BINGHAMTON
UNIVERSITY
State University of New York

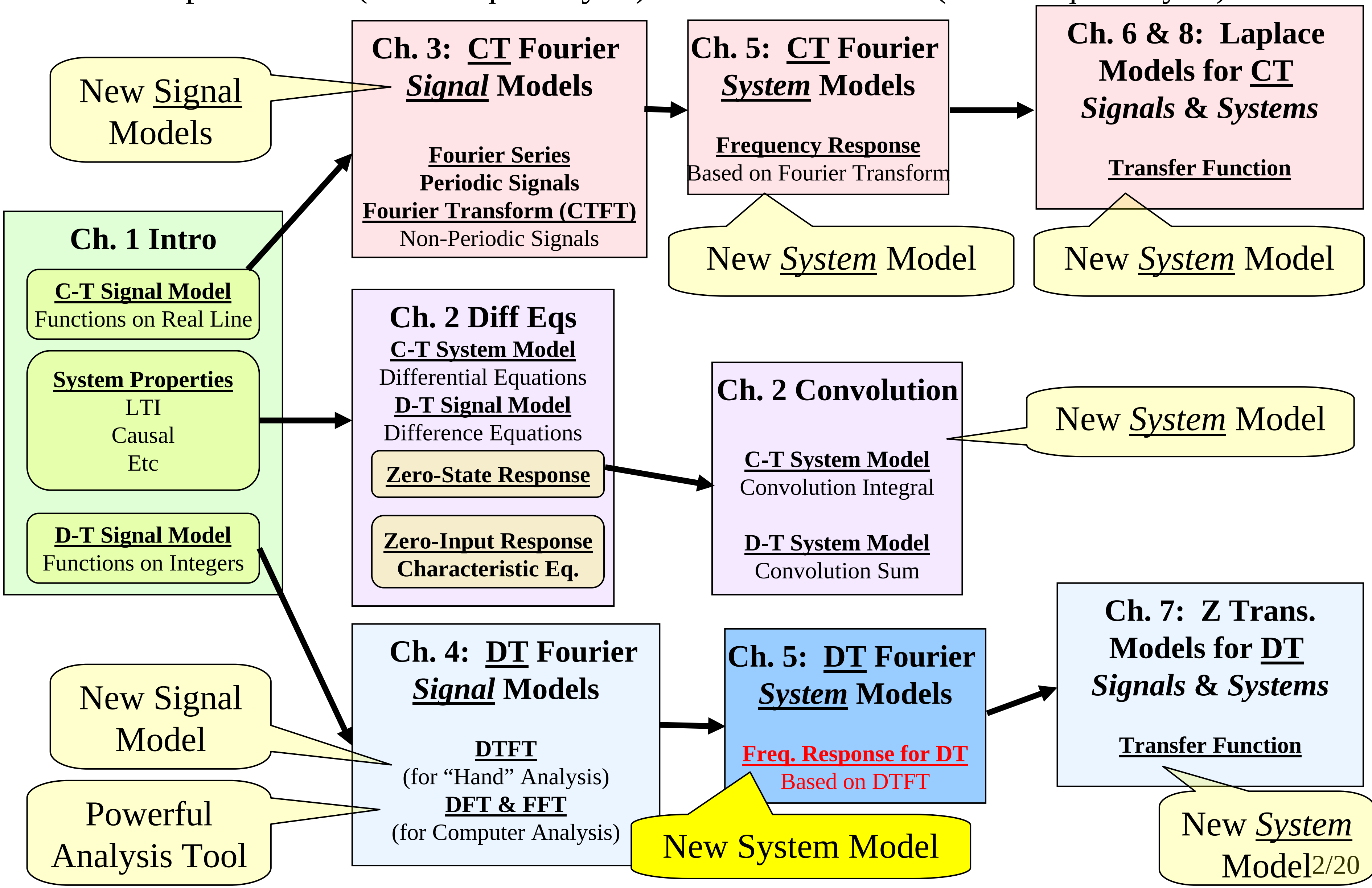
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #26

- D-T Systems: DTFT Analysis of DT Systems
- Reading Assignment: Sections 5.5 & 5.6 of Kamen and Heck

Course Flow Diagram

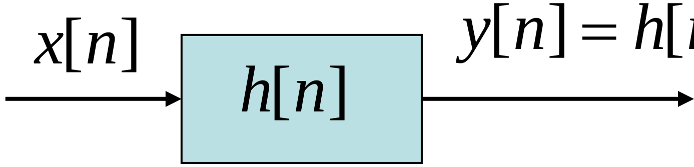
The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



5.5: System analysis via DTFT

We now return to Ch. 5 for its DT coverage!

