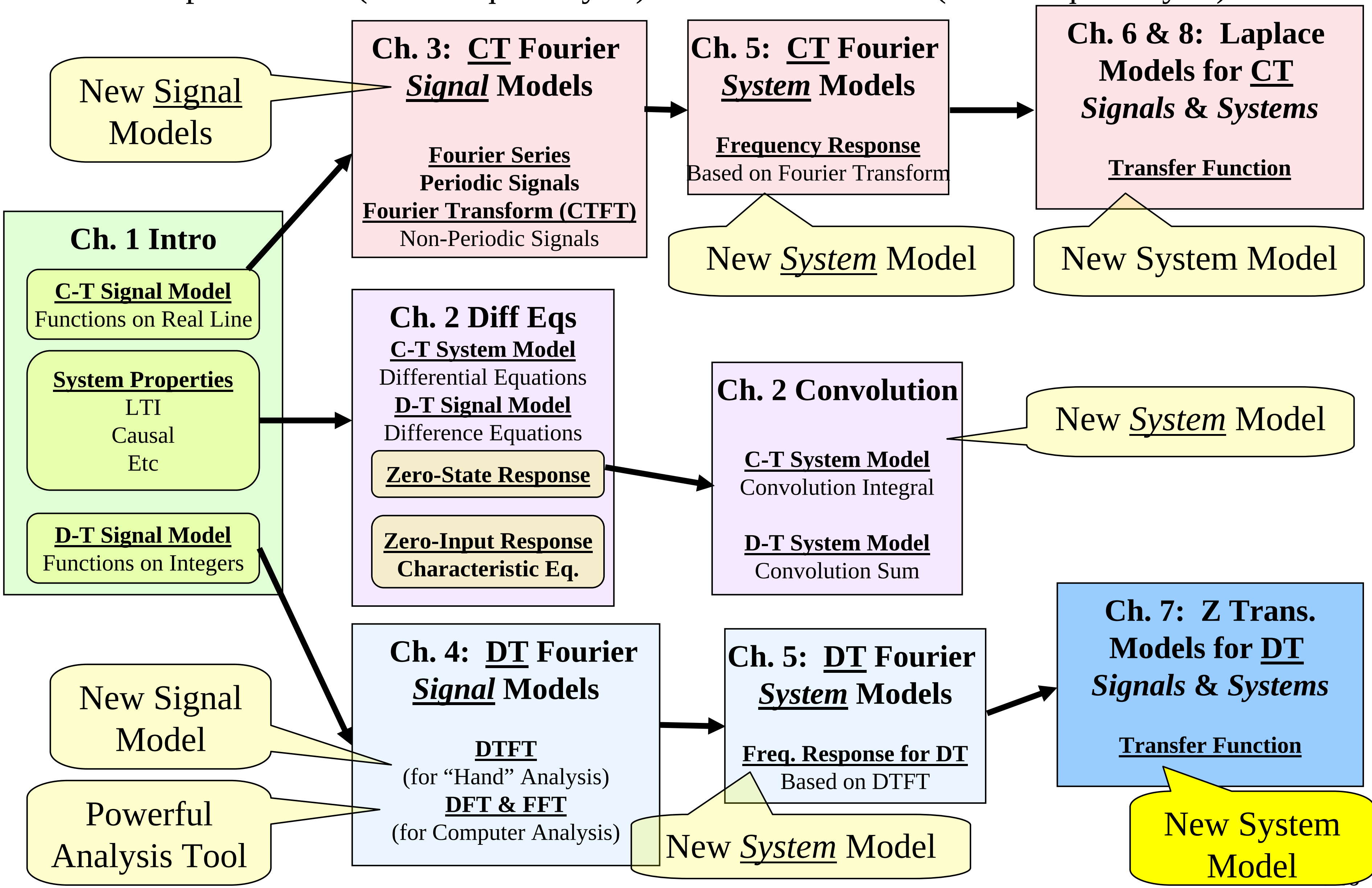
# EECE 301
# Signals & Systems
# Prof. Mark Fowler

## Note Set #34

- D-T Systems: Z-Transform … Solving Difference Eqs. & Transfer Func.
- Reading Assignment: Sections 7.4 – 7.5 of Kamen and Heck

# Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



# ZT For Difference Eqs.

Given a difference equation that models a D-T system we may want to solve it:

-with IC's — Apply ZT to the Difference Equation

-with IC's of zero — Use the Transfer Function Approach

**Note… the ideas here are very much like what we did with the Laplace Transform for CT systems.**

We'll consider the ZT/Difference Eq. approach first…

## Solving a First-order Difference Equation using the ZT

Given: $y[n] + ay[n-1] = bx[n]$

$\text{IC} = y[-1]$

$x[n]$ for $n = 0, 1, 2, \ldots$

Solve for: $y[n]$ for $n$ = 0, 1, 2,…

Take ZT of differential equation: $Z\{y[n] + ay[n-1]\} = Z\{bx[n]\}$

Use Linearity of ZT

$$Z\{y[n]\} + aZ\{y[n-1]\} = bZ\{x[n]\}$$

$Y(z)$

$X(z)$

Need Right-Shift Property… but which one???

Because of the non-zero IC we need to use the *non*-causal form:

$$Z\{y[n-1]\} = z^{-1}Y(z) + y[-1]$$

Using these results gives: $$Y(z) + a\left[z^{-1}Y(z) + y[-1]\right] = bX(z)$$

Which is an algebraic equation that can be solved for $Y(z)$:

$$Y(z) = \frac{-ay[-1]}{1 + az^{-1}} + \frac{b}{1 + az^{-1}} X(z)$$

Not the best form for doing Inverse ZT… we want things in terms of $z$ not $z^{-1}$

Multiply each term by $z/z$

$$Y(z) = -ay[-1]\frac{z}{z + a} + \frac{bz}{z + a} X(z)$$

On ZT Table

Part due to input signal modified by **Transfer Function**

$$H(z) = \frac{bz}{z + a}$$

$$y[n] = -ay[-1](-a)^n u[n] + Z^{-1}\{H(z)X(z)\}$$

If $|a| < 1$ this dies out as $n \uparrow$, its an IC-driven transient

If the ICs are zero, this is all we have!!!

## Ex.: Solving a Difference Equation using ZT: 1st-Order System w/ Step Input

For $x[n] = u[n] \quad \leftrightarrow \quad X(z) = \dfrac{z}{z-1}$

Then using our general results we just derived we get:

$$Y(z) = \frac{-ay[-1]z}{z+a} + \left(\frac{bz}{z+a}\right)\left(\frac{z}{z-1}\right)$$

For now assume that $a \neq -1$ so we don't have a repeated root.

Then doing Partial Fraction Expansion we get (and we have to do the PFE by hand because we don't know *a*… but it is not that hard!!!)

$$Y(z) = \frac{-ay[-1]z}{z+a} + \frac{\left(\frac{ab}{a+1}\right)z}{z+a} + \frac{\left(\frac{b}{a+1}\right)z}{z-1}$$

Now using ZT Table we get:

$$y[n] = -ay[-1](-a)^n + \frac{b}{a+1}\left[a(-a)^n + (1)^n\right] \qquad n = 0, 1, 2, \ldots$$

**IC-Driven Transient: decays if system is stable**

**Input-Driven Output… 2 Terms:**

**1st term decays (Transient)**
**2nd term persists (Steady State)**

# Solving a Second-order Difference Equation using the ZT

The Given Difference Equation: $$y[n]+a_1 y[n-1]+a_2 y[n-2]=b_0 x[n]+b_1 x[n-1]$$

Assume that the input is causal

Assume you are given ICs: $y[-1]$ & $y[-2]$

**Find the system response $y[n]$ for $n$ = 0, 1, 2, 3, …**

Take the ZT using the non-causal right-shift property:

$$Y(z)+a_1\left(z^{-1}Y(z)+y[-1]\right)+a_2\left(z^{-2}Y(z)+z^{-1}y[-1]+y[-2]\right)$$
$$=b_0 X(z)+b_1 z^{-1}X(z)$$

$$Y(z)=\frac{-a_1 y[-1]-a_2 y[-1]z^{-1}-a_2 y[-2]}{1+a_1 z^{-1}+a_2 z^{-2}}+\underbrace{\frac{b_0+b_1 z^{-1}}{1+a_1 z^{-1}+a_2 z^{-2}}}_{H(z)}X(z)$$

