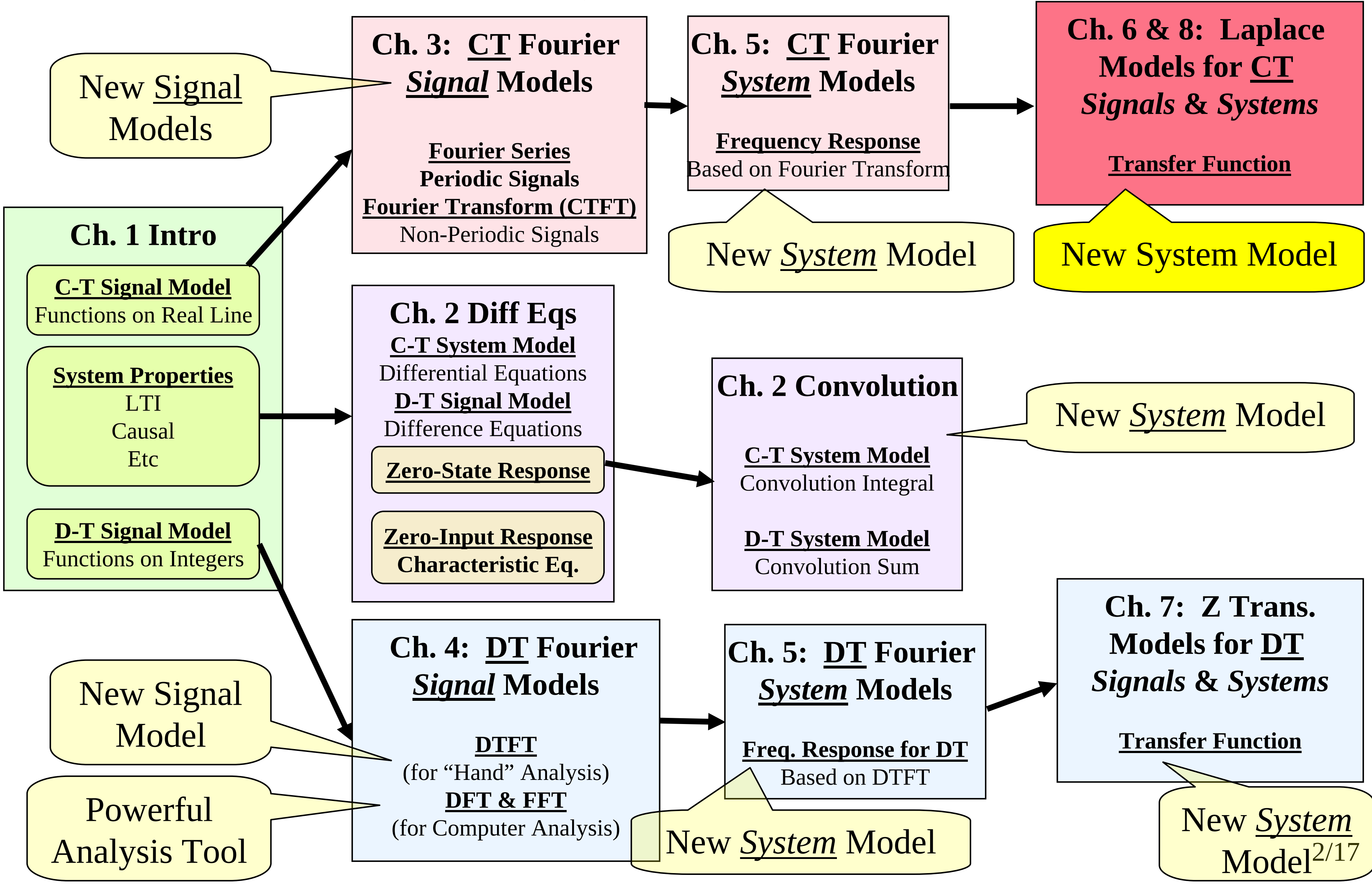
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #29

- C-T Systems: Laplace Transform… Transfer Function
- Reading Assignment: Section 6.5 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



