BINGHAMTON UNIVERSITY
State University of New York

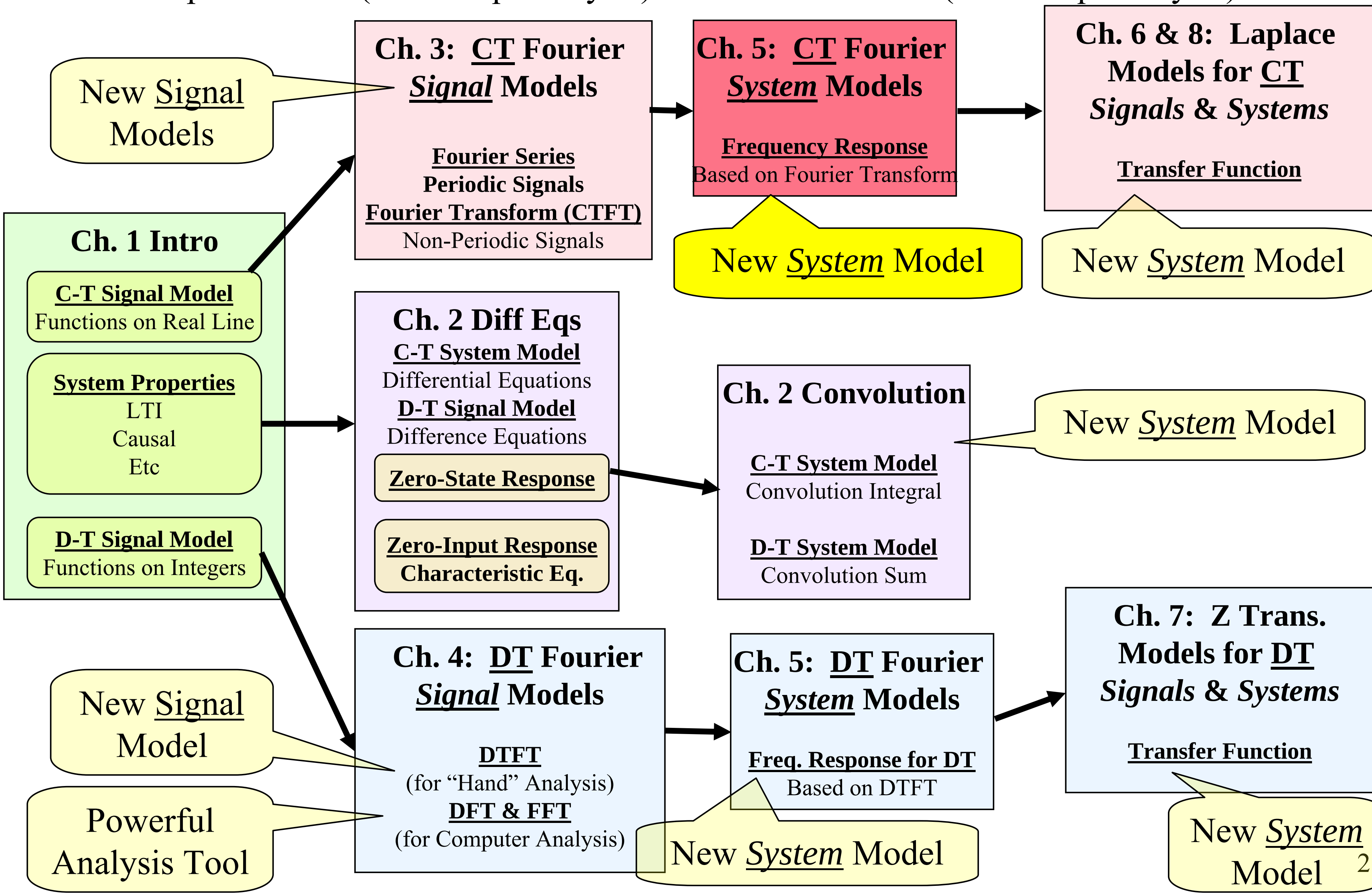
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #21

- C-T to D-T Conversion: Sampling of C-T signals
- Reading Assignment: Section 5.4 of Kamen and Heck
- We study this now for two reasons:
  - The analysis uses C-T Frequency-Domain System Analysis Methods
  - Next we will study Fourier Transform ideas for D-T signals and this gives a good transition

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



5.4 Sampling

The Connection Between:

Continuous Time

&

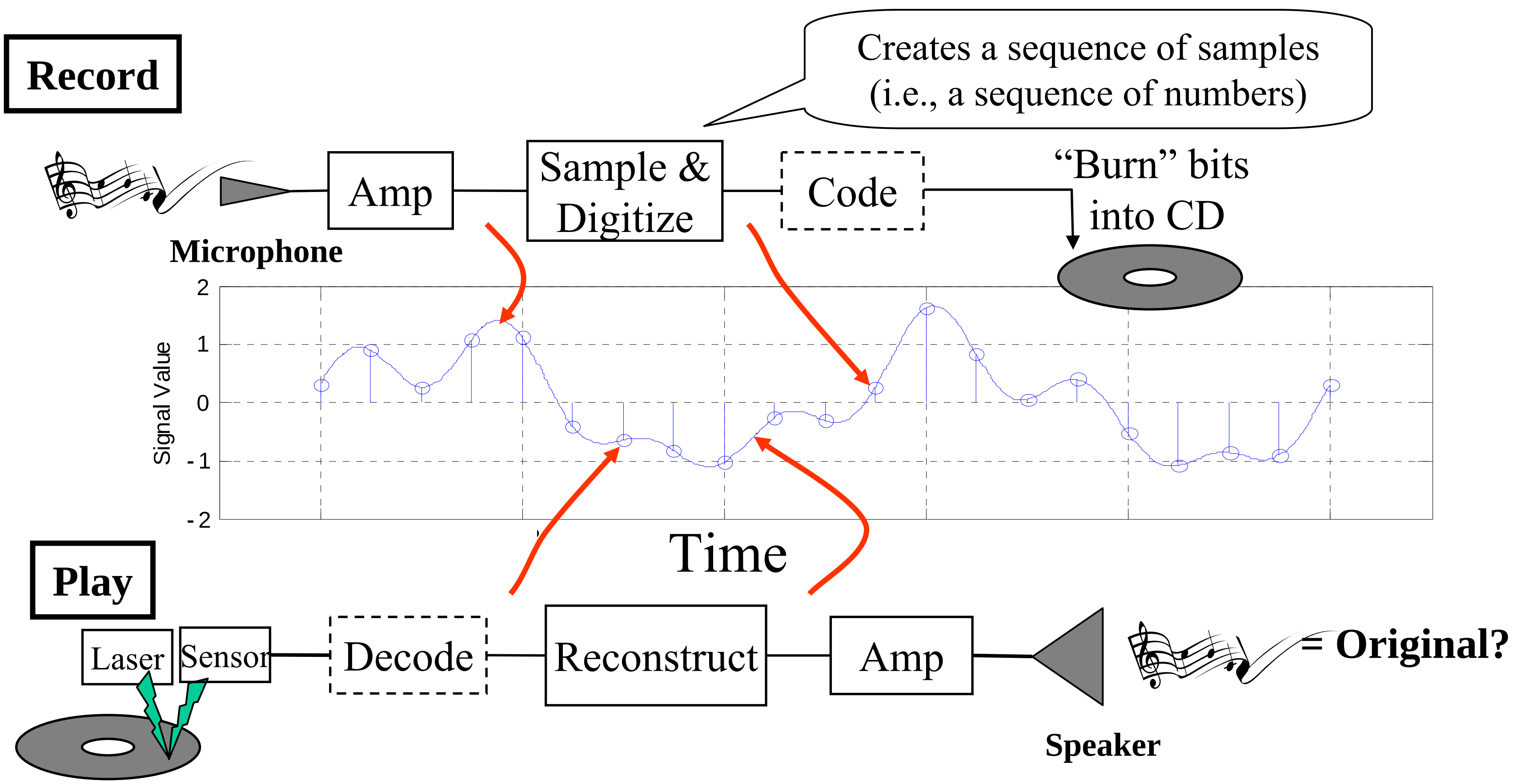
Discrete Time

Warning: I don't really like how the book covers this! It is not that it is wrong… it just fails to make the correct connection between the mathematics and physical reality!!!!

Follow these notes and you'll get it!!!

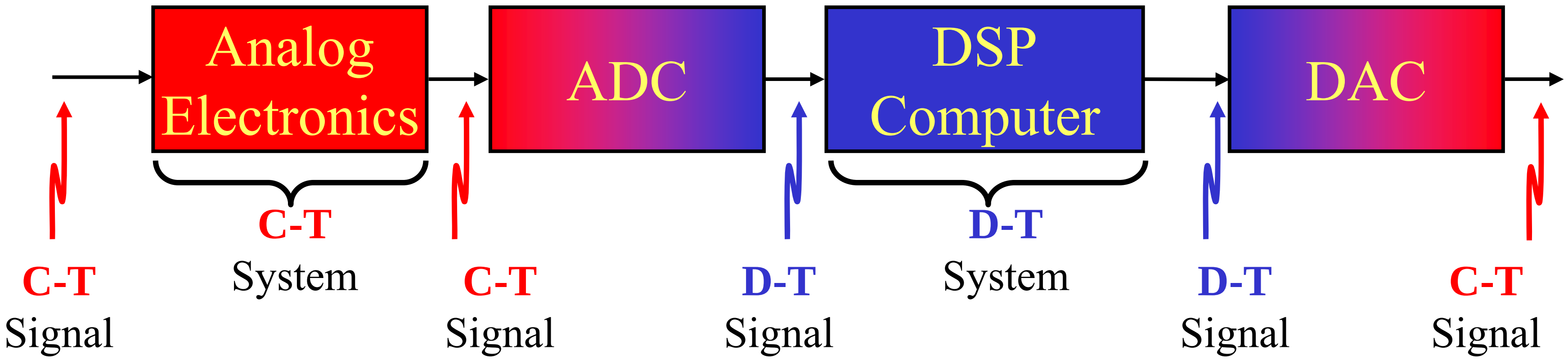
Sampling is Key Part of CD Scheme

- Sampled & Digitized music on a Compact Disc
  - **What ensures that we can "perfectly" reconstruct the music signal from its samples???!!!!**

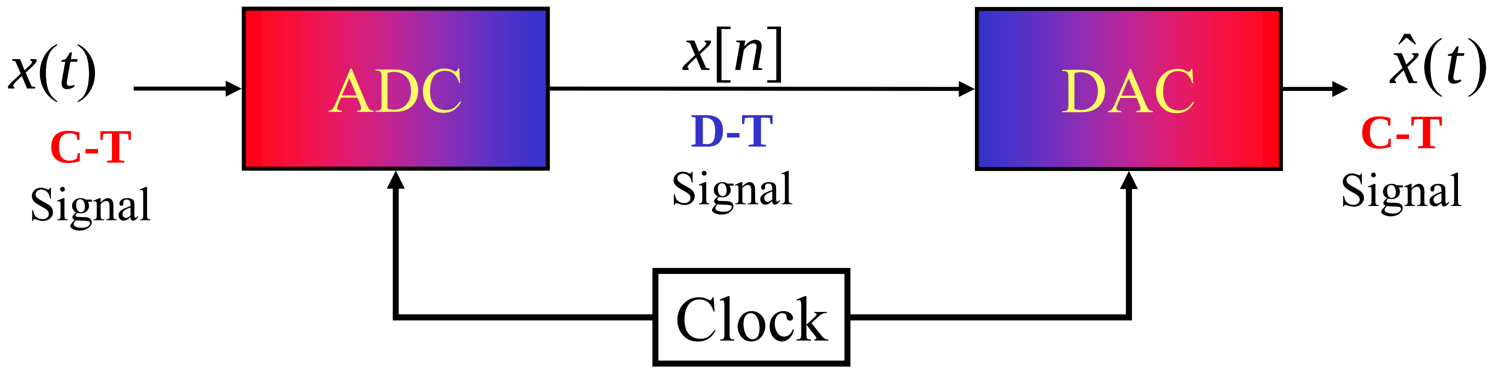


Sampling is Key Part of Many Systems

- Systems that use Digital Signal Proc. (DSP) generally
  - get a **continuous-time signal** from a sensor
  - a **cont.-time system** modifies the signal
  - an "analog-to-digital converter" (ADC) samples the signal to create a **discrete-time signal**
  - A **discrete-time system** to do the Digital Signal Processing
  - and then (if desired) convert back to analog using a "digital-to-analog converter (DAC)