Back in Ch. 2, we saw that a D-T system in "zero state" has an output-input relation of:



$$y[n] = h[n] * x[n] = \sum_{i=-\infty}^{\infty} h[i]x[n-i]$$

Recall that in Ch. 5 we saw how to use frequency domain methods to analyze the input-output relationship for the C-T case.

We now do a similar thing for D-T

Define the "Frequency Response" of the D-T system

$$\boxed{H(\Omega) = \underbrace{\sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}}_{\text{DTFT of } h[n]}}$$

Perfectly parallel to the same idea for CT systems!!!

From Table of DTFT properties: $\boxed{x[n] * h[n] \leftrightarrow X(\Omega)H(\Omega)}$

So we have:

$$x[n] \xrightarrow{} \boxed{\begin{matrix} h[n] \\ H(\Omega) \end{matrix}} \xrightarrow{} y[n] = h[n] * x[n]$$

$X(\Omega)$

$$\boxed{Y(\Omega) = X(\Omega)H(\Omega)}$$

So…

$$\boxed{\begin{aligned} |Y(\Omega)| &= |X(\Omega)||H(\Omega)| \\ \angle Y(\Omega) &= \angle X(\Omega) + \angle H(\Omega) \end{aligned}}$$

So…in general we see that the system frequency response re-shapes the input DTFT's magnitude and phase.

$\Rightarrow$ System can:

-emphasize some frequencies

-de-emphasize other frequencies

Perfectly parallel to the same ideas for CT systems!!!

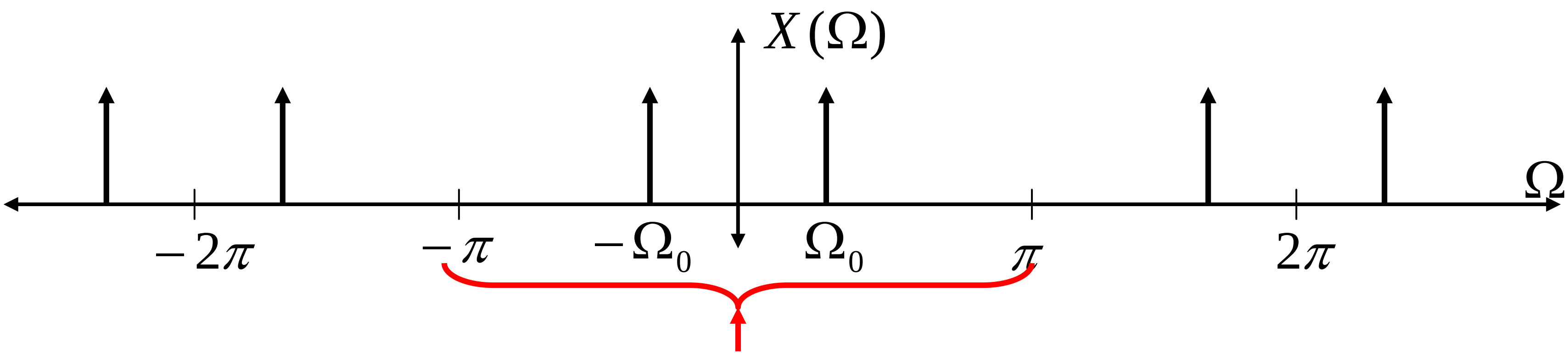
The above shows how to use DTFT to do general DT system analyses … virtually all of your insight from the CT case carries over!

Now lets look at the special case: **Response to Sinusoidal Input**

$$x[n] = A\cos(\Omega_0 n + \theta) \qquad n = -3, -2, -1, 0, 1, 2, 3, \ldots$$

From DTFT Table:

$$X(\Omega) = \begin{cases} A\pi\left[e^{-j\theta}\delta(\Omega+\Omega_0) + e^{j\theta}\delta(\Omega-\Omega_0)\right] & -\pi < \Omega < \pi \\ \text{periodic elsewhere} & \end{cases}$$



We only need to focus our attention here

$$Y(\Omega) = H(\Omega)X(\Omega) \quad \Rightarrow \quad y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\Omega)X(\Omega)e^{j\Omega n}\,d\Omega$$

So what does $Y(\Omega)$ look like?

$$Y(\Omega) = \begin{cases} A\pi\left[H(-\Omega_0)e^{-j\theta}\delta(\Omega+\Omega_0) + H(\Omega_0)e^{j\theta}\delta(\Omega-\Omega_0)\right] & -\pi < \Omega < \pi \\ \text{periodic elsewhere} & \end{cases}$$

$$\text{Now} \quad H(\Omega_0) = \left|H(\Omega_0)\right| e^{j\angle H(\Omega_0)}$$

$$H(-\Omega_0) = \underbrace{\left|H(\Omega_0)\right| e^{-j\angle H(\Omega_0)}}_{\text{Used symmetry properties}}$$

$$\boxed{Y(\Omega) = \begin{cases} A\pi\left|H(\Omega_0)\right|\left[e^{-j(\theta+\angle H(\Omega_0))}\delta(\Omega+\Omega_0) + e^{+j(\theta+\angle H(\Omega_0))}\delta(\Omega-\Omega_0)\right] & -\pi < \Omega < \pi \\ \text{periodic elsewhere} & \end{cases}}$$

