EECE 301
Signals & Systems
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Note Set #34

• D-T Systems: Z-Transform … Solving Difference Eqs. & Transfer Func.
• Reading Assignment: Sections 7.4 – 7.5 of Kamen and Heck
Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).
ZT For Difference Eqs.

Given a difference equation that models a D-T system we may want to solve it:

- with IC’s
- with IC’s of zero

Apply ZT to the Difference Equation
Use the Transfer Function Approach

Note… the ideas here are very much like what we did with the Laplace Transform for CT systems.

We’ll consider the ZT/Difference Eq. approach first…
Solving a First-order Difference Equation using the ZT

Given: \( y[n] + ay[n - 1] = bx[n] \)

\[ IC = y[-1] \]

\[ x[n] \text{ for } n = 0, 1, 2, \ldots \]

Solve for: \( y[n] \) for \( n = 0, 1, 2, \ldots \)

Take ZT of differential equation:

\[ Z\{y[n] + ay[n - 1]\} = Z\{bx[n]\} \]

Use Linearity of ZT

\[ Z\{y[n]\} + aZ\{y[n - 1]\} = bZ\{x[n]\} \]

Need Right-Shift Property… but which one???

Because of the non-zero IC we need to use the non-causal form:

\[ Z\{y[n - 1]\} = z^{-1}Y(z) + y[-1] \]
Using these results gives:

\[ Y(z) + a\left[z^{-1}Y(z) + y[-1]\right] = bX(z) \]

Which is an algebraic equation that can be solved for \( Y(z) \):

\[ Y(z) = \frac{-ay[-1]}{1 + az^{-1}} + \frac{b}{1 + az^{-1}} X(z) \]

Not the best form for doing Inverse ZT… we want things in terms of \( z \) not \( z^{-1} \)

Multiply each term by \( z/z \)

\[ Y(z) = -ay[-1] \left(\frac{z}{z + a}\right) + b\left(\frac{z}{z + a}\right) X(z) \]

On ZT Table

Part due to input signal modified by **Transfer Function**

\[ H(z) = \frac{bz}{z + a} \]

\[ y[n] = -ay[-1](-a)^n u[n] + Z^{-1}\{H(z)X(z)\} \]

If \( |a| < 1 \) this dies out as \( n \uparrow \), its an IC-driven transient

If the ICs are zero, this is all we have!!!
**Ex.: Solving a Difference Equation using ZT: 1st-Order System w/ Step Input**

For \( x[n] = u[n] \) \( \leftrightarrow \) \( X(z) = \frac{z}{z-1} \)

Then using our general results we just derived we get:

\[
Y(z) = -ay[-1]z + \left( \frac{bz}{z+a} \right) \left( \frac{z}{z-1} \right)
\]

For now assume that \( a \neq -1 \) so we don’t have a repeated root.

Then doing Partial Fraction Expansion we get (and we have to do the PFE by hand because we don’t know \( a \)… but it is not that hard!!!)

\[
Y(z) = -ay[-1]z + \left( \frac{ab}{a+1} \right)z + \left( \frac{b}{a+1} \right)z
\]

Now using ZT Table we get:

\[
y[n] = -ay[-1](-a)^n + \frac{b}{a+1} \left[ a(-a)^n + (1)^n \right] \quad n = 0, 1, 2, \ldots
\]

**IC-Driven Transient:**
decays if system is stable

**Input-Driven Output… 2 Terms:**
1\(^{\text{st}}\) term decays (Transient)
2\(^{\text{nd}}\) term persists (Steady State)
Solving a Second-order Difference Equation using the ZT

The Given Difference Equation: \[ y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] \]

Assume that the input is causal

Assume you are given ICs: \( y[-1] \) \& \( y[-2] \)

**Find the system response \( y[n] \) for \( n = 0, 1, 2, 3, \ldots \)**

Take the ZT using the non-causal right-shift property:

\[
Y(z) + a_1 \left( z^{-1} Y(z) + y[-1] \right) + a_2 \left( z^{-2} Y(z) + z^{-1} y[-1] + y[-2] \right) = b_0 X(z) + b_1 z^{-1} X(z)
\]

\[
Y(z) = -\frac{a_1 y[-1] + a_2 y[-1] z^{-1} + a_2 y[-2]}{1 + a_1 z^{-1} + a_2 z^{-2}} + \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} X(z)
\]

**Due to IC’s… decays if system is stable**

\( H(z) \)

Due to input – will have transient part and steady-state part
Let’s take a look at the IC-Driven transient part:

\[
Y_{zi}(z) = \frac{-a_1 y[-1] - a_2 y[-1] z^{-1} - a_2 y[-2]}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{A - B z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}
\]

Multiply top and bottom by \( z^2 \):

\[
Y_{zi}(z) = \frac{A z^2 + B z}{z^2 + a_1 z + a_2}
\]

Now to do an inverse ZT on this requires a bit of trickery…
Take the bottom two entries on the ZT table and form a linear combination:

\[
C_1 a^n \cos(\Omega_0 n) u[n] \leftrightarrow \frac{C_1 z^2 + a (C_2 \sin(\Omega_0) - C_1 \cos(\Omega_0)) z}{z^2 - 2a \cos(\Omega_0) z + a^2} + C_2 a^n \sin(\Omega_0 n) u[n]
\]

\[
a = \sqrt{a_2} \quad \Omega_0 = \cos^{-1} \left[ \frac{-a_1}{2 \sqrt{a_2}} \right] \]
\[
C_1 = A \quad C_2 = \frac{B}{a \sin(\Omega_0)} - \frac{C_1 \cos(\Omega_0)}{\sin(\Omega_0)}
\]