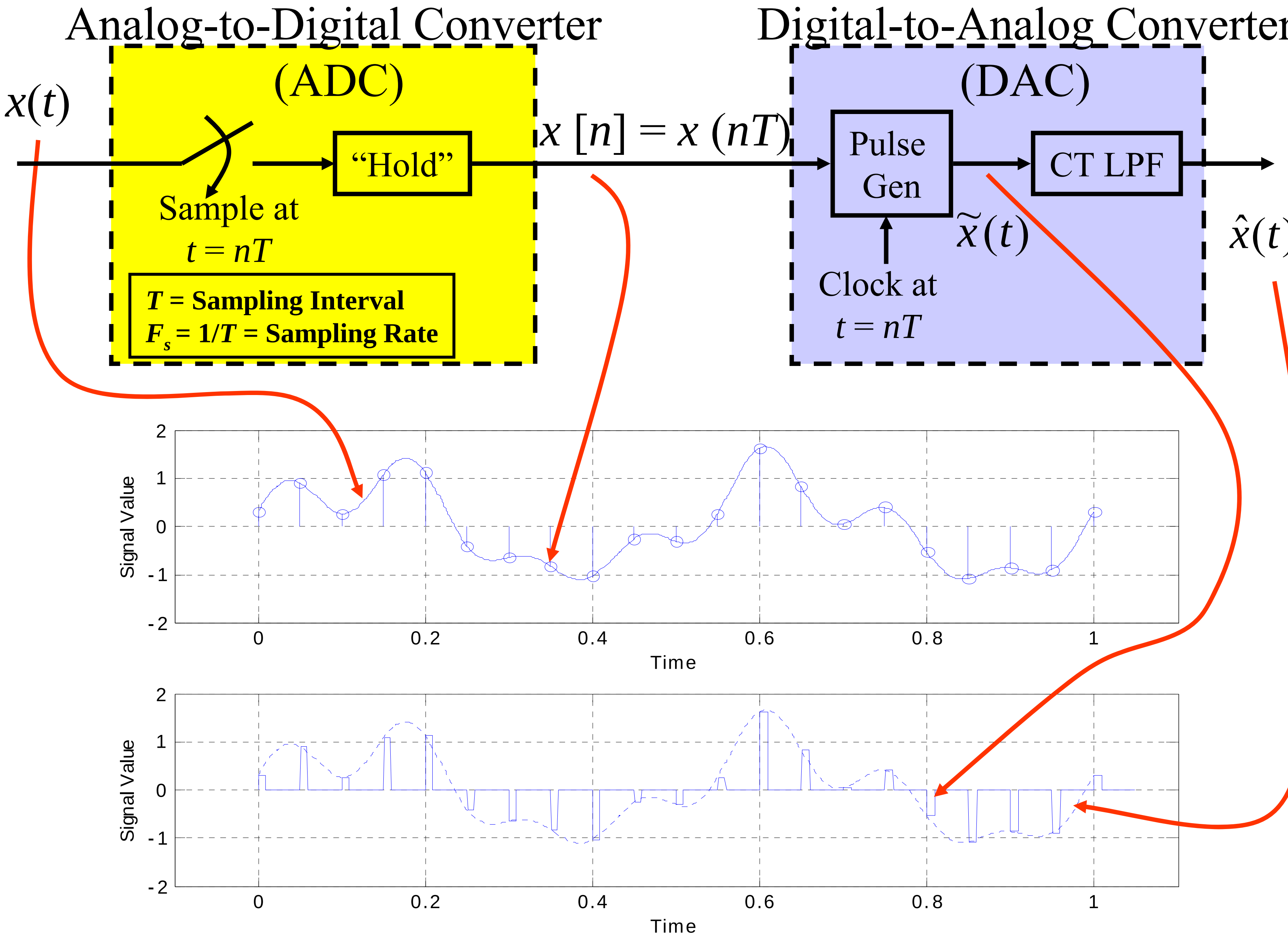
<u>If Sampling is "Valid"…</u>
<u>We Should be Able to</u>
<u>"Perfectly" Reconstruct from Samples</u>



Can we make: $\hat{x}(t) = x(t)$???

If we *can*… *then* we can <u>process</u> the <u>samples</u> $x[n]$ as an alternative to processing $x(t)$!!!

Practical Sampling-Reconstruction Set-Up



- You learn the circuits in an electronics class
- Here we focus on the "why," so we need math models
- We start in a little different place than the book but we end up with the same result (but a little easier to see how/why)

Math Model for Sampling (ADC)

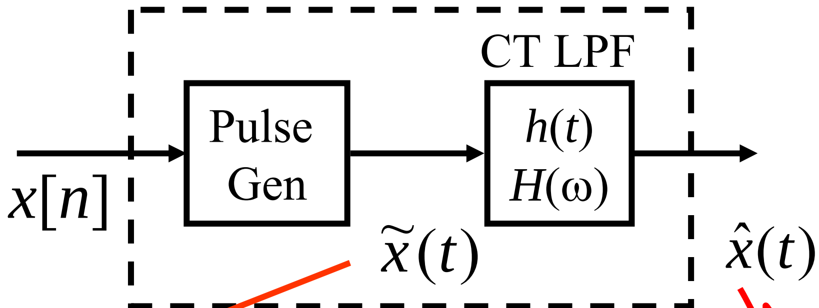
- Math Modeling the ADC is **easy**….
  - $x[n] = x(nT)$, so the n^{th} sample is the value of $x(t)$ at $t = nT$

$$\boxed{x[n] = x(t)\big|_{t=nT} = x(nT)}$$

Note: the book uses an "impulse sampling" model for the ADC… but that has no connection to a physical ADC… we'll see later that it does have a physical connection to the physical DAC!