From DTFT Table we see this is the DTFT of a cosine signal with:

$$\boxed{\begin{aligned} \text{Amplitude} &= A\left|H(\Omega_0)\right| \\ \text{Phase} &= \theta + \angle H(\Omega_0) \end{aligned}}$$

So…

$$\boxed{y[n] = \left|H(\Omega_0)\right| A\cos(\Omega_0 n + \theta + \angle H(\Omega_0))}$$

$$A\cos(\Omega_0 n + \theta) \longrightarrow \boxed{H(\Omega)} \longrightarrow \underbrace{\left|H(\Omega_0)\right| A\cos(\Omega_0 n + \theta + \angle H(\Omega_0))}$$

System changes amplitude and phase of sinusoidal input

Perfectly parallel to the same ideas for CT systems!!!

Example 5.8 (error in book)

Suppose you have a system described by $H(\Omega) = 1 + e^{-j\Omega}$

And you put the following signal into it

$$x[n] = 2 + 2\sin\left(\frac{\pi}{2}n\right) = 2\cos(0n) + 2\sin\left(\frac{\pi}{2}n\right)$$

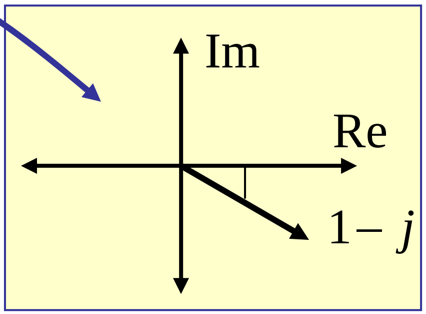
Cosine with Ω = 0

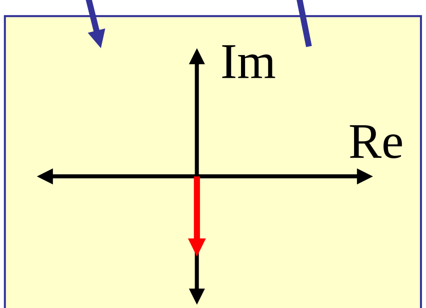
Find the output.

So we need to know the system's frequency response at only 2 frequencies.

$$H(0) = 1 + e^{-j0} = 2$$

$$H\left(\frac{\pi}{2}\right) = 1 + e^{-j\frac{\pi}{2}} = \boxed{1 - j = \sqrt{2}e^{-j\frac{\pi}{4}}}$$





Since the system is linear we can consider each of the input terms separately….

$$y_1[n] = H(0)2\cos(0 \cdot n) = 2 \cdot 2 = 4$$

$$y_2[n] = \left| H\left(\frac{\pi}{2}\right) \right| 2\sin\left(\frac{\pi}{2}n + \angle H\left(\frac{\pi}{2}\right)\right) = 2\sqrt{2}\sin\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)$$

And then add them to get the complete response…

$$\boxed{y[n] = 4 + 2\sqrt{2}\sin\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)}$$

Note: In the above example we used

$$A\sin(\Omega_0 n+\theta) \longrightarrow \boxed{H(\Omega)} \longrightarrow \left|H(\Omega_0)\right| A\sin(\Omega_0 n+\theta+\angle H(\Omega_0))$$

…which is the “sine” version of our result above for a cosine input

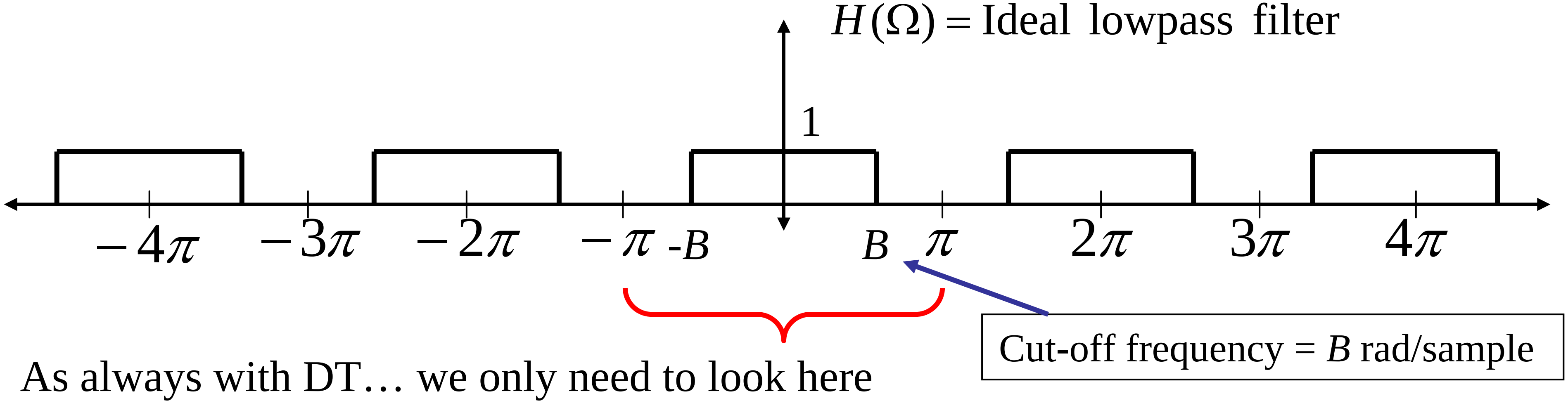
Q: Why does that follow?