**Due to IC's… decays if system is stable**

**Due to input – will have transient part and steady-state part**

Let's take a look at the IC-Driven transient part:

$$Y_{zi}(z) = \frac{-a_1 y[-1] - a_2 y[-1] z^{-1} - a_2 y[-2]}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{A - Bz^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Multiply top and bottom by $z^2$:

$$Y_{zi}(z) = \frac{Az^2 + Bz}{z^2 + a_1 z + a_2}$$

Now to do an inverse ZT on this requires a bit of trickery…

Take the bottom two entries on the ZT table and form a linear combination:

$$C_1 a^n \cos(\Omega_o n) u[n] + C_2 a^n \sin(\Omega_o n) u[n] \leftrightarrow \frac{C_1 z^2 + a\left(C_2 \sin(\Omega_o) - C_1 \cos(\Omega_o)\right) z}{z^2 - 2a\cos(\Omega_o) z + a^2}$$

$$a = \sqrt{a_2} \qquad \Omega_0 = \cos^{-1}\left[\frac{-a_1}{2\sqrt{a_2}}\right]$$

$$C_1 = A \qquad C_2 = \frac{B}{a\sin(\Omega_0)} - C_1 \frac{\cos(\Omega_0)}{\sin(\Omega_0)}$$

**Compare & Identify**

Finally, by a trig ID we know that

$$C_1 a^n \cos(\Omega_o n)u[n] + C_2 a^n \sin(\Omega_o n)u[n] = Ca^n \cos(\Omega_o n + \theta)u[n]$$

So… all of this machinery leads to the insight that the IC-Driven transient of a second-order system will look like this:

$$y_{zi}[n] = Ca^n \cos(\Omega_o n + \theta)u[n]$$

…where:

1. The frequency $\Omega_0$ and exponential $a$ are set by the Characteristic Eq.
2. The amplitude $C$ and the phase $\theta$ are set by the ICs

$$a = \sqrt{a_2} \qquad \Omega_0 = \cos^{-1}\left[\frac{-a_1}{2\sqrt{a_2}}\right]$$

Note: If $|a_2| < 1$ then we get a decaying response!!

## Solving a N$^{th}$-order Difference Equation using the ZT

$$y[n]+\sum_{i=1}^{N}a_i y[n-i]=\sum_{i=0}^{M}b_i x[n-1]$$

Contains $x[n]$, $x[n\text{-}1]$,…

If this system is causal, we won't have $x[n+1]$, $x[n+2]$, etc. here

$$A(z)=z^N+a_1 z^{N-1}+\ldots+a_{N-1}z+a_N$$

$$B(z)=b_0 z^N+b_1 z^{N-1}+\ldots+b_M z^{N-M}$$

$$C(z)=\text{depends on the IC's}$$

Transforming gives:

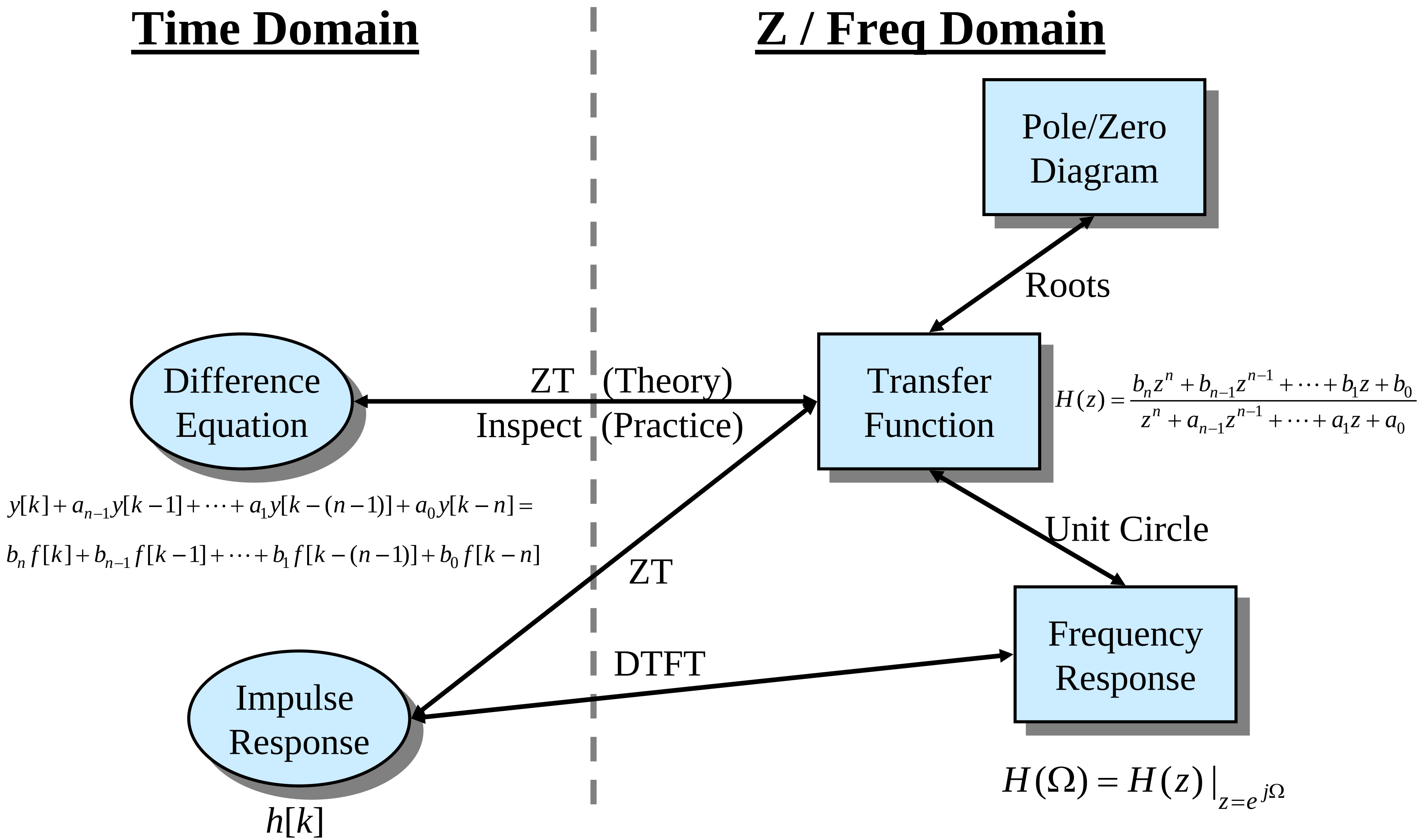
$$Y(z)=\frac{C(z)}{A(z)}+\frac{B(z)}{A(z)}X(z)$$

**Transient and steady state part due to input**

$H(z)$ – transfer function

**Transient part due to ICs**

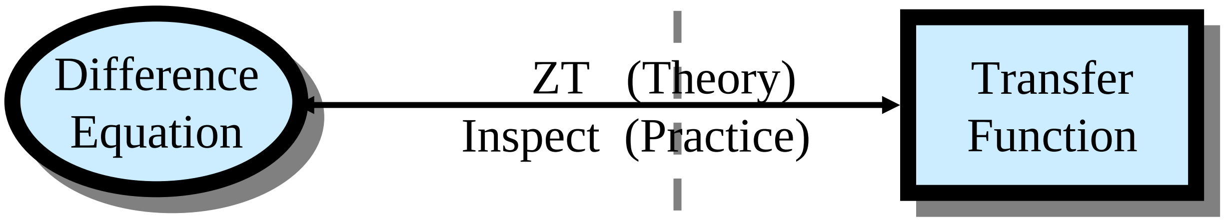
# Discrete-Time System Relationships



# Example System Relationships

## Time Domain

## Z / Freq Domain




Input-Output Form

$$y(n) - \alpha y(n-1) = \beta x(n)$$

$$Y(z)\left[1 - \alpha z^{-1}\right] = \beta X(z)$$

Recursion Form

$$y(n) = \beta x(n) + \alpha y(n-1)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\beta}{1 - \alpha z^{-1}}$$