Math Model for Reconstruction (DAC)

- Math Model for the DAC consists of two parts:
  - converting a DT sequence (of numbers) into a CT pulse train
  - “smoothing” out the pulse train using a lowpass filter

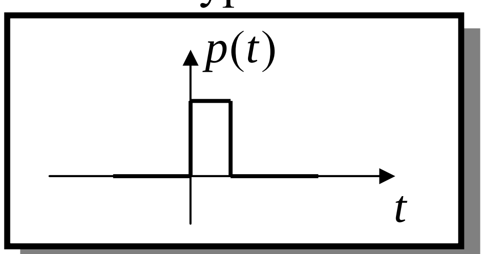


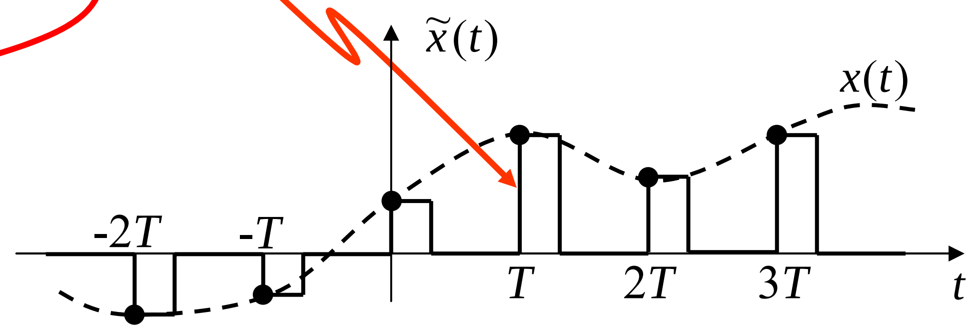
$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\underbrace{p(t-nT)}$$

$$\hat{x}(t) = \tilde{x}(t) * h(t)$$

$$\hat{X}(\omega) = \tilde{X}(\omega)H(\omega)$$

“Prototype” Pulse





"Impulse Sampling" Model for DAC

Now we have a good model that handles quite well what REALLY happens inside a DAC… but we simplify it !!!!

To Ease Analysis: Use $p(t) = \delta(t)$

Why???? 1. Because delta functions are EASY to analyze!!!
2. Because it leads to the best possible results (see later!)
3. We can easily account for real-life pulses later!!

$$\boxed{p(t) = \delta(t)} \Rightarrow \boxed{\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)}$$

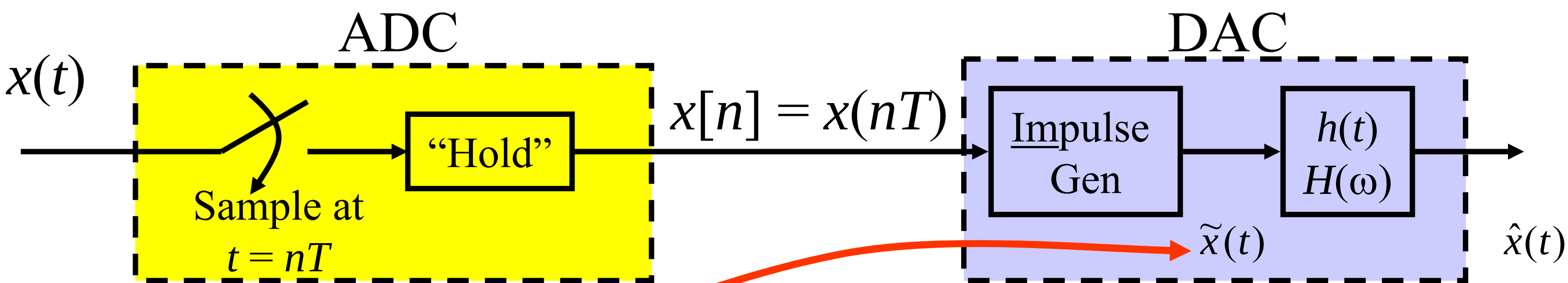
In this form… this is called the "Impulse Sampled" signal.

Now.. Using property of delta function we can also write…

$$\tilde{x}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Sampling Analysis (p. 1)

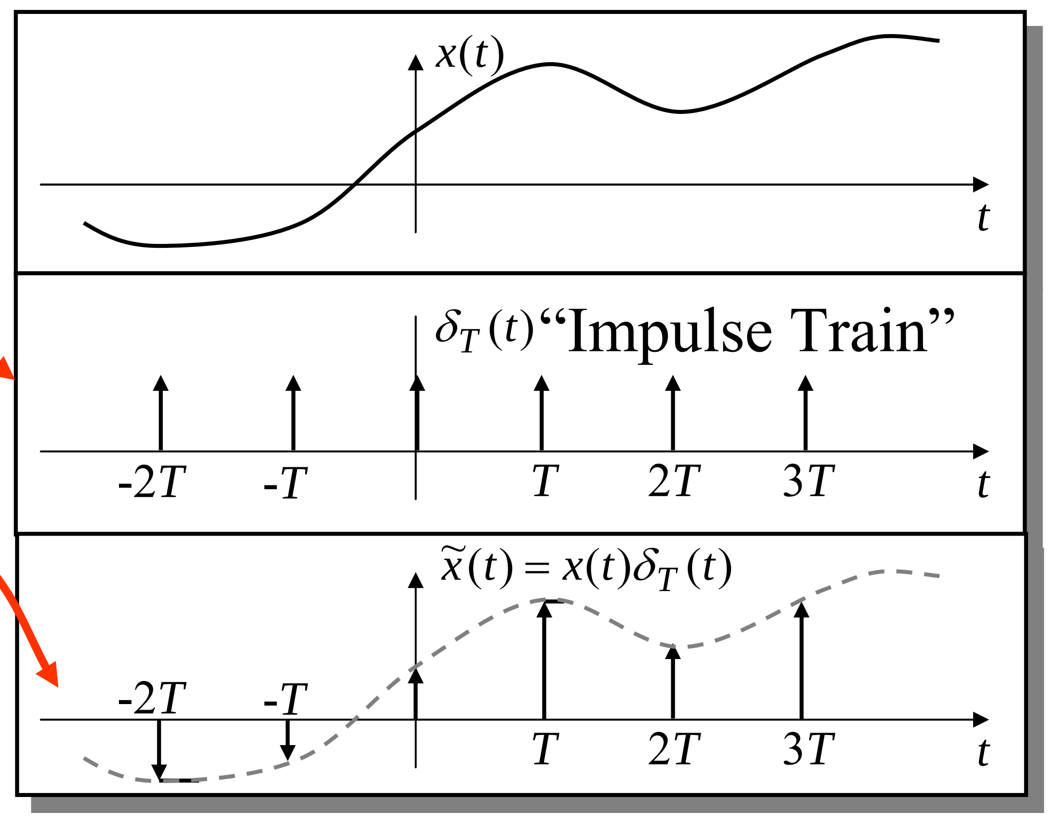
Analysis will be done using the Impulse Sampling Math Model



$$\tilde{x}(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$= x(t)\delta_T(t)$$

Impulse Sampled Signal



Note: we are using the "impulse sampling" model in the DAC not the ADC!!!

