EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #29

• C-T Systems: Laplace Transform… Transfer Function
• Reading Assignment: Section 6.5 of Kamen and Heck
Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).
6.5 Transfer Function

We’ve seen that the system output’s LT is:

\[ Y(s) = \frac{C(s)}{A(s)} + \frac{B(s)}{A(s)} X(s) \]

Part due to IC’s Part due to Input

So, if the system is in zero-state then we only get the second term:

\[ Y(s) = \frac{B(s)}{A(s)} X(s) \]

Define: \( H(s) \triangleq \frac{B(s)}{A(s)} \)

\[ H(s) = \text{“transfer function”} \]

\[ \Rightarrow \text{System effect in zero-state case is completely set by the transfer function} \]
Note: If the system is described by a linear, constant coefficient differential equation, we can get \( H(s) \) by inspection!! Let’s see how…

Recall:

\[
x^{(n)}(t) \leftrightarrow s^n X(s) - s^{n-1} x(0) - s^{n-2} \dot{x}(0) \cdots x^{(n-1)}(0)
\]

With zero ICs we have that each higher derivative corresponds to just another power of \( s \).

We can then apply this idea to get the Transfer Function…

To illustrate…Take the LT of a Diff. Eq. under the zero-state case:

\[
\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_1 \frac{dx(t)}{dt} + b_0 x(t)
\]

\[
s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = b_1 s X(s) + b_0 X(s)
\]

Solve for \( Y(s) \) and identify the \( H(s) \):

\[
Y(s) = \left[ \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \right] X(s)
\]

The condition under which we get the TF
So… now it is possible to directly identify the TF $H(s)$ from the Diff. Eq.: 

\[
\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_1 \frac{dx(t)}{dt} + b_0 x(t)
\]

\[
H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}
\]
But, we have also seen that for the zero-state case the system output is:

\[ y(t) = \int_0^t h(\lambda)x(t - \lambda)\,d\lambda = h(t) * x(t) \]

These limits arise by assuming \( h(t) \) and \( x(t) \) are causal.

But…We have an LT property for convolution that says:

\[ Y(s) = H(s)X(s) \iff y(t) = h(t) * x(t) \]

where: \( H(s) = \mathcal{L}\{h(t)\} \)

Transfer Function = \( \mathcal{L}\{\text{Impulse Response}\} \)

So, we have two ways to get \( H(s) \)
- Inspect the Diff. Eq. and identify the transfer function \( H(s) \)
- Take the LT of the impulse response \( h(t) \)

This gives an easy way to get the impulse response from a Diff. Eq.:
- Identify the \( H(s) \) from he Diff. Eq. and then find the ILT of that
Recall that the transfer function does essentially the same thing that the frequency response does…

\[ Y(s) = H(s)X(s) \quad \leftrightarrow \quad y(t) = h(t) * x(t) \]

\[ Y(\omega) = H(\omega)X(\omega) \quad \leftrightarrow \quad y(t) = h(t) * x(t) \]

Recall: If the ROC of \( H(s) \) includes the \( j\omega \) axis, then

\[ H(\omega) = H(s) \bigg|_{s=j\omega} \]

This is the connection between
The transfer function and frequency response.

Recall that the LT is a generalized, more-powerful version of the FT… this result just says that we can do the same thing with \( H(s) \) that we did with \( H(\omega) \), but we can do it for a larger class of systems…

There are some systems for which we can use either method… those are the ones for which the ROC of \( H(s) \) includes the \( j\omega \) axis.
So… we know that $H(s)$ is completely described by the Diff. Eq…. Therefore we should expect that we can tell a lot about a system by looking at the structure of the transfer function $H(s)$… This structure is captured in the idea of “Poles” and “Zeros”…

**Poles and Zeros of a system**

Given a system with Transfer Function:

$$H(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \ldots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0}$$

We can factor $B(s)$ and $A(s)$: (Recall: $A(s) =$ characteristic polynomial)

$$H(s) = \frac{b_M (s - z_1)(s - z_2)\ldots(s - z_M)}{(s - p_1)(s - p_2)\ldots(s - p_N)}$$

Assume any common factors in $B(s)$ and $A(s)$ have been cancelled out

**Note:**

- $H(s)\big|_{s=z_i} = 0$ \quad $i = 1, 2, \ldots, M$
  - $\{z_i\}$ are called "zeros of $H(s)$"