Compare & Identify
Finally, by a trig ID we know that

\[ C_1 a^n \cos(\Omega_o n)u[n] + C_2 a^n \sin(\Omega_o n)u[n] = Ca^n \cos(\Omega_o n + \theta)u[n] \]

So… all of this machinery leads to the insight that the IC-Driven transient of a second-order system will look like this:

\[ y_{zi}[n] = Ca^n \cos(\Omega_o n + \theta)u[n] \]

…where:

1. The frequency \( \Omega_0 \) and exponential \( a \) are set by the Characteristic Eq.
2. The amplitude \( C \) and the phase \( \theta \) are set by the ICs

Note: If \( |a_2| < 1 \) then we get a decaying response!!
Solving a $N^{\text{th}}$-order Difference Equation using the ZT

$$y[n] + \sum_{i=1}^{N} a_i y[n - i] = \sum_{j=0}^{M} b_j x[n - 1]$$

Contains $x[n], x[n-1], \ldots$

If this system is causal, we won’t have $x[n+1], x[n+2], \text{etc. here}$

$$A(z) = z^N + a_1 z^{N-1} + \ldots + a_{N-1} z + a_N$$

$$B(z) = b_0 z^N + b_1 z^{N-1} + \ldots + b_M z^{N-M}$$

C(z) = depends on the IC's

Transforming gives:

$$Y(z) = \frac{C(z)}{A(z)} + \frac{B(z)}{A(z)} X(z)$$

Transient and steady state part due to input

$H(z)$ – transfer function

Transient part due to ICs
Discrete-Time System Relationships

**Time Domain**
- Difference Equation
- Impulse Response
  - $y[k] + a_{n-1}y[k-1] + \cdots + a_1y[k-(n-1)] + a_0y[k-n] = b_n f[k] + b_{n-1} f[k-1] + \cdots + b_1 f[k-(n-1)] + b_0 f[k-n]$  
  - $h[k]$  

**Z / Freq Domain**
- Transfer Function
  - $H(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$
- Pole/Zero Diagram
- Frequency Response
  - $H(\Omega) = H(z) \bigg|_{z = e^{j\Omega}}$
- Roots
- Unit Circle
- DTFT
- ZT (Theory)
- Inspect (Practice)
Example System Relationships

**Time Domain**
- Difference Equation
- Input-Output Form
  \[ y(n) - \alpha y(n-1) = \beta x(n) \]
- Recursion Form
  \[ y(n) = \beta x(n) + \alpha y(n-1) \]

**Z / Freq Domain**
- Transfer Function
  \[ Y(z)[1 - \alpha z^{-1}] = \beta X(z) \]
- 
  \[ H(z) = \frac{Y(z)}{X(z)} = \frac{\beta}{1 - \alpha z^{-1}} \]

\[ H(z) = \frac{\beta z}{z - \alpha} \]
Example System (cont.)

Z / Freq Domain

\[ H(z) = \frac{\beta}{1 - \alpha z^{-1}} \]

\[ H(\Omega) = \frac{\beta}{1 - \alpha e^{-j\Omega}} \]

\[ \Omega \in (-\pi, \pi] \]
Example System (cont.)

Plotting Transfer Function

\[
H(\Omega) = \frac{\beta}{1 - \alpha e^{-j\Omega}} = \frac{\beta}{1 - [\alpha \cos(\Omega) - j\alpha \sin(\Omega)]} = \frac{\beta}{[1 - \alpha \cos(\Omega)] + j\alpha \sin(\Omega)}
\]

\[
|H(\Omega)| = \frac{\beta}{\sqrt{[1 - \alpha \cos(\Omega)]^2 + [\alpha \sin(\Omega)]^2}}
\]

\[
\angle H(\Omega) = -\tan^{-1}\left(\frac{\alpha \sin(\Omega)}{1 - \alpha \cos(\Omega)}\right)
\]

Euler’s Equation

Group into Real & Imag

Standard Eqs for Mag. & Angle
Example System (cont.)

**Z/Freq Domain**

- **Transfer Function**
  \[ H(z) = \frac{\beta}{1 - \alpha z^{-1}} = \frac{\beta z}{z - \alpha} \]

- **Pole/Zero Diagram**

  - **Zero**
    Num. = 0
    When \( z = 0 \)
  - **Pole**
    Den. = 0
    When \( z = \alpha \)

  - If \( \alpha < 1 \): **Inside UC**
  - If \( \alpha > 1 \): **Outside UC**

- **Unit Circle**
  - \( \text{Re}\{z\} \)
  - \( \text{Im}\{z\} \)
Example System (cont.)

Time Domain

Impulse Response

$$h(n) = \beta \alpha^n$$

$$h(n) = \int_{-\pi}^{\pi} \frac{\beta}{1 - \alpha e^{-j\Omega}} d\Omega$$

Use Table if Possible

$$H(\Omega) = \frac{\beta}{1 - \alpha e^{-j\Omega}}$$

$$\Omega \in (-\pi, \pi]$$

Z / Freq Domain

Transfer Function

$$H(z) = \frac{\beta}{1 - \alpha z^{-1}}$$

Unit Circle

Inv. ZT Table

Frequency Response

Inv. DTFT

DTFT

ZT