A: It is a special case of the cosine result that is easy to see:

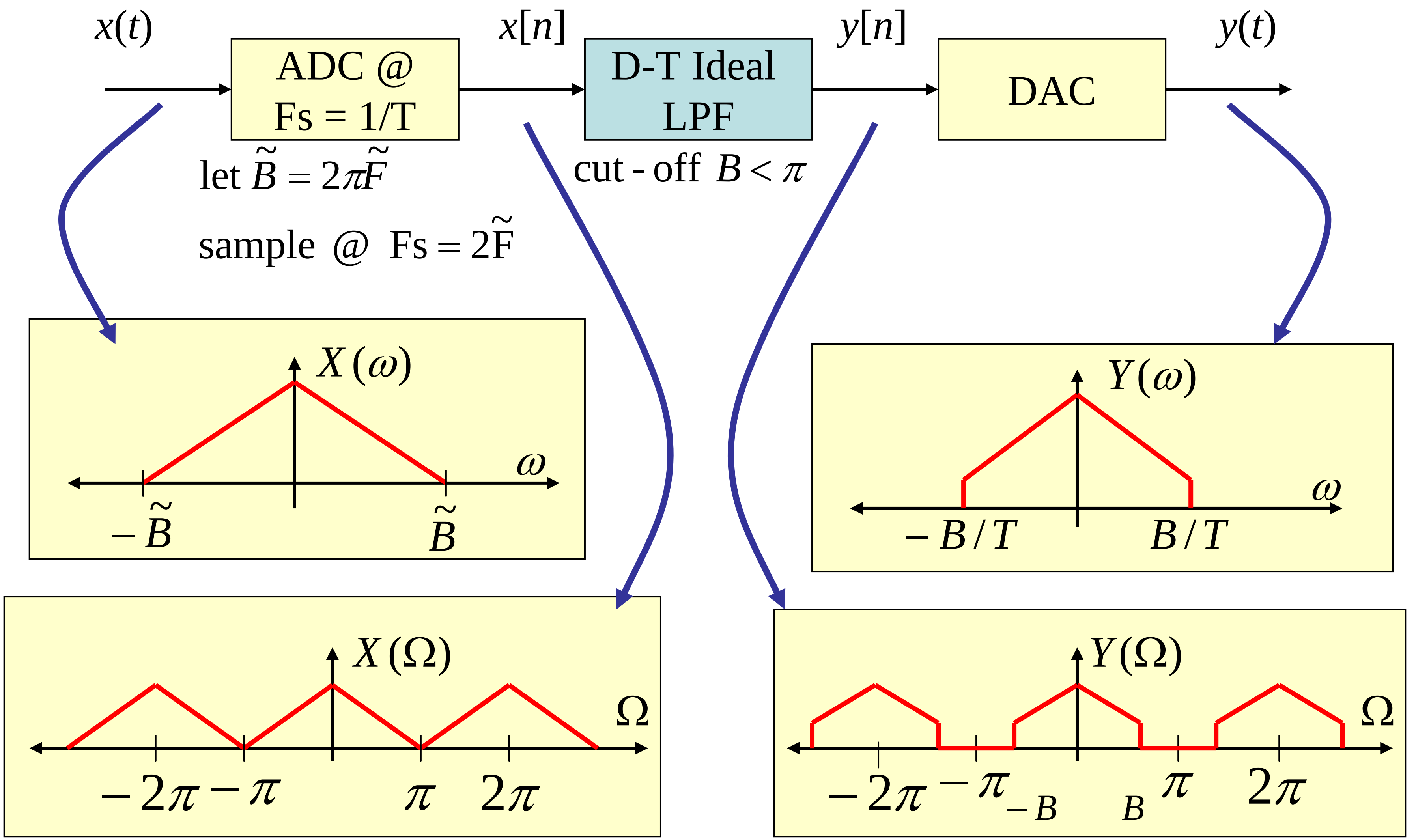
- convert $\sin(\Omega_0 n + \theta)$ into a cosine form
- apply the cosine result
- convert cosine output back into sine form

Analysis of Ideal D-T lowpass Filter (LPF)

Just as in the CT case… we can specify filters. We looked at the ideal lowpass filter for the CT case… here we look at it for the DT case.