New Signal Models

Ch. 3: CT Fourier *Signal* Models

Fourier Series
Periodic Signals
Fourier Transform (CTFT)
Non-Periodic Signals

Ch. 5: CT Fourier *System* Models

Frequency Response
Based on Fourier Transform

New *System* Model

Ch. 6 & 8: Laplace Models for CT *Signals & Systems*

Transfer Function

New System Model

Ch. 1 Intro

C-T Signal Model
Functions on Real Line

System Properties
LTI
Causal
Etc

D-T Signal Model
Functions on Integers

Ch. 2 Diff Eqs
C-T System Model
Differential Equations
D-T Signal Model
Difference Equations

Zero-State Response

Zero-Input Response
Characteristic Eq.

Ch. 2 Convolution

C-T System Model
Convolution Integral

D-T System Model
Convolution Sum

New *System* Model

New Signal Model

Powerful Analysis Tool

Ch. 4: DT Fourier *Signal* Models

DTFT
(for “Hand” Analysis)
DFT & FFT
(for Computer Analysis)

Ch. 5: DT Fourier *System* Models

Freq. Response for DT
Based on DTFT

New *System* Model

Ch. 7: Z Trans. Models for DT *Signals & Systems*

Transfer Function

New *System* Model

6.5 Transfer Function

We've seen that the system output's LT is:

$$Y(s) = \underbrace{\frac{C(s)}{A(s)}}_{\text{Part due to IC's}} + \underbrace{\frac{B(s)}{A(s)}X(s)}_{\text{Part due to Input}}$$

So, if the system is in zero-state then we only get the second term:

$$\boxed{\begin{gathered} Y(s) = \frac{B(s)}{A(s)}X(s) \\ \Downarrow \\ Y(s) = H(s)X(s) \end{gathered}}$$

Define: $\boxed{H(s) \overset{\Delta}{=} \frac{B(s)}{A(s)}}$

***H*(*s*) = "transfer function"**

⇒ System effect in zero-state case is completely set by the transfer function

Note: If the system is described by a linear, constant coefficient differential equation, we can get $H(s)$ by inspection!! Let's see how…

Recall: $$x^{(n)}(t) \quad \leftrightarrow \quad s^n X(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0) \cdots x^{(n-1)}(0)$$

With zero ICs we have that each higher derivative corresponds to just another power of *s*.

We can then apply this idea to get the Transfer Function…

The condition under which we get the TF

To illustrate…Take the LT of a Diff. Eq. under the zero-state case:

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_1\frac{dx(t)}{dt} + b_0 x(t)$$

$$s^2Y(s) + a_1 sY(s) + a_0 Y(s) = b_1 sX(s) + b_0 X(s)$$

Solve for $Y(s)$ and identify the $H(s)$:

$$Y(s) = \underbrace{\left[\frac{b_1 s + b_0}{s^2 + a_1 s + a_0}\right]}_{=H(s)} X(s)$$

So… now it is possible to directly identify the TF $H(s)$ from the Diff. Eq.:

$$\boxed{\underbrace{\frac{d^2y(t)}{dt^2}+a_1\frac{dy(t)}{dt}+a_0y(t)}=\underbrace{b_1\frac{dx(t)}{dt}+b_0x(t)}}$$

$$\boxed{H(s)=\frac{\boxed{b_1s+b_0}}{\boxed{s^2+a_1s+a_0}}}$$

But, we have also seen that for the zero-state case the system output is:

$$y(t) = \int_0^t h(\lambda)x(t-\lambda)d\lambda = h(t) * x(t)$$

These limits arise by assuming $h(t)$ and $x(t)$ are causal

But…We have an LT property for convolution that says:

$$\boxed{Y(s) = H(s)X(s) \quad \leftrightarrow \quad y(t) = h(t) * x(t)}$$

where: $\boxed{H(s) = \mathcal{L}\{h(t)\}}$

$$\boxed{\text{Transfer Function} = \mathcal{L}\{\text{Impulse Response}\}}$$

So, we have two ways to get $H(s)$

- Inspect the Diff. Eq. and identify the transfer function $H(s)$
- Take the LT of the impulse response $h(t)$

