BINGHAMTON UNIVERSITY
State University of New York

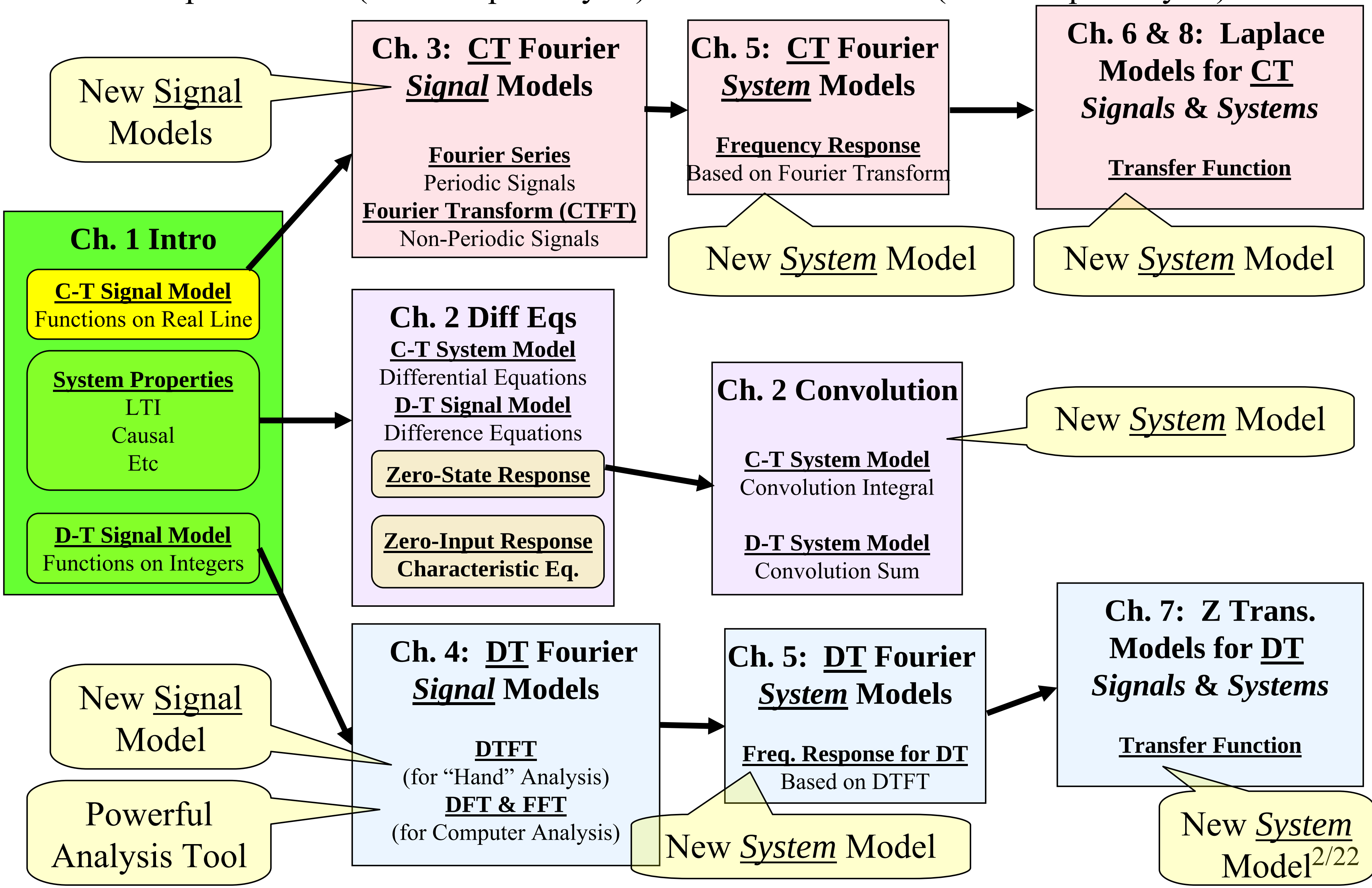
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #2

- What are Continuous-Time Signals???
- Reading Assignment: Section 1.1 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