This slide shows how a DT filter might be employed… but ideal filters can't be built in practice. We'll see later a few practical DT filters.



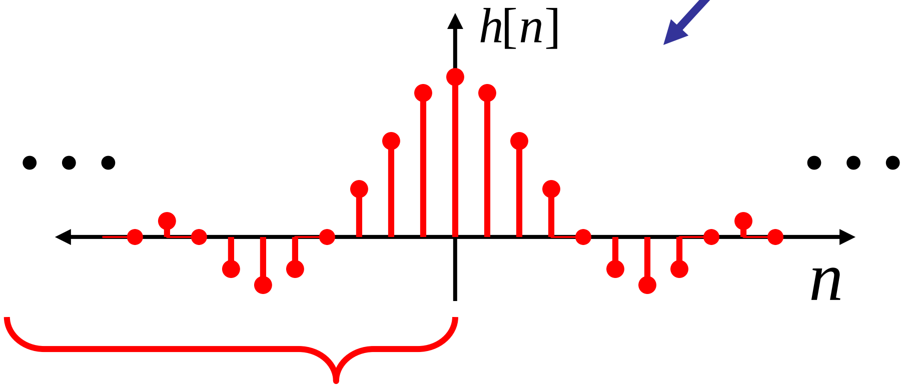
Whole System (ADC – D-T filter – DAC) acts like an equivalent C-T system

<u>Why can't an ideal LPF exist in practice??</u>

We know the frequency response of the ideal LPF… so find its impulse response:

From DTFT Table :

$$h[n] = \frac{B}{\pi}\,\mathrm{sinc}\!\left(\frac{B}{\pi} n\right)$$



Key Point: $h[n]$ is <u>non</u>-zero here

⇒ starts <u>before</u> the impulse that "makes it" is even "applied"!

⇒ Can't build an Ideal LPF

(Same thing is true in C-T)

Causal Lowpass Filter (But not Ideal)

In practice, the best we can do is try to approximate the ideal LPF

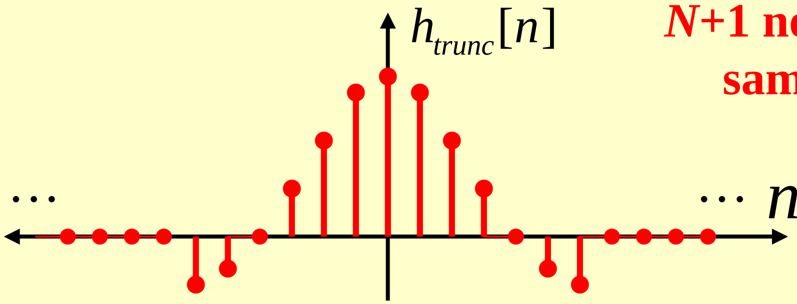
If you go on to study DSP you'll learn how to design filters that do a good job at this approximation

Here we'll look at two "seat of the pants" approaches to get a good LPF

Approach #1 Truncate & Shift Ideal $h[n]$

Truncate

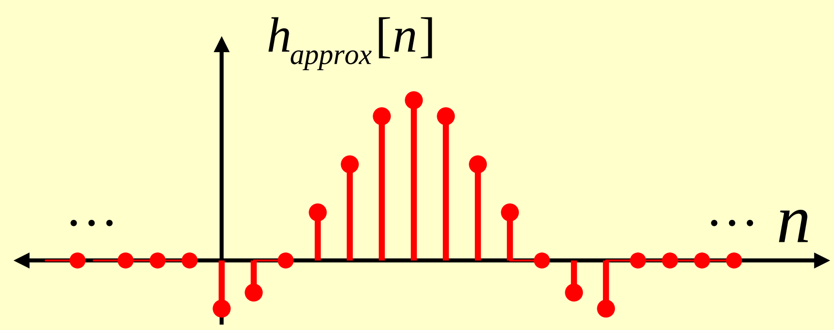
$$h_{trunc}[n] = \begin{cases} h_{ideal}[n], & \frac{-N}{2} \le n \le \frac{N}{2} \\ 0, & otherwise \ \ (N \ \ even) \end{cases}$$