This gives an easy way to get the impulse response from a Diff. Eq.:

- Identify the $H(s)$ from he Diff. Eq. and then find the ILT of that

Note that the transfer function does essentially the same thing that the frequency response does…

$$\boxed{Y(s) = H(s)X(s) \quad \leftrightarrow \quad y(t) = h(t) * x(t)}$$

$$\boxed{Y(\omega) = H(\omega)X(\omega) \quad \leftrightarrow \quad y(t) = h(t) * x(t)}$$

Recall: If the ROC of $H(s)$ includes the $j\omega$ axis, then

$$\boxed{H(\omega) = H(s)\Big|_{s=j\omega}}$$

This is the connection between The transfer function and frequency response.

Recall that the LT is a generalized, more-powerful version of the FT… this result just says that we can do the same thing with $H(s)$ that we did with $H(\omega)$, but we can do it for a larger class of systems…

There are some systems for which we can use either method… those are the ones for which the ROC of $H(s)$ includes the $j\omega$ axis.

So… we know that $H(s)$ is completely described by the Diff. Eq.… Therefore we should expect that we can tell a lot about a system by looking at the structure of the transfer function $H(s)$… This structure is captured in the idea of “Poles” and “Zeros”…

<u>Poles and Zeros of a system</u>

Given a system with Transfer Function:

$$H(s)=\frac{b_M s^M + b_{M-1}s^{M-1} + \ldots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + a_0}\quad \begin{array}{l} \leftarrow B(s) \\ \leftarrow A(s) \end{array}$$

We can factor $B(s)$ and $A(s)$: (Recall: $A(s)$ = characteristic polynomial)

$$H(s)=\frac{b_M(s-z_1)(s-z_2)\ldots(s-z_M)}{(s-p_1)(s-p_2)\ldots(s-p_N)}$$

Assume any common factors in $B(s)$ and $A(s)$ have been cancelled out

<u>Note:</u> $H(s)\big|_{s=z_i}=0 \qquad i=1,2,\ldots,M$

$\{z_i\}$ are called "zeros of $H(s)$"

$H(s)\big|_{s=p_i}=\infty \qquad i=1,2,\ldots,N$

$\{p_i\}$ are called "poles of $H(s)$"

Note: p_i are the roots of the char. polynomial

Note that knowing the sets $\{z_i\}_{i=1}^{M}$ & $\{p_i\}_{i=1}^{N}$

tells us what $H(s)$ is: (up to the multiplicative scale factor b_M)

-b_M is like a gain (i.e. amplification)

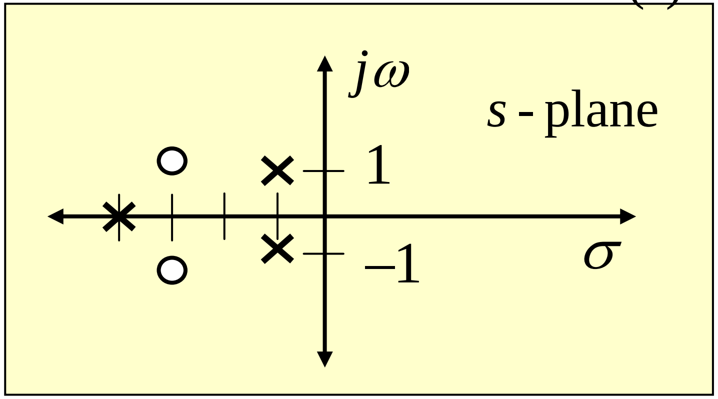
Pole-Zero Plot

This gives us a graphical view of the system's behavior

Example: $$H(s) = \frac{2s^2 + 12s + 20}{s^3 + 6s^2 + 10s + 8} = \frac{2(s+3-j)(s+3+j)}{(s+4)(s+1-j)(s+1+j)}$$

