EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #28
• C-T Systems: Laplace Transform… Solving Differential Equations
• Reading Assignment: Section 6.4 of Kamen and Heck
Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).
6.4 Using LT to solve Differential Equations

In Ch. 2 we saw that the solution to a linear differential equation has two parts:

\[ y_{total}(t) = y_{zs}(t) + y_{zi}(t) \]

Here we’ll see how to get \( y_{total}(t) \) using LT… … get both parts with one tool!!!
**First-order case:** Let’s see this for a 1\textsuperscript{st}-order Diff. Eq. with a causal input and a non-zero initial condition just before the causal input is applied.

The 1\textsuperscript{st}-order Diff. Eq. describes: a simple RC or RL circuit.

The causal input means: we switch on some input at time $t = 0$.

The initial condition means: just before we switch on the input the capacitor has a specified voltage on it (i.e., it holds some charge).

**Input:** Time-Varying Voltage (e.g., guitar, microphone, etc.)

**Output:** Time-Varying Voltage

Assume that for $t<0$ this has been switched on for “a long time”

Thus… the cap is fully charged to $V_{IC}$ volts
This circuit is then described by this Diff. Eq.:

\[
\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)
\]

With IC \( y(0^-) = V_{IC} \)

\( x(t) = 0, \quad t < 0 \)

Cap voltage… just before \( x(t) \) “turns on”

For this ex. we’ll solve the general 1st-order Diff. Eq.:

\[
\frac{dy(t)}{dt} + ay(t) = bx(t)
\]

Now the **key steps** in using the LT are:

- take the LT of both sides of the Differential Equation…
- use the LT properties where appropriate…
- solve the resulting Algebraic Equation for \( Y(s) \)
- find the inverse LT of the resulting \( Y(s) \)

Laplace Transform:

Differential Equation… turns into an… Algebraic Equation

Hard to solve

Easy to solve
We now apply these steps to the 1st-order Diff. Eq.:

\[ \mathcal{L}\left\{ \frac{dy(t)}{dt} + ay(t) \right\} = \mathcal{L}\{bx(t)\} \]

Apply LT to both sides

\[ \mathcal{L}\left\{ \frac{dy(t)}{dt} \right\} + a\mathcal{L}\{y(t)\} = b\mathcal{L}\{x(t)\} \]

Use Linearity of LT

\[ \left[sY(s) - y(0^-)\right] + aY(s) = bX(s) \]

Use Property for LT of Derivative… accounting for the IC

\[ Y(s) = \frac{y(0^-)}{s + a} + \frac{b}{s + a}X(s) \]

Solve algebraic equation for \( Y(s) \)

Part of sol’n driven by IC

"Zero-Input Sol’n"

Part of sol’n driven by input

"Zero-State Sol’n"

Note that \( 1/(s+a) \) plays a role in both parts…

Hey! \( s+a \) is the Characteristic Polynomial!!

Now… the “hard” part is to find the inverse LT of \( Y(s) \)
Example: RC Circuit

Now we apply these general ideas to solving for the output of the previous RC circuit with a unit step input…. \( x(t) = u(t) \)

\[
\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)
\]

\[
Y(s) = \frac{y(0^-)}{s + 1/RC} + \left[ \frac{1}{s + 1/RC} \right] \frac{1}{RC} X(s)
\]

This “transfers” the input \( X(s) \) to the output \( Y(s) \)

We’ll see this later as “The Transfer Function”

Now… we need the LT of the input…

From the LT table we have:

\[
x(t) = u(t) \iff X(s) = \frac{1}{s}
\]

\[
Y(s) = \frac{y(0^-)}{s + 1/RC} + \left[ \frac{1}{s + 1/RC} \right] \frac{1}{RC} \frac{1}{s}
\]

Now we have “just a function of \( s \)” to which we apply the ILT…
So now applying the ILT we have:

\[ \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ \frac{y(0^-)}{s + 1/RC} + \frac{1/RC}{(s + 1/RC)s} \right\} \]

Apply LT to both sides

\[ y(t) = \mathcal{L}^{-1}\left\{ \frac{y(0^-)}{s + 1/RC} \right\} + \mathcal{L}^{-1}\left\{ \frac{1/RC}{(s + 1/RC)s} \right\} \]

Linearity of LT

This part (zero-input sol’n) is easy…
Just look it up on the LT Table!!