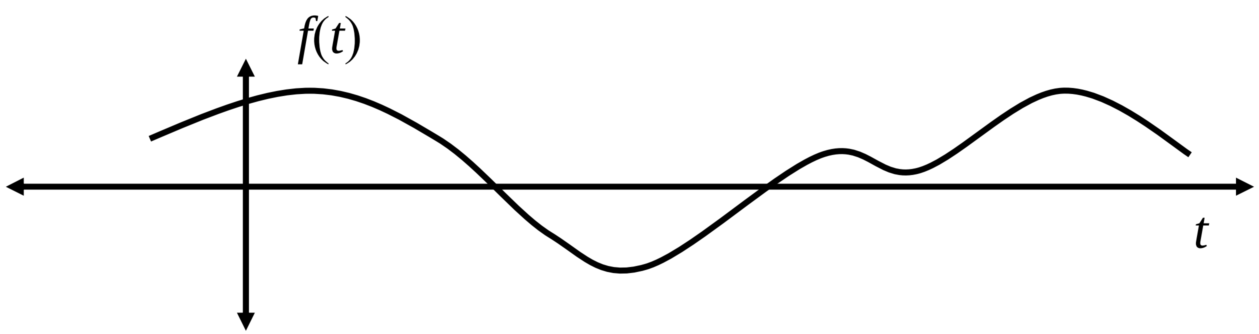
<u>1.1 Continuous-Time Signal</u>

Our first math model for a signal will be a "function of time"

<u>Continuous Time (C-T) Signal:</u>

A C-T signal is defined on the continuum of time values. <u>That is:</u>

$f(t)$ for $t \in \Re$ Real line

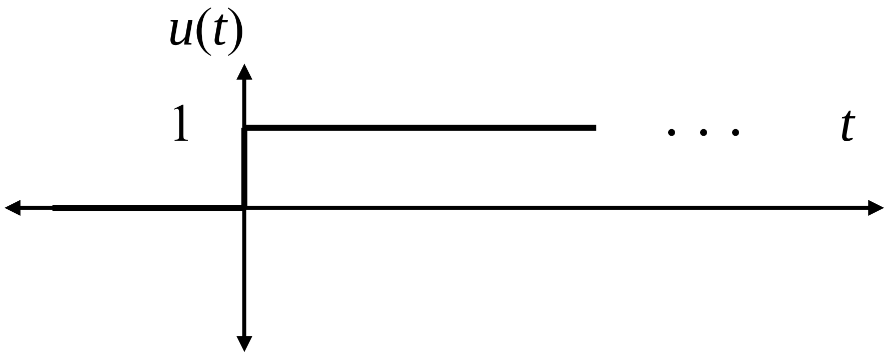


Step & Ramp Functions

These are common textbook signals but are also common test signals, especially in control systems.

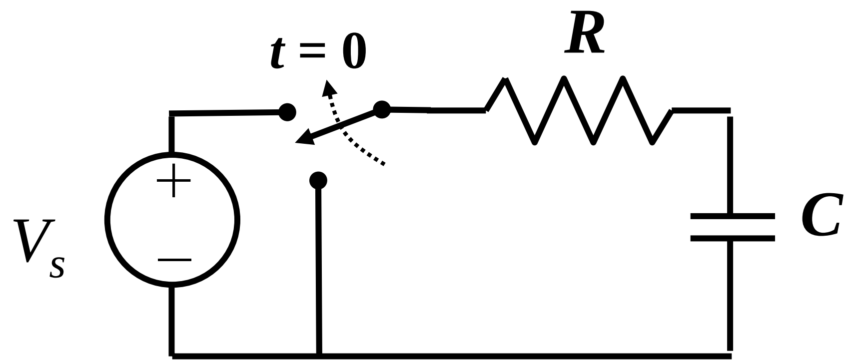
Unit Step Function *u*(*t*)

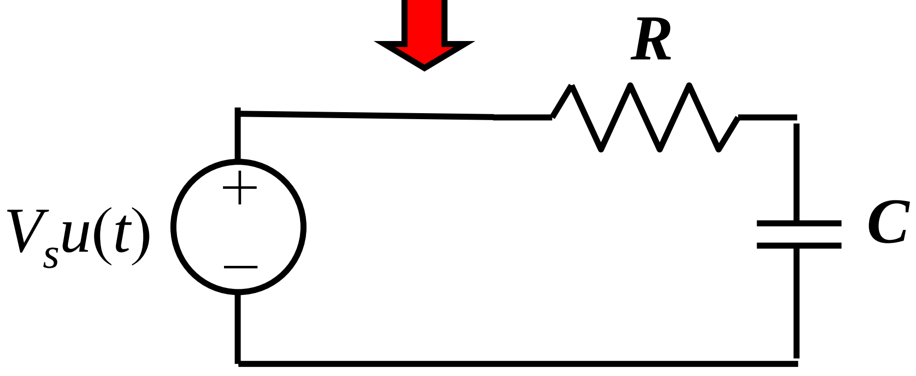
$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