Real coefficients ⇒ complex conjugate pairs

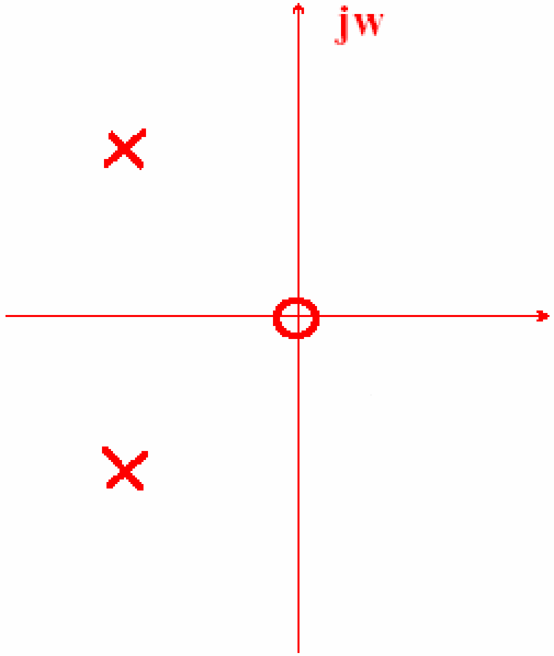
Pole-Zero Plot for this $H(s)$



x denotes a pole

o denotes a zero

From the Pole-Zero Plots we can Visualize the TF function on the s-plane:



Plot of |H(s)| vs. s
(i.e., plot of |H(s)| over the s-plane)