This part (zero-state sol’n) is harder…
It is NOT on the LT Table!!

\[ \mathcal{L}^{-1}\left\{ \frac{y(0^-)}{s + 1/RC} \right\} = y(0^-)e^{-(t/RC)}u(t) \]

So… the part of the sol’n due to the IC (zero-input sol’n) decays down from the IC voltage
Now let’s find the other part of the solution… the zero-state sol’n… the part that is driven by the input:

\[ y(t) = \mathcal{L}^{-1} \left\{ \frac{y(0^-)}{s + 1/RC} \right\} + \mathcal{L}^{-1} \left\{ \frac{1/RC}{(s + 1/RC)s} \right\} \]

We can **factor** this function of \( s \) as follows:

\[ \mathcal{L}^{-1} \left\{ \frac{1/RC}{(s + 1/RC)s} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{s + 1/RC} \right\} \]

Now… each of these terms **is** on the LT table:

\[ = \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{s + 1/RC} \right\} \]

\[ = u(t) - e^{-(t/RC)} u(t) \]

\[ = [1 - e^{-(t/RC)}] u(t) \]
So the zero-state response of this system is: 

\[ 1 - e^{-(t/RC)} \] \( u(t) \)

Now putting this zero-state response together with the zero-input response we found gives:

\[ y(t) = y(0^-) e^{-(t/RC)} u(t) + \left[ 1 - e^{-(t/RC)} \right] u(t) \]

Notice that:

The IC Part "Decays Away"

but...

The Input Part "Persists"
Here is an example for $RC = 0.5 \text{ sec}$ and the initial $V_{IC} = 5 \text{ volts}$:
**Second-order case**

Circuits with two energy-storing devices (C & L, or 2 Cs or 2 Ls) are described by a second-order Differential Equation…

\[
\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_1 \frac{dx(t)}{dt} + b_0 x(t)
\]

w/ ICs \( \dot{y}(0^-) \) & \( y(0^-) \)

Assume Causal Input

\( x(t) = 0 \quad t < 0 \)

\( x(0^-) = 0 \)

We solve the 2\textsuperscript{nd}-order case using the same steps:

Take LT of Diff. Equation:

\[
[s^2Y(s) - y(0^-)s - \dot{y}(0^-)] + a_1 [sY(s) - y(0^-)] + a_0 Y(s) = b_1 sX(s) + b_0 X(s)
\]

From 2\textsuperscript{nd} derivative property, accounting for ICs

From 1\textsuperscript{st} derivative property, accounting for ICs

From 1\textsuperscript{st} derivative property, causal signal
Solve for $Y(s)$:

$$Y(s) = \frac{y(0^-)s + y'(0^-) + a_1y(0^-)}{s^2 + a_1s + a_0} + \left[ \frac{b_1s + b_0}{s^2 + a_1s + a_0} \right] X(s)$$

Part of sol’n driven by IC

“Zero-Input Sol’n”

Note this shows up in both places… it is the Characteristic Equation

Part of sol’n driven by input

“Zero-State Sol’n”

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Note: The role the **Characteristic Equation** plays here!

It just pops up in the LT method!

The same happened for a 1st-order Diff. Eq…

…and it happens for all orders

Like before…

to get the solution in the time domain find the Inverse LT of $Y(s)$
To get a feel for this let’s look at the zero-input solution for a 2nd-order system:

\[
Y_{zi}(s) = \frac{y(0^-)s + \dot{y}(0^-) + a_1y(0^-)}{s^2 + a_1s + a_0} = \frac{y(0^-)s + [\dot{y}(0^-) + a_1y(0^-)]}{s^2 + a_1s + a_0}
\]

which has… either a 1st-order or 0th-order polynomial in the numerator and…

… a 2nd-order polynomial in the denominator

For such scenarios there are **Two LT Pairs that are Helpful**:

For… \(0 < |\zeta| < 1\)

\[
Ae^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2} t\right) u(t)
\]

where: \(A = \frac{\alpha}{\omega_n \sqrt{1 - \zeta^2}}\)

\[
Ae^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2} t + \phi\right) u(t)
\]

where: \(A = \beta \sqrt{\frac{(\alpha - \zeta\omega_n)^2}{\omega_n^2 (1 - \zeta^2)}} + 1\)

\(\phi = \tan^{-1}\left(\frac{\omega_n \sqrt{1 - \zeta^2}}{\alpha - \zeta\omega_n}\right)\)

\[
\frac{\alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

\[
\frac{s + \alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

These are not in your book’s table… but they are on the table on my website!

Otherwise… Factor into two terms
Note the effect of the ICs:

\[ Y_{zi}(s) = \frac{y(0^-)s + \dot{y}(0^-) + a_1y(0^-)}{s^2 + a_1s + a_0} = \frac{y(0^-)s + [\dot{y}(0^-) + a_1y(0^-)]}{s^2 + a_1s + a_0} \]

\[ Ae^{-\zeta \omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2} t\right) u(t) \]

This form gives \( y_{zi}(0) = 0 \) as set by the IC

\[ \frac{\alpha}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

If \( y(0^-) = 0 \)

\[ Ae^{-\zeta \omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2} t + \phi\right) u(t) \]

\[ \frac{s + \alpha}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

Otherwise
Example of using this type of LT pair: Let $y(0^-) = 2, \dot{y}(0^-) = 4$

Then

$$Y_{zi}(s) = \frac{2s + (4 + a_12)}{s^2 + a_1s + a_0} = 2\left[\frac{s + (2 + a_1)}{s^2 + a_1s + a_0}\right]$$