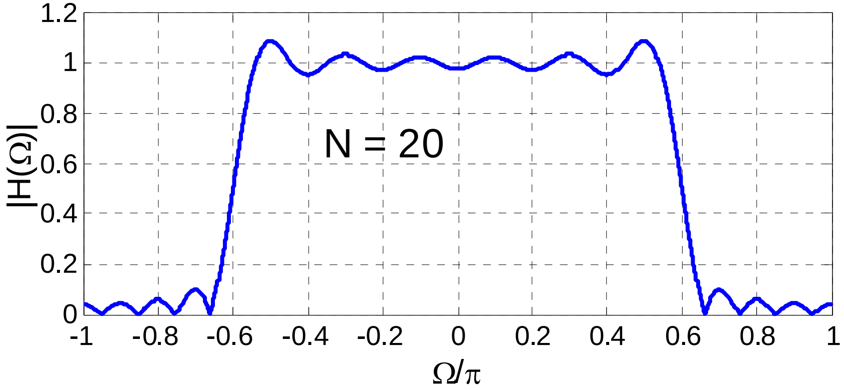
***N*+1 non-zero samples**

Shift

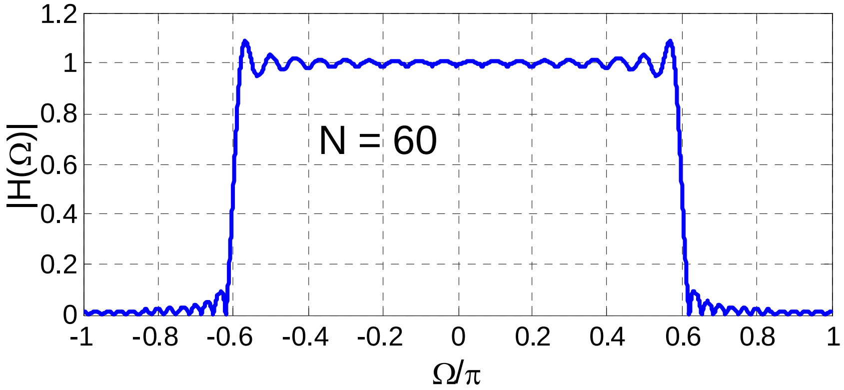
$$h_{approx}[n] = h_{trunc}\left[n - \frac{N}{2}\right]$$



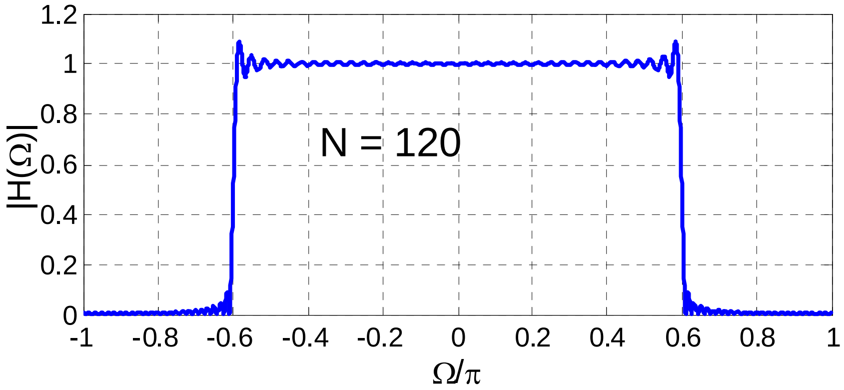
Let's see how well these work…



Frequency Response of a filter truncated to **21 samples**



Frequency Response of a filter truncated to **61 samples**



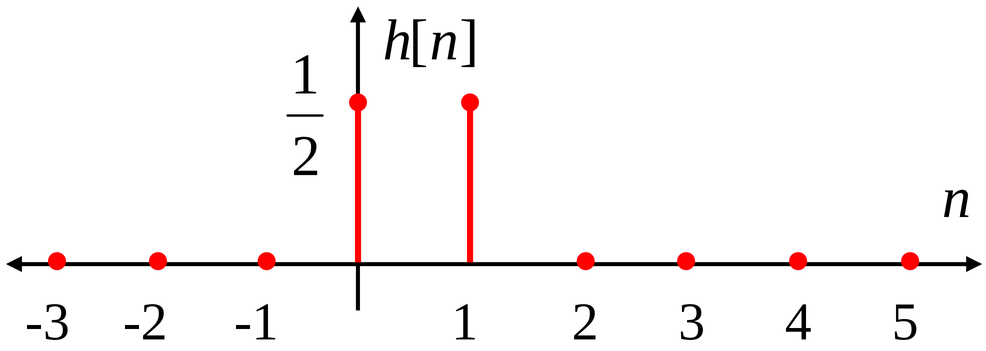
Frequency Response of a filter truncated to **121 samples**

Some general insight: Longer lengths for the truncated impulse response Gives better approximation to the ideal filter response!!