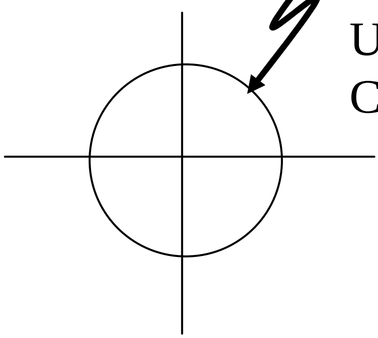
$$H(z) = \frac{\beta z}{z - \alpha}$$

# Example System (cont.)

## Z / Freq Domain

$$z = e^{j\Omega}$$
$$z^{-1} = e^{-j\Omega}$$

On Unit Circle



$$H(z) = \frac{\beta}{1 - \alpha z^{-1}}$$

$$z^{-1} = e^{-j\Omega}$$

$$H(\Omega) = \frac{\beta}{1 - \alpha e^{-j\Omega}}$$

$$\Omega \in (-\pi, \pi]$$

Transfer Function

Unit Circle

Frequency Response

# Example System (cont.)

## Plotting Transfer Function

$$H(\Omega) = \frac{\beta}{1-\alpha e^{-j\Omega}} = \frac{\beta}{1-\left[\alpha\cos(\Omega) - j\alpha\sin(\Omega)\right]}$$

Euler's Equation

$$= \frac{\beta}{\left[1-\alpha\cos(\Omega)\right] + j\alpha\sin(\Omega)}$$

Group into Real & Imag

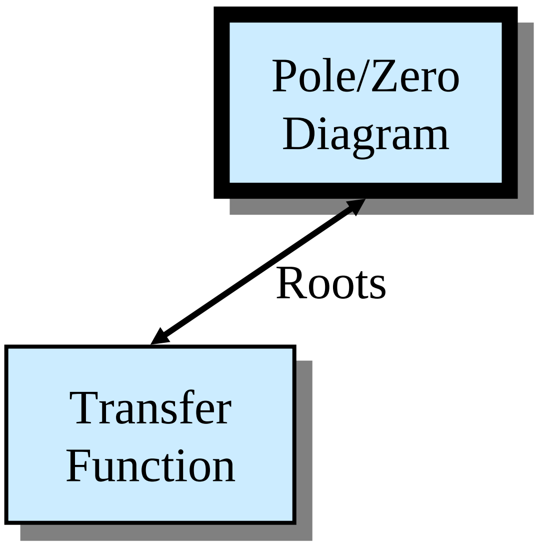
$$\left|H(\Omega)\right| = \frac{\beta}{\sqrt{\left[1-\alpha\cos(\Omega)\right]^2 + \left[\alpha\sin(\Omega)\right]^2}}$$

$$\angle H(\Omega) = -\tan^{-1}\left[\frac{\alpha\sin(\Omega)}{1-\alpha\cos(\Omega)}\right]$$

Standard Eqs for Mag. & Angle

# Example System (cont.)

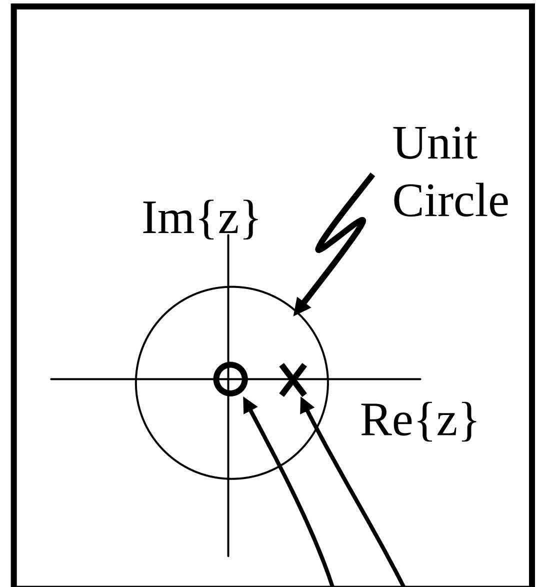
## Z/Freq Domain




$$H(z) = \frac{\beta}{1-\alpha z^{-1}} = \frac{\beta z}{z-\alpha}$$

Zero
Num. = 0
When z = 0

Pole
Den. = 0
When z = α




If $\alpha < 1$: **In**side UC
If $\alpha > 1$: **Out**side UC

# Example System (cont.)