Sampling Analysis (p. 2)

Goal = Determine Under What Conditions We Get:

Reconstructed CT Signal = *Original* CT Signal

$$\hat{x}(t) = x(t)$$

Approach: 1. Find the FT of the signal $\tilde{x}(t)$

2. Use Freq. Response of Filter to get $\hat{X}(\omega) = \tilde{X}(\omega)H(\omega)$

3. Look to see what is needed to make $\hat{X}(\omega) = X(\omega)$

Sampling Analysis (p. 3)

Step #1: Hmmm… well $\delta_T(t)$ is periodic with period T so we COULD expand it as a Fourier series:

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk2\pi F_s t}$$

Period = T sec
Fund. Freq = F_s = 1/T Hz

So… what are the FS coefficients???

Only one delta inside a single period

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t) e^{-jk2\pi F_s t} = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi F_s t}$$

$$= \frac{1}{T}\left[e^{-jk2\pi F_s t}\right]_{t=0} = \frac{1}{T}$$

By sifting property of the delta function!!!

So… an alternate model for $\delta_T(t)$ is

$$\delta_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk2\pi F_s t}$$

Sampling Analysis (p. 4)

So we now have….

$$\tilde{x}(t) = x(t)\delta_T(t)$$

$$= x(t)\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk2\pi F_s t}\right]$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} x(t) e^{jk2\pi F_s t}$$

Use FS Result

By frequency shift property of FT… each term is a frequency shifted version of the original signal!!!

So using the frequency shift property of the FT gives:

$$\boxed{\tilde{X}(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f + kF_s)}$$

Extremely Important Result… the basis of all understanding of sampling!!!

$$\tilde{X}(f) = \frac{1}{T}\left[\cdots + X(f-2F_s) + X(f-F_s) + \boxed{X(f)} + X(f+F_s) + X(f+2F_s) + \cdots\right]$$

Original FT

Shifted Replicas

Sampling Analysis (p. 5)

So… the **BIG Thing** we've just found out is that:

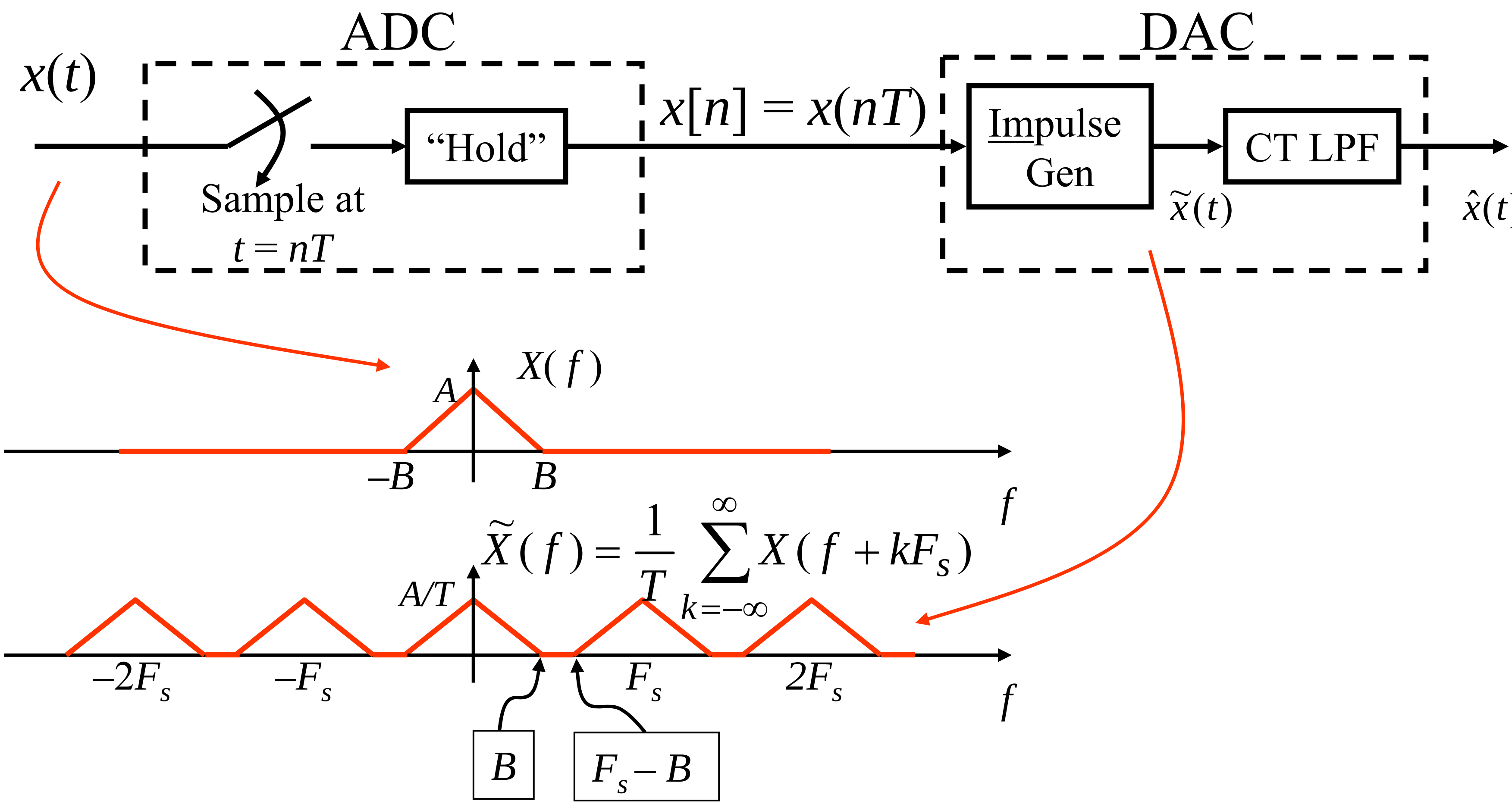
the impulse sampled signal (inside the DAC) has a FT that consists of the original signal's FT and frequency-shifted version of it (where the frequency shifts are by integer multiples of the sampling rate F_s)

This result allows us to see how to make sampling work …

By "work" we mean: how to ensure that even though we only have samples of the signal, we can still get perfect reconstruction of the original signal…. at least in theory!!

The figure on the next page shows how….