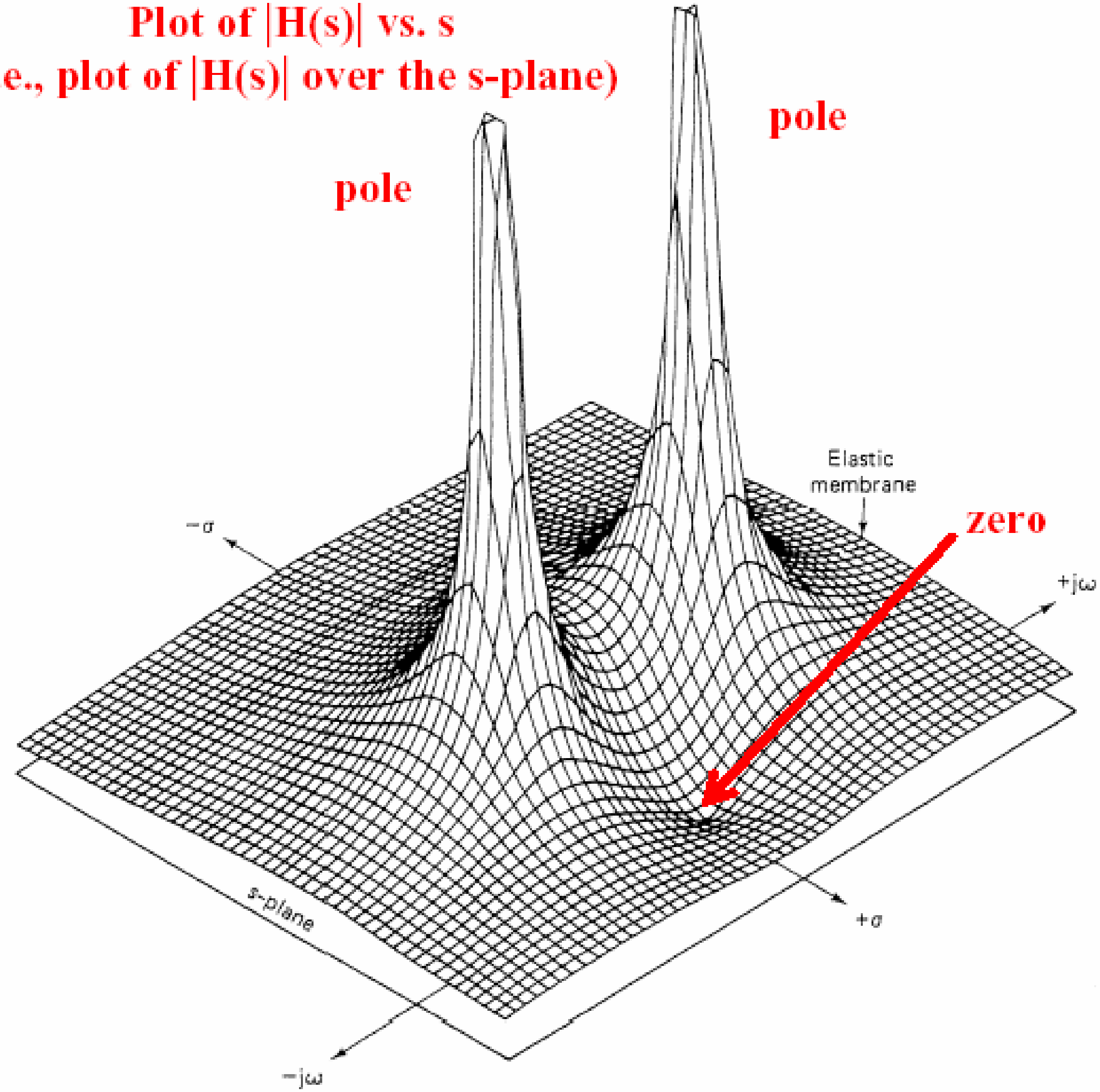


Fig. 9-4. Shape of elastic membrane for a pair of poles and a zero.

From our Visualization of the TF function on the s-plane we can see the Freq. Resp.:

To get the frequency response H(w) from the transfer function H(s) we replace s by jw… this is graphically equivalent to "cutting along the jw axis"

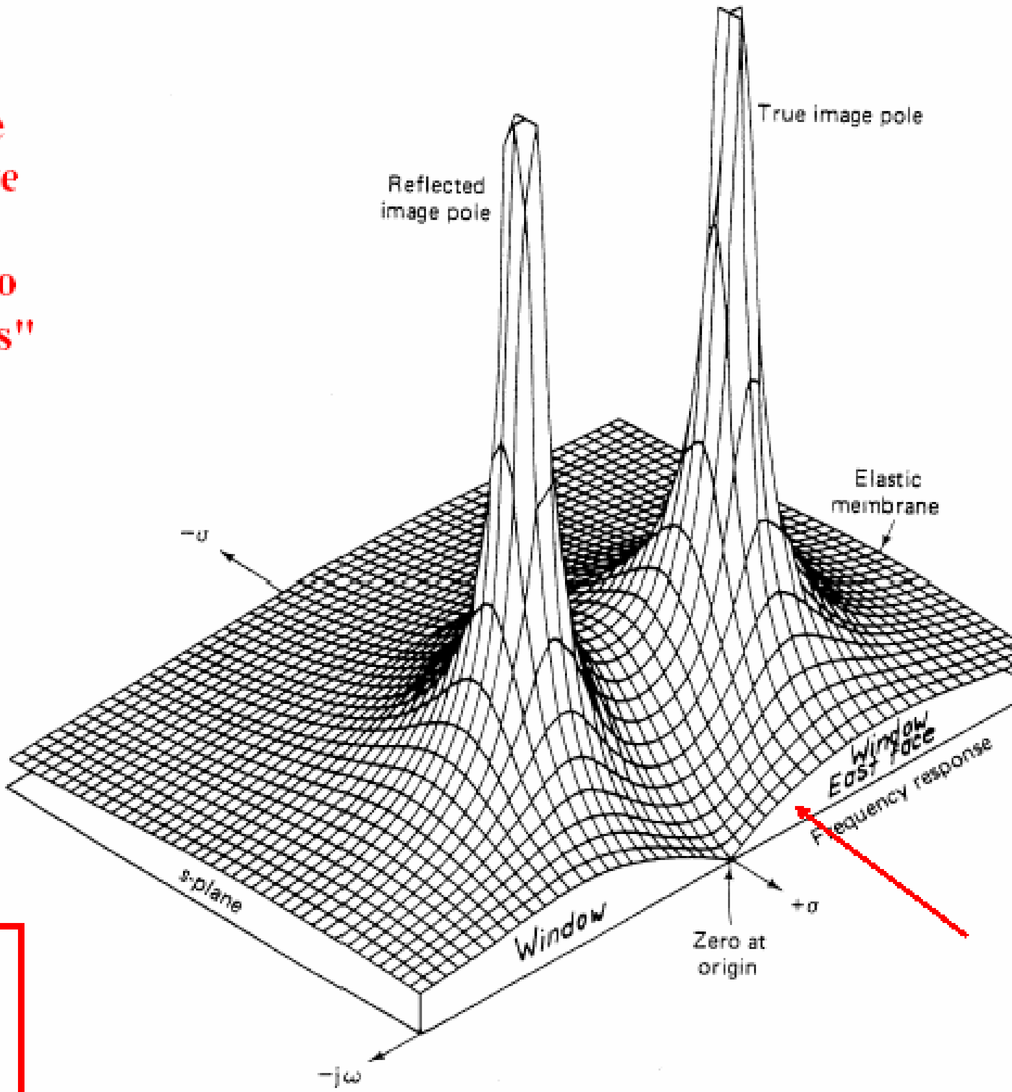
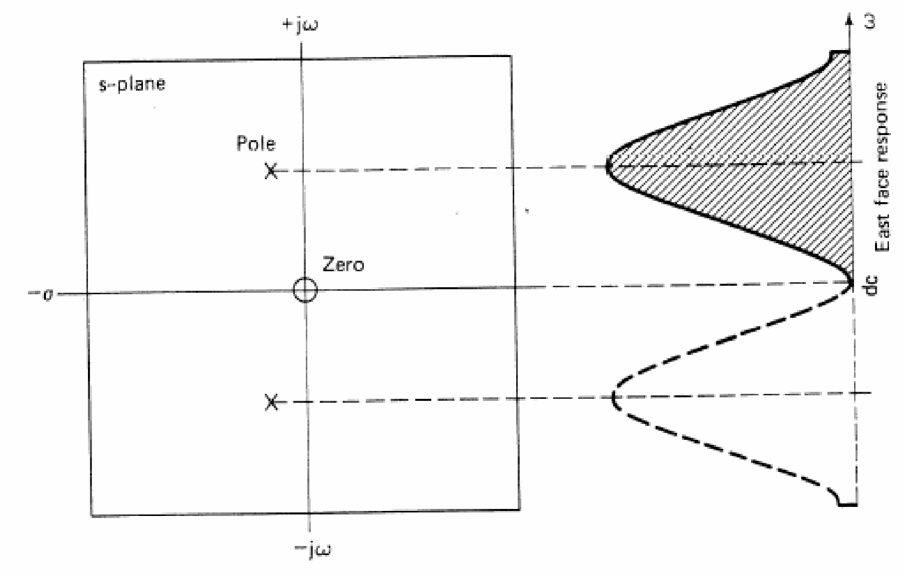