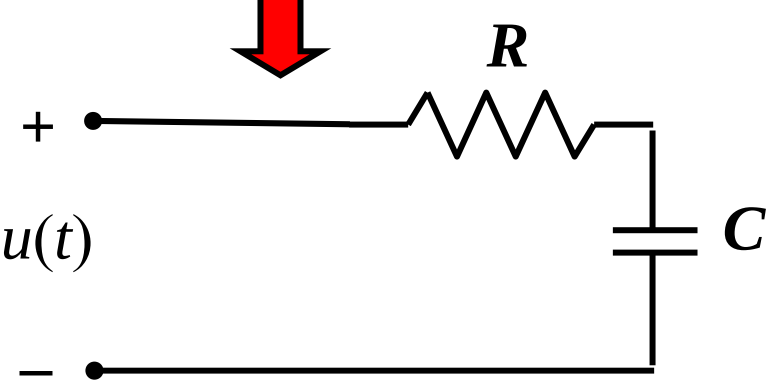


Note: A step of height A can be made from $Au(t)$

The unit step signal can model the act of switching on a DC source…

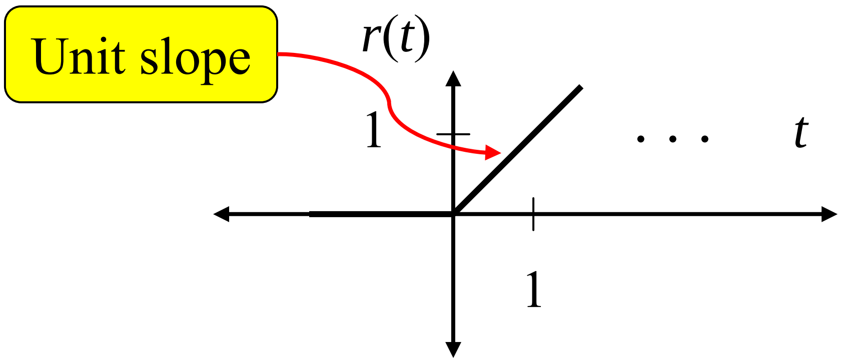






<u>Unit Ramp Function *r*(*t*)</u>

$$r(t) = \begin{cases} t, \; t \geq 0 \\ 0, \; t < 0 \end{cases}$$



<u>Note:</u> A ramp with slope m can be made from: $mr(t)$

$$mr(t) = \begin{cases} mt, \; t \geq 0 \\ 0, \; t < 0 \end{cases}$$

Relationship between $u(t)$ & $r(t)$

What is $\int_{-\infty}^{t} u(\lambda)d\lambda$?

Depends on t value
$\Rightarrow$function of t: $f(t)$

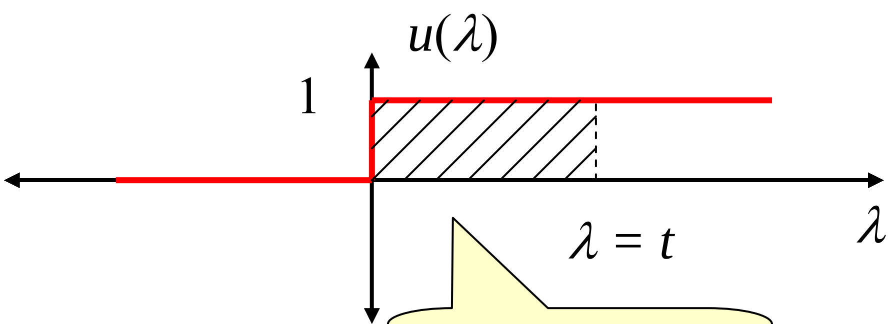
$\Rightarrow \quad f(t) = \int_{-\infty}^{t} u(\lambda)d\lambda$ What is $f(t)$?

-Write unit step as a function of λ

-Integrate up to $\lambda = t$

i.e., Find Area

-How does area change as t changes?