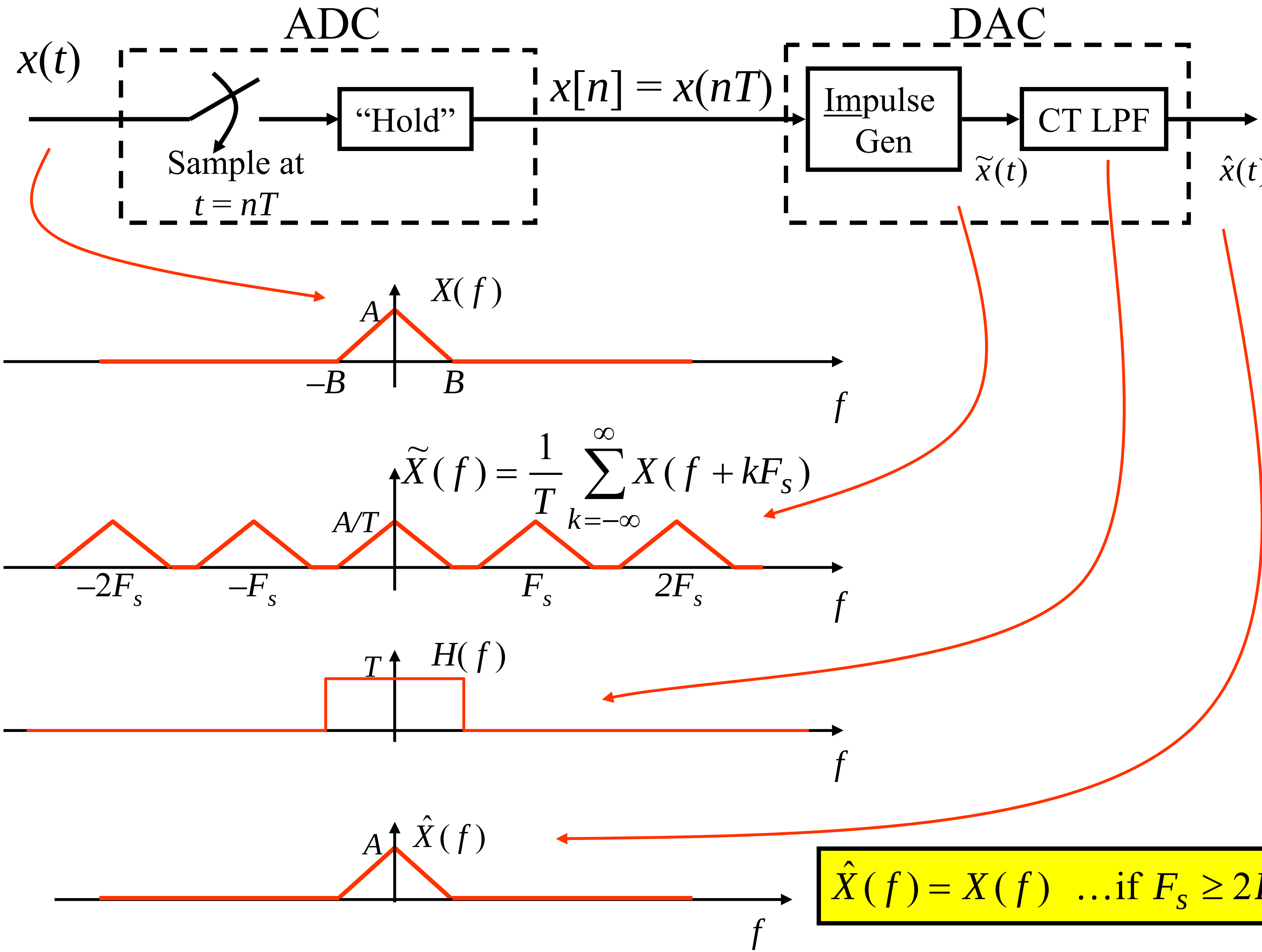
Sampling Analysis (p. 6)



To ensure that the replicas don't overlap the original….
we need $F_s - B \geq B$ or equivalently… $\boxed{F_s \geq 2B}$

<u>When there is no overlap</u>, the original spectrum is left "unharmed" and <u>can be recovered using a CT LPF</u> (as seen on the next page).

Sampling Analysis (p. 7)



$$\boxed{\hat{X}(f) = X(f) \quad \ldots\text{if } F_s \geq 2B}$$

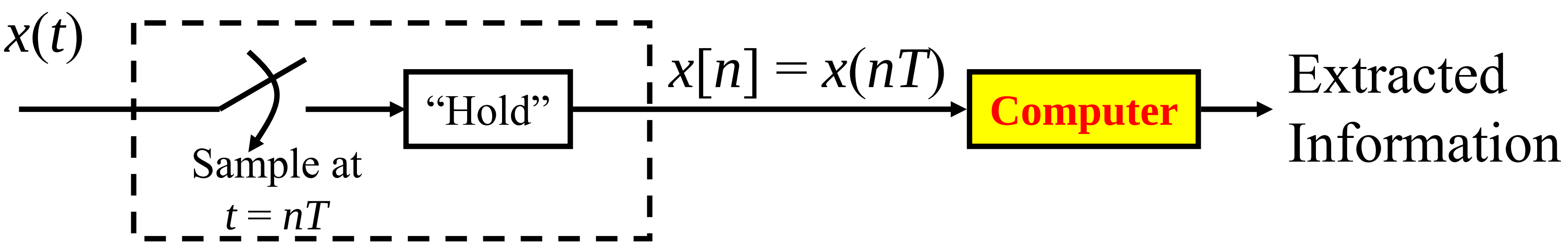
Sampling Analysis (p. 8)

What this analysis says:

Sampling Theorem: A **bandlimited** signal with BW = B Hz is completely defined by its samples as long as they are taken at a rate $F_s \geq 2B$.

Impact: To extract the info from a **bandlimited** signal we only need to operate on its (properly taken) samples

➔ **Then can use a computer to process signals!!!**



This math result (published in the late 1940s!) is the foundation of: …CD's, MP3's, digital cell phones, etc….