Fig. 9-5. Pole-zero diagram showing the east face.

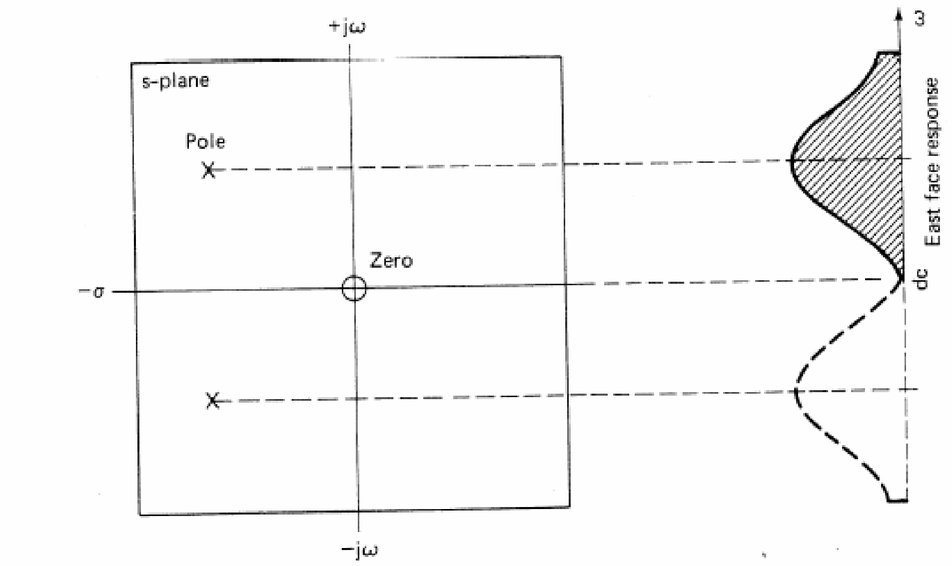
shows the "cut" along the jw axis

Plots from my favorite book on Op Amps: "Operational Amplifier: Characteristics and Applications" by Robert G. Irvine, Prentice-Hall, 1981