Area = $f(t)$

$$\Rightarrow \quad f(t) = \int_{-\infty}^{t} u(\lambda)d\lambda = 1 \cdot t = t = r(t)$$

$$\Rightarrow \quad \boxed{r(t) = \int_{-\infty}^{t} u(\lambda)d\lambda}$$

"Running Integral of step = ramp"

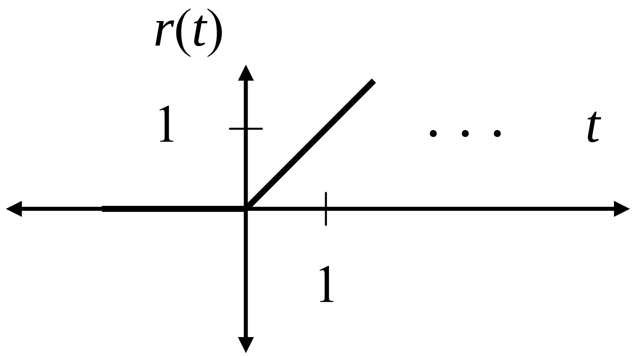
Also note: For $r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$

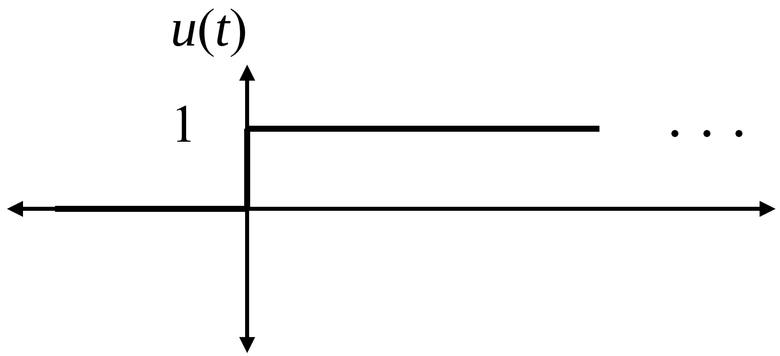
we have:

$$\frac{dr(t)}{dt} = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

Not defined at $t = 0$!

Overlooking this, we can roughly say $u(t) = \dfrac{dr(t)}{dt}$





Time Shifting Signals

Time shifting is an operation on a signal that shows up in many areas of signals and systems:

- Time delays due to propagation of signals
  - acoustic signals propagate at the speed of sound
  - radio signals propagate at the speed of light
- Time delays can be used to "build" complicated signals
  - We'll see this later

Time Shift: If you know $x(t)$, what does $x(t-t_0)$ look like?

For example… If $t_0 = 2$:

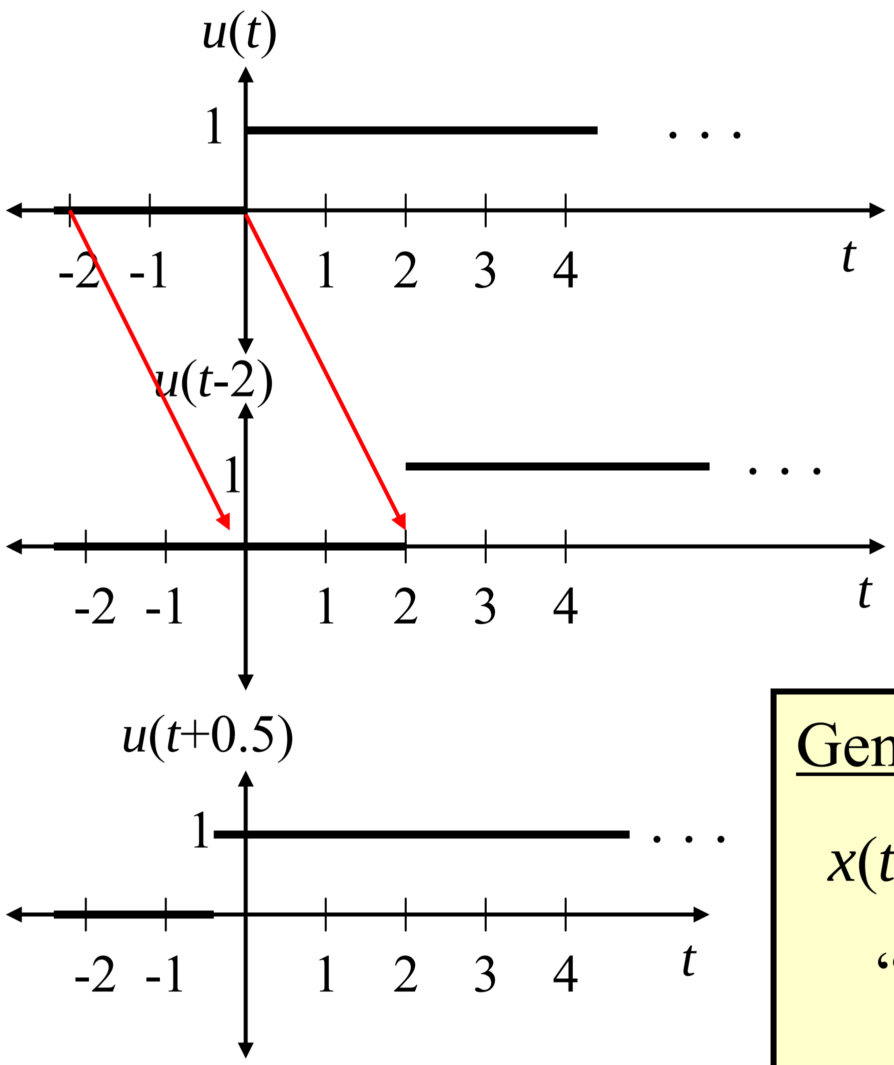
$x(0-2) = x(-2)$

At $t = 0$, $x(t-2)$ takes the value of $x(t)$ at $t = -2$

$x(1-2) = x(-1)$

At $t = 1$, $x(t-2)$ takes the value of $x(t)$ at $t = -1$

Example of Time Shift of the Unit Step $u(t)$:



General View:

$x(t \pm t_0)$ for $t_0 > 0$

"$+t_0$" gives Left shift (Advance)

"$-t_0$" gives Right shift (Delay)

The Impulse Function

Other Names: Delta Function, Dirac Delta Function

One of the most important functions for understanding systems!!

Ironically…it does not exist in practice!!

⇒ It is a theoretical tool used to understand what is important to know about systems!

But… it leads to ideas that are used all the time in practice!!

There are three views we'll take of the delta function:

Rough View: a pulse with:

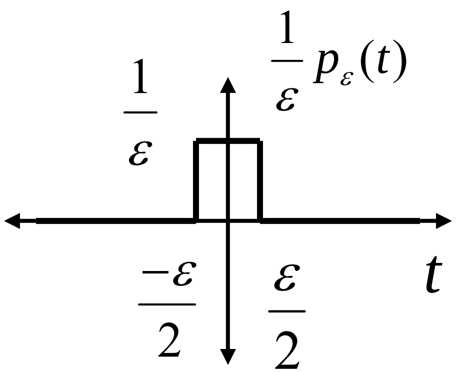
Infinite height
Zero width
Unit area

"A *really* narrow, *really* tall pulse that has unit area"

Slightly Less-Rough View:

$$\boxed{\delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} p_\varepsilon(t)}$$

Here we define $\frac{1}{\varepsilon} p_\varepsilon(t)$ as:



Beware of Fig 1.4 in the book… it does not show the real $\delta(t)$…
So its vertical axis should NOT be labeled with $\delta(t)$

Pulse having… <u>height of 1/ε</u> and <u>width of ε</u>

… which therefore has… <u>area of 1</u> (1 = ε ×1/ε)

So <u>as ε gets smaller</u> the pulse gets <u>higher</u> and <u>narrower</u> but always has <u>area of 1</u>…

<u>In the limit</u> it “becomes” the delta function

<u>Precise Idea:</u> $\delta(t)$ is not an ordinary function… It is defined in terms of its behavior inside an integral:

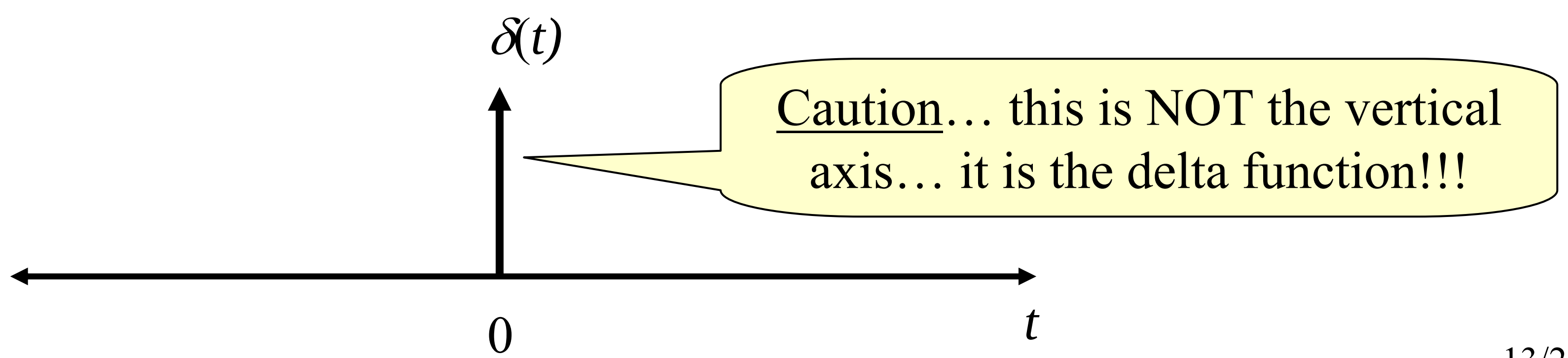
The delta function $\delta(t)$ is defined as something that satisfies the following two conditions:

$$\delta(t) = 0, \quad \text{for any } t \neq 0$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)dt = 1, \quad \text{for any } \varepsilon > 0$$

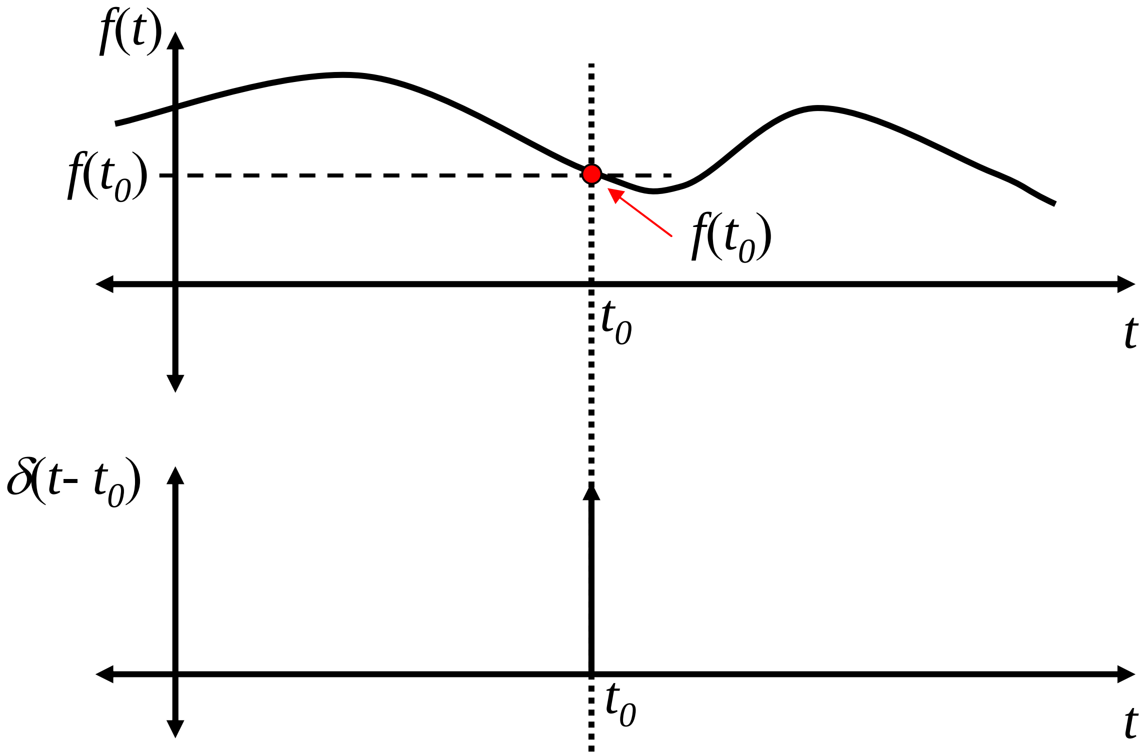
We show $\delta(t)$ on a plot using an arrow…

(conveys infinite height and zero width)



The **Sifting Property** is the most important property of $\delta(t)$:

$$\boxed{\int_{t_0-\varepsilon}^{t_0+\varepsilon} f(t)\delta(t-t_0)dt = f(t_0) \quad \forall\varepsilon > 0}$$



Integrating the product of $f(t)$ and $\delta(t-t_o)$ returns a single number… the value of $f(t)$ at the "location" of the shifted delta function