Some Sampling Terminology

F_s is called the sampling rate. Its unit is samples/sec which is often "equivalently" expressed as Hz

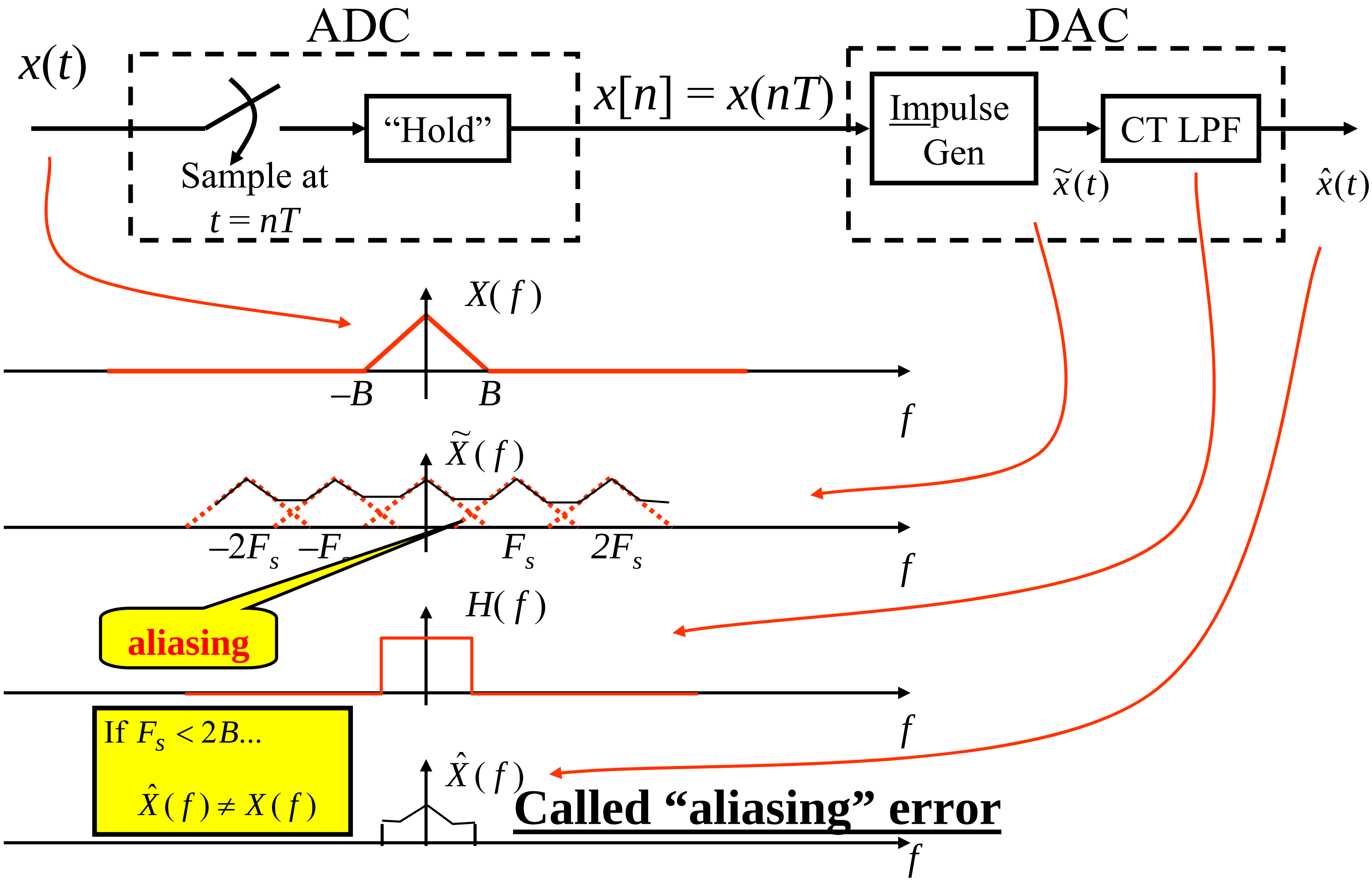
The minimum sampling rate of F_s = 2B samples/sec is called the Nyquist Rate.

Sampling at the Nyquist rate is called Critical Sampling.

Sampling faster than the Nyquist rate is called Over Sampling

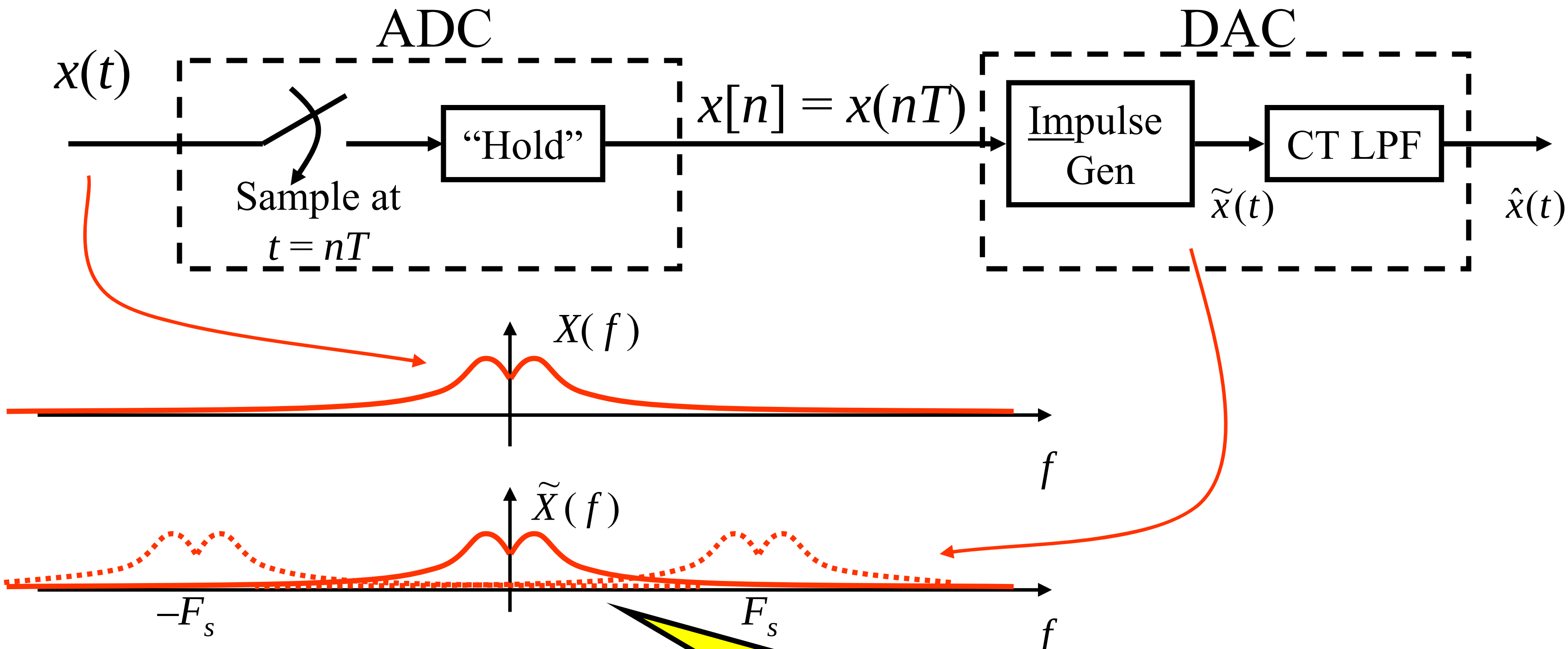
Note: Critical sampling is only possible if an IDEAL lowpass filter is used…. so in practice we generally need to choose a sampling rate somewhat above the Nyquist rate (e.g., $2.2B$); the choice depends on the application.