Can also look at a pole-zero plot and see the effects on Freq. Resp.



(a) Frequency response for pole close to jf axis



(b) Frequency response for pole far from jf axis

Fig. 9-6. Frequency response versus pole location.

As the pole moves closer to the $j\omega$ axis it has a stronger effect on the frequency response $H(\omega)$. Poles close to the $j\omega$ axis will yield sharper and taller bumps in the frequency response.

By being able to visualize what $|H(s)|$ will look like based on where the poles and zeros are, an engineer gains the ability to know what kind of transfer function is needed to achieve a desired frequency response… then through accumulated knowledge of electronic circuits (requires experience accumulated AFTER graduation) the engineer can devise a circuit that will achieve the desired effect.

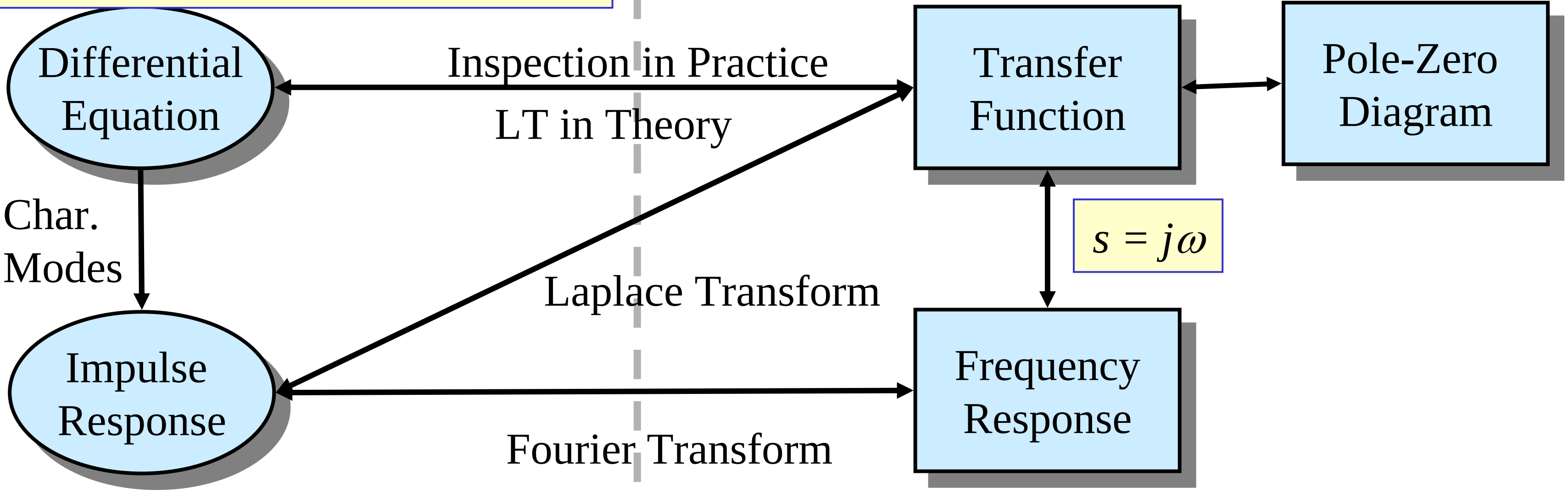
Continuous-Time System Relationships

Time Domain

Freq Domain

$$\frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) =$$
$$b_M\frac{d^M x(t)}{dt^M} + b_{M-1}\frac{d^{M-1}x(t)}{dt^{M-1}} + \cdots + b_1\frac{dx(t)}{dt} + b_0 x(t)$$

$$H(s) = \frac{b_M s^M + b_{M-1}s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}$$



Differential Equation

Inspection in Practice

LT in Theory

Transfer Function

Pole-Zero Diagram

Char. Modes

$s = j\omega$

Laplace Transform

Impulse Response

Frequency Response

Fourier Transform

$h(t)$

$H(\omega) = H(s)|_{s=j\omega}$

This Chart provides a "Roadmap" to the CT System Relationships!!!