- $H(s)\big|_{s=p_i} = \infty$ \quad $i = 1, 2, \ldots, N$
  - $\{p_i\}$ are called "poles of $H(s)$"

Note: $p_i$ are the roots of the char. polynomial
Note that knowing the sets $\{z_i\}_{i=1}^M$ and $\{p_i\}_{i=1}^N$ tells us what $H(s)$ is: (up to the multiplicative scale factor $b_M$)

$-b_M$ is like a gain (i.e. amplification)

Pole-Zero Plot

This gives us a graphical view of the system’s behavior

Example: $H(s) = \frac{2s^2 + 12s + 20}{s^3 + 6s^2 + 10s + 8} = \frac{2(s + 3 - j)(s + 3 + j)}{(s + 4)(s + 1 - j)(s + 1 + j)}$

Real coefficients $\Rightarrow$ complex conjugate pairs

Pole-Zero Plot for this $H(s)$

$\text{x denotes a pole}$

$\text{o denotes a zero}$
From the Pole-Zero Plots we can visualize the TF function on the s-plane:

Plot of $|H(s)|$ vs. $s$  
(i.e., plot of $|H(s)|$ over the s-plane)

Fig. 9-4. Shape of elastic membrane for a pair of poles and a zero.
From our Visualization of the TF function on the s-plane we can see the Freq. Resp.:

To get the frequency response $H(w)$ from the transfer function $H(s)$ we replace $s$ by $jw$... this is graphically equivalent to "cutting along the $jw$ axis"


Fig. 9-5. Pole-zero diagram showing the east face.
As the pole moves closer to the $j\omega$ axis it has a stronger effect on the frequency response $H(\omega)$. Poles close to the $j\omega$ axis will yield sharper and taller bumps in the frequency response.

By being able to visualize what $|H(s)|$ will look like based on where the poles and zeros are, an engineer gains the ability to know what kind of transfer function is needed to achieve a desired frequency response... then through accumulated knowledge of electronic circuits (requires experience accumulated AFTER graduation) the engineer can devise a circuit that will achieve the desired effect.
Continuous-Time System Relationships

**Time Domain**

\[
\frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =
\]

\[
b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)
\]

**Freq Domain**

\[
H(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}
\]

This Chart provides a “Roadmap” to the CT System Relationships!!!
In practice you may need to start your work in any spot on this diagram…

1. From the differential equation you can get:
   a. Transfer function, then the impulse response, the pole-zero plot, and if allowable you can get the frequency response

2. From the impulse response you can get:
   a. Transfer function, then the Diff. Eq., the pole-zero plot, and if allowable you can get the frequency response

3. From the Transfer Function you can get:
   a. Diff. Eq., the impulse response, the pole-zero plot, and if allowable you can get the frequency response

4. From the Frequency Response you can get:
   a. Transfer function, then the Diff. Eq., the pole-zero plot, and the impulse response

5. From the Pole-Zero Plot you can get:
   a. (up to a scaling factor) Transfer function, then the Diff. Eq., the impulse response, and possible the Frequency Response
Similarly, we can get the transfer function using the \( s \)-domain impedances:

\[
Z_R(s) = R \quad Z_C(s) = \frac{1}{sC} \quad Z_L(s) = sL
\]
Example 6.37  (Assume zero-state)

We must assume this to find the transfer function

Now we treat everything here as if we have a “DC Circuit”… which leads to simple algebraic manipulation rather than differential equations!

For this circuit the easiest approach is to use the Voltage Divider

\[
Y(s) = \frac{1/Cs}{R + Ls + (1/Cs)} X(s) \Rightarrow \frac{1/LC}{s^2 + (R/L)s + (1/LC)} X(s)
\]

A “standard” form:
- a ratio of two polynomials in \( s \)
- unity coefficient on the highest power in the denominator
Some comments:

For RLC circuits, the ROC always includes $j\omega$

Once you start including linear amplifiers with gain $> 1$ this may not be true

If you include non-linear devices $\Rightarrow$ the system becomes non-linear

But, you may be able to “linearize” the system over a small operating range

E.g. – A transistor can be used to build a (nearly) linear amplifier even though the transistor is itself a non-linear device