"Aliasing" Analysis: What if samples are *not* taken fast enough???



To enable error-free reconstruction, a signal bandlimited to B Hz <u>must</u> be sampled faster than 2B samples/sec

"Aliasing" Analysis: What if the signal is NOT BANDLIMITED???



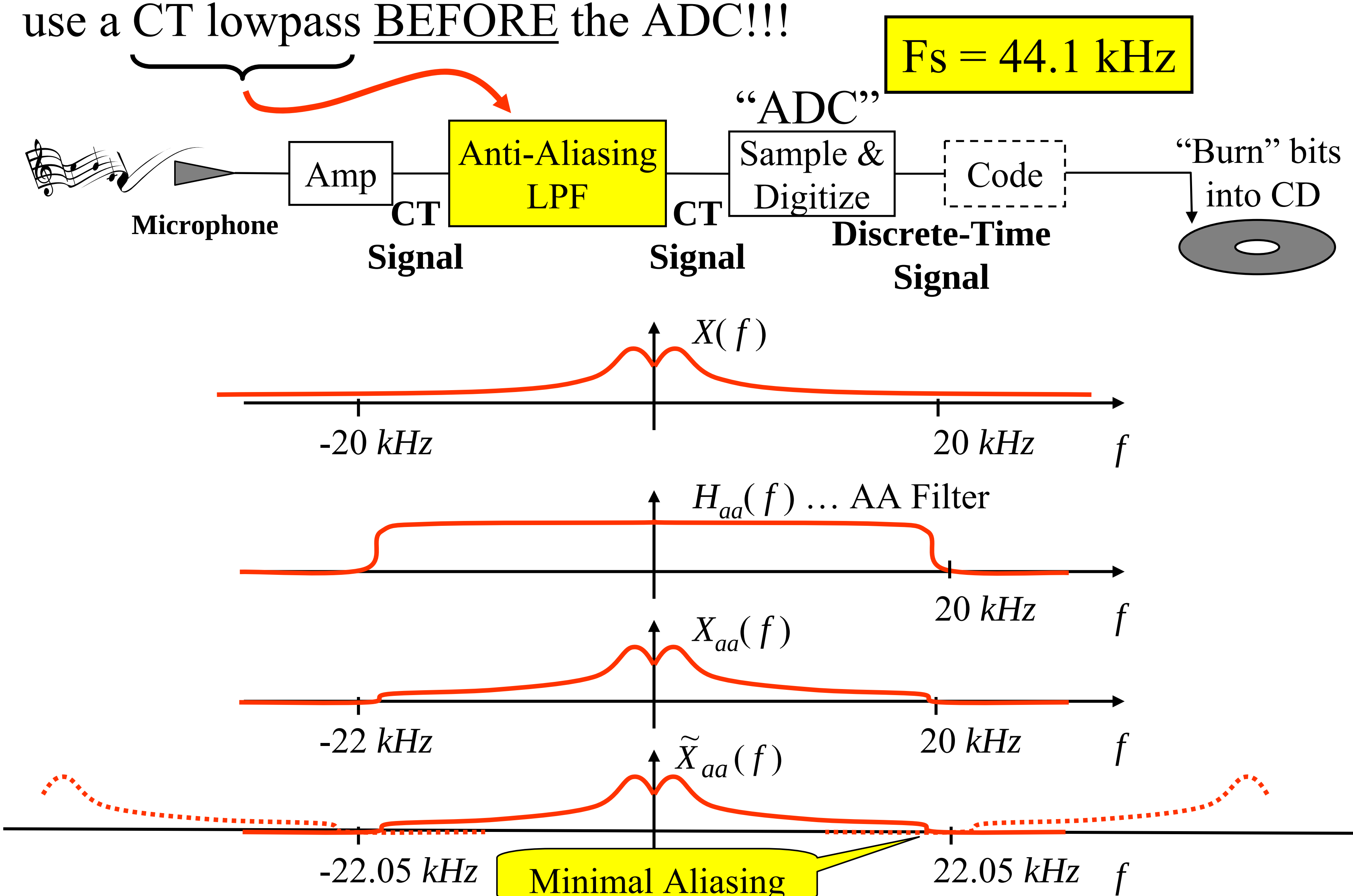
<u>For Non-BL Signal</u> Aliasing always happens regardless of F_s value

<u>All</u> practical signal are <u>Non</u>-BL!!!!

… so we choose F_s to minimize the aliasing to an level acceptable for the specific application

Practical Sampling: Use of Anti-Aliasing Filter

In practice it is important to avoid excessive aliasing. So we use a CT lowpass BEFORE the ADC!!!



Summary of Sampling

- Math Model for Impulse Sampling says
  - The FT of the impulse sampled signal has spectral replicas spaced F_s Hz apart
  - This math result drives all of the insight into practical aspects
- Theory says for a BL'd Signal with BW = B Hz
  - It is completely defined by samples taken at a rate $F_s \geq 2B$
  - Then… Perfect reconstruction can be achieved using an ideal LPF reconstruction filter (i.e., the filter inside the DAC)
- Theory says for a Practical Signal…
  - Practical signals aren't bandlimited… so use an Anti-Aliasing lowpass filter BEFORE the ADC
  - Because the A-A LPF is not ideal there will still be some aliasing
    - Design the A-A LPF to give acceptably low aliasing error for the expected types of signals