Approach #2: Moving Average Filters

Here is a very simple, low quality LPF:

$$h[n] = \begin{cases} \frac{1}{2}, & n = 0,1 \\ 0, & otherwise \end{cases}$$



To see how well this works as a lowpass filter we find its frequency response:

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n} = \frac{1}{2} \times e^{-j\Omega 0} + \frac{1}{2} \times e^{-j\Omega 1}$$

$$= \frac{1}{2}\left[1 + e^{-j\Omega}\right]$$

By definition of the DTFT

Only 2 non-zero terms in the sum

Now, to see what this looks like we find its magnitude….

$$H(\Omega) = \frac{1}{2}\left[1 + \boxed{e^{-j\Omega}}\right]$$

Euler!

$$= \frac{1}{2}\left[(1 + \boxed{\cos(\Omega)) - j\sin(\Omega)}\right]$$

It is now in rect. form…

$$|H(\Omega)| = \sqrt{\left[\tfrac{1}{2}(1+\cos(\Omega))\right]^2 + \left(-\tfrac{1}{2}\sin(\Omega)\right)^2}$$

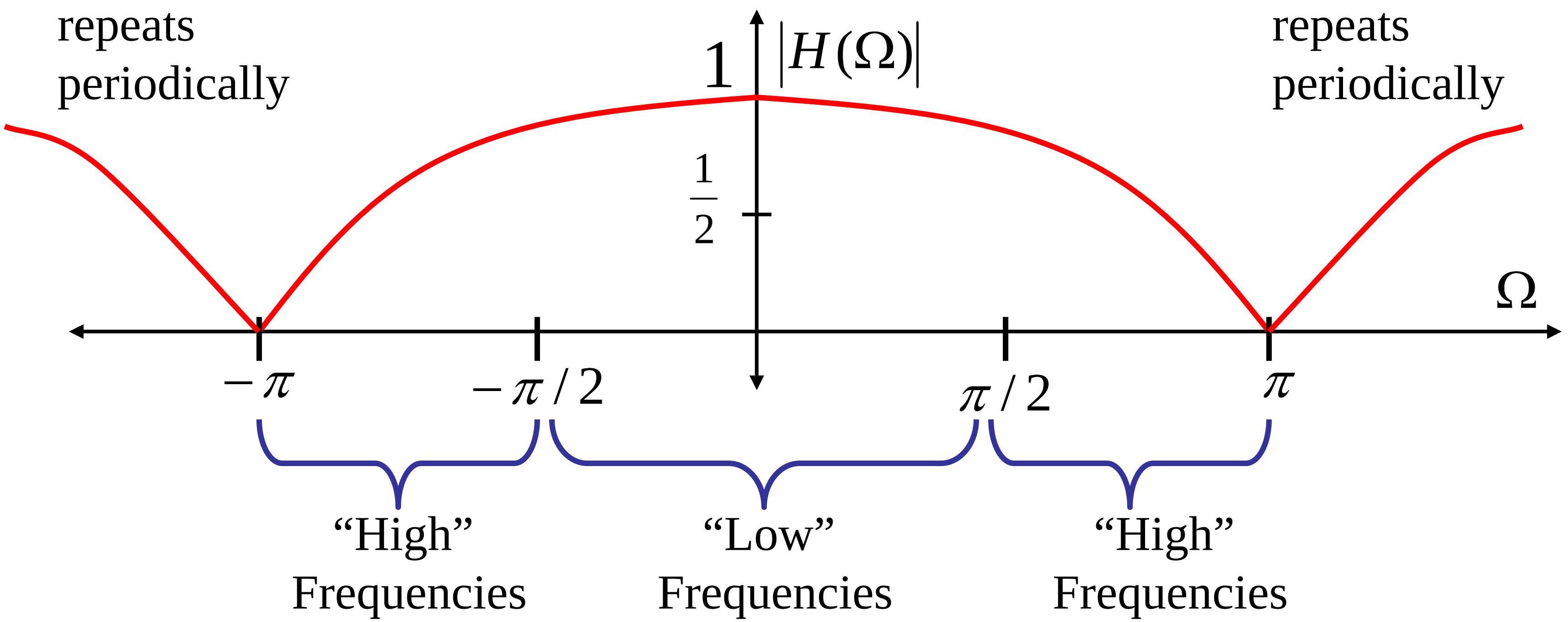
$$= \frac{1}{2}\sqrt{1 + 2\cos(\Omega) + \underbrace{\cos^2(\Omega) + \sin^2(\Omega)}_{=1}}$$

Trig. ID

$$= \frac{\sqrt{2}}{2}\sqrt{1+\cos(\Omega)} = \frac{1}{\sqrt{2}}\sqrt{1+\cos(\Omega)}$$

Now.. Plot this to see if it is a good LPF!

Here's a plot of this filter's freq. resp. magnitude:



Well…this does attenuate high frequencies but doesn't really "<u>stop</u>" them!