Now assume that for our system we have: $a_0 = 100$ & $a_1 = 4$

Then

$$Y_{zi}(s) = 2\left[\frac{s + 6}{s^2 + 4s + 100}\right]$$

Compare to LT:

$$\frac{s + \alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

And identify:

$$\alpha = 6, \beta = 2$$

$$\omega_n^2 = 100 \Rightarrow \omega_n = 10$$

$$2\zeta\omega_n = 4 \Rightarrow \zeta = 4 / 2\omega_n = 4 / 20 = 0.2$$

Pulled a 2 out from each term in Num. to get form just like in LT Pair.
So now we use these parameters in the time-domain side of the LT pair:

\[
\alpha = 6 \quad \beta = 2 \\
\omega_n = 10 \\
\zeta = 0.2
\]

\[
A = \beta \sqrt{\frac{(\alpha - \zeta \omega_n)^2}{\omega_n^2(1 - \zeta^2)}} + 1 = 2 \sqrt{\frac{(6 - 0.2 \times 10)^2}{100(1 - 0.2^2)}} + 1 = 2.16 \text{ volts}
\]

\[
\phi = \tan^{-1}\left(\frac{\omega_n \sqrt{1 - \zeta^2}}{\alpha - \zeta \omega_n}\right) = \tan^{-1}\left(\frac{10 \sqrt{1 - 0.2^2}}{6 - 0.2 \times 10}\right) = 1.18 \text{ rad}
\]

\[
y_{zi}(t) = 2.16e^{-2t} \sin[9.80t + 1.18] u(t)
\]

Assuming output is a voltage!

Notice that the zero-input solution for this 2nd-order system oscillates…
1st-order systems can’t oscillate…
2nd- and higher-order systems can oscillate but might not!!
Here is what this zero-input solution looks like:

\[ y_{zi}(t) = 2.16e^{-2t} \sin[9.80t + 1.18] u(t) \]

Notice that it satisfies the ICs!!

\[ y(0^-) = 2 \quad \dot{y}(0^-) = 4 \]
\textbf{N}^\text{th}-\text{Order Case}

\textbf{Diff. eq of the system}

\[
\frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_M \frac{dx^M(t)}{dt^M} + b_1 \frac{dx(t)}{dt} + b_0 x(t)
\]

For \(M \leq N\) and \(\frac{d^i x(t)}{dt^i}\bigg|_{t=0^-} = 0\) \(i = 0, 1, 2, \ldots, M - 1\)

Taking LT and re-arranging gives:

\[
Y(s) = \frac{IC(s)}{A(s)} + \frac{B(s)}{A(s)} X(s)
\]

LT of the solution (i.e. the LT of the system output)

where

\[
\begin{align*}
A(s) &= s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + a_0 & \text{“output-side” polynomial} \\
B(s) &= b_M s^M + \ldots + b_1 s + b_0 & \text{“input-side” polynomial} \\
IC(s) &= \text{polynomial in } s \text{ that depends on the ICs}
\end{align*}
\]

Recall: For 2\textsuperscript{nd} order case: \(IC(s) = y(0^-)s + \left[\dot{y}(0^-) + a_1 y(0^-)\right]\)
Consider the case where the LT of $x(t)$ is rational: $X(s) = \frac{N_X(s)}{D_X(s)}$

Then…

$$Y(s) = \frac{IC(s)}{A(s)} + \frac{B(s)}{A(s)} X(s) = \frac{IC(s)}{A(s)} + \frac{B(s) N_X(s)}{A(s) D_X(s)}$$

This can be expanded like this:

$$Y(s) = \frac{IC(s)}{A(s)} + \frac{E(s)}{A(s)} + \frac{F(s)}{D_X(s)}$$

for some resulting polynomials $E(s)$ and $F(s)$

So… for a system with $H(s) = \frac{B(s)}{A(s)}$ and input with $X(s) = \frac{N_X(s)}{D_X(s)}$ and initial conditions you get:

$$Y(s) = \frac{IC(s)}{A(s)} + \frac{E(s)}{A(s)} + \frac{F(s)}{D_X(s)}$$

Decays in time domain if roots of system char. poly. $A(s)$ have negative real parts
If all IC’s are zero (zero state) \( C(s) = 0 \)

Then:

\[
Y(s) = \frac{B(s)}{A(s)} X(s) \equiv H(s)
\]

Called “Transfer Function” of the system… see Sect. 6.5

Zero-State Response

\[
Y(s) = \frac{E(s)}{A(s)} + \frac{F(s)}{D_X(s)}
\]

Transient Response  Steady-State Response
Summary Comments:
1. From the differential equation one can easily write the $H(s)$ by inspection!
2. The denominator of $H(s)$ is the characteristic equation of the differential equation.
3. The roots of the denominator of $H(s)$ determine the form of the solution…
   …recall partial fraction expansions

**BIG PICTURE:** The roots of the characteristic equation drive the nature of the system response… we can now see that via the LT.

We now see that there are three contributions to a system’s response:

1. The part driven by the ICs
   a. This will decay away if the Ch. Eq. roots have negative real parts

2. A part driven by the input that will decay away if the Ch. Eq. roots have negative real parts … “Transient Response”

3. A part driven by the input that will persist while the input persists… “Steady State Response”