In practice you may need to start your work in any spot on this diagram…

1. From the differential equation you can get:
   a. Transfer function, then the impulse response, the pole-zero plot, and if allowable you can get the frequency response
2. From the impulse response you can get:
   a. Transfer function, then the Diff. Eq., the pole-zero plot, and if allowable you can get the frequency response
3. From the Transfer Function you can get:
   a. Diff. Eq., the impulse response, the pole-zero plot, and if allowable you can get the frequency response
4. From the Frequency Response you can get:
   a. Transfer function, then the Diff. Eq., the pole-zero plot, and the impulse response
5. From the Pole-Zero Plot you can get:
   a. (up to a scaling factor) Transfer function, then the Diff. Eq., the impulse response, and possible the Frequency Response

In Practice we often get the transfer function from a circuit… Here's an example:

Recall: We get the frequency response from a circuit by using frequency dependent impedances…

$$Z_R(\omega) = R \qquad Z_C(\omega) = \frac{1}{j\omega C} \qquad Z_L(\omega) = j\omega L$$

…and then doing circuit analysis.

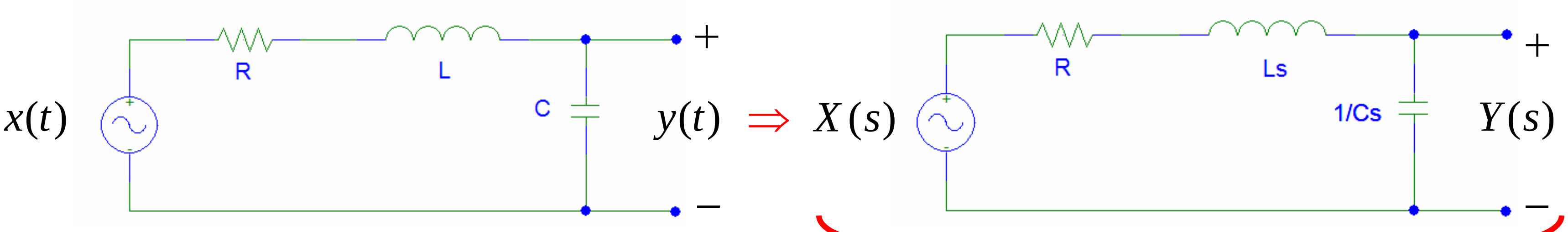
Similarly, we can get the transfer function using the *s*-domain impedances:

$$\boxed{Z_R(s) = R \qquad Z_C(s) = \frac{1}{sC} \qquad Z_L(s) = sL}$$

<u>**Example 6.37**</u>

(Assume zero-state)

We must assume this to find the transfer function



Now we treat everything here as if we have a "DC Circuit"… which leads to simple algebraic manipulation rather than differential equations!

For this circuit the easiest approach is to use the <u>Voltage Divider</u>

$$Y(s) = \left[\frac{1/Cs}{R + Ls + (1/Cs)}\right] X(s) \qquad \Rightarrow \qquad = \underbrace{\left[\frac{1/LC}{s^2 + (R/L)s + (1/LC)}\right]}_{H(s)} X(s)$$

A "standard" form:
- a ratio of two polynomials in *s*
- unity coefficient on the highest power in the denominator

Some comments:

For RLC circuits, the ROC always includes $j\omega$

Once you start including linear amplifiers with gain > 1 this may not be true

If you include non-linear devices ⇒ the system becomes non-linear

But, you may be able to "linearize" the system over a small operating range

E.g. – A transistor can be used to build a (nearly) linear amplifier even though the transistor is itself a non-linear device