It is a low pass filter but not a very good one!

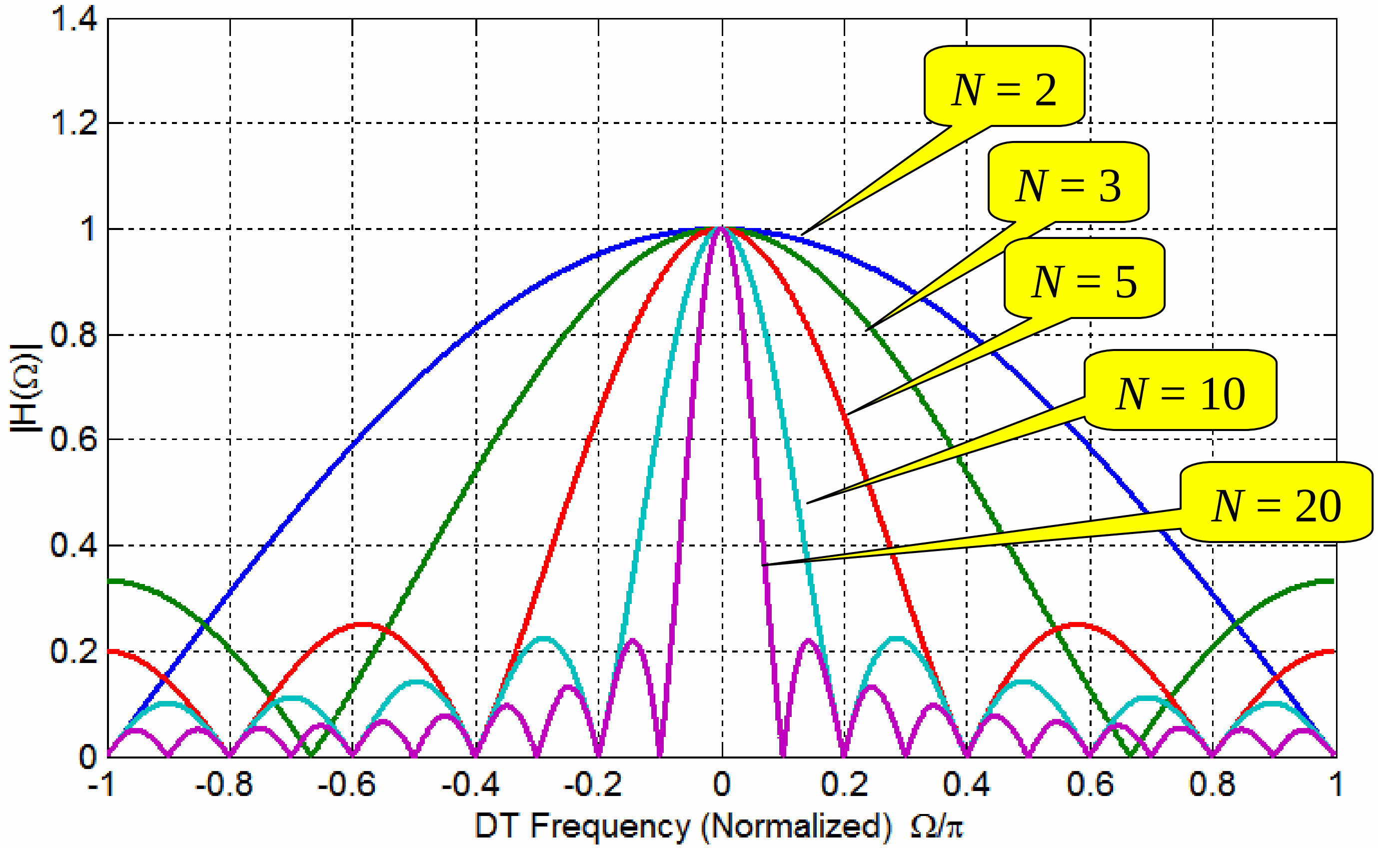
How do we make a better LPF???

We could try longer moving average filters:

$$h[n] = \begin{cases} \dfrac{1}{N}, & n = 0, 1, 2, \cdots, N-1 \\ 0, & otherwise \end{cases}$$

Plots of various Moving Average Filters…

$$h[n] = \begin{cases} \dfrac{1}{N}, & n = 0, 1, 2, \cdots, N-1 \\ 0, & otherwise \end{cases}$$



We see that increasing the length of the “all-ones” moving average filter causes the passband to get narrower… but the quality of the filter doesn’t get better… so we generally need other types of filters.

Comments on these Filter Design Approaches

The two approaches to DT filters we've seen here are simplistic approaches

There are now very powerful methods for designing REALLY good DT filters… we'll look at some of these later in this course.

A complete study of such issues must be left to a senior-level course in DSP!!