As long as the integral's limits surround the "location" of the delta… otherwise it returns zero

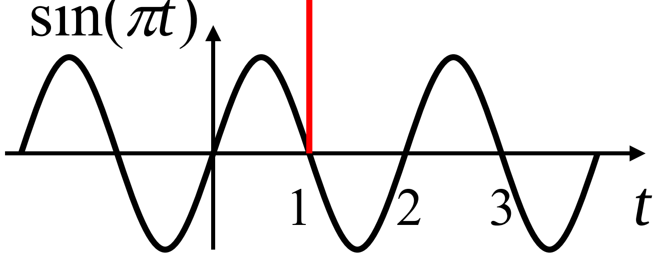
Steps for applying sifting property:

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \underbrace{f(t)}\delta(\overbrace{t-t_0})\,dt = f(t_0)$$

Step 1: Find variable of integration
Step 2: Find the argument of $\delta(\bullet)$
Step 3: Find the value of the variable of integration that causes the argument of $\delta(\bullet)$ to go to zero.
Step 4: If value in Step 3 lies inside limits of integration… Take everything that is multiplying $\delta(\bullet)$ and evaluate it at the value found in step 3; Otherwise… "return" zero

Example #1:

$$\int_{-4}^{7} \sin(\pi t)\delta(t-1)\,dt = ?$$



Step 1: t Step 2: $t - 1$
Step 3: $t - 1 = 0 \quad \Rightarrow \quad t = 1$
Step 4: $t = 1$ lies in $[-4,7]$ so evaluate… $\sin(\pi\times 1) = \sin(\pi) = 0$

$$\boxed{\int_{-4}^{7} \sin(\pi t)\delta(t-1)\,dt = 0}$$

Example #2: $$\int_0^2 \sin(\pi t)\delta(t-2.5)dt = ?$$

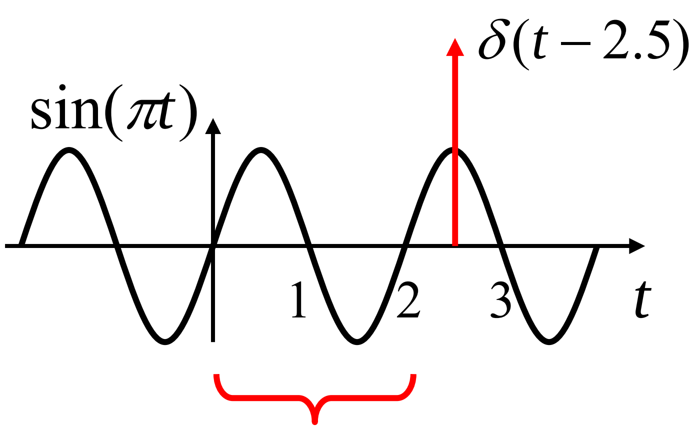
Step 1: Find variable of integration: t

Step 2: Find the argument of $\delta(\bullet)$: $t-2.5$

Step 3: Find the value of the variable of integration that causes the argument of $\delta(\bullet)$ to go to zero: $t-2.5=0 \Rightarrow t=2.5$

Step 4: If value in Step 3 lies inside limits of integration… No!

Otherwise… "return" zero…



$$\boxed{\int_0^2 \sin(\pi t)\delta(t-2.5)dt = 0}$$

Range of Integration Does NOT "include delta function"

Example #3: $\int_{-4}^{7} \sin(\omega t)(t-3)^2 \delta(3t+4)dt = ?$

Because of this… handle slightly differently!

Step 0: Change variables: let $\tau = 3t$ ➔ $d\tau = 3dt$ ➔ limits: $\tau_L = 3(-4)$ $\tau_L = 3(7)$

$$\int_{-12}^{21} \frac{1}{3}\sin(\omega\tau/3)(\tau/3-3)^2 \delta(\tau+4)d\tau = ?$$

Step 1: Find variable of integration: τ

Step 2: Find the argument of $\delta(\bullet)$: $\tau + 4$

Step 3: Find the value of the variable of integration that causes the argument of $\delta(\bullet)$ to go to zero: $\tau = -4$

Step 4: If value in Step 3 lies inside limits of integration… Yes!

Take everything that is multiplying $\delta(\bullet)$: $(1/3)\sin(\omega\tau/3)(\tau/3 - 3)^2$

…and evaluate it at the value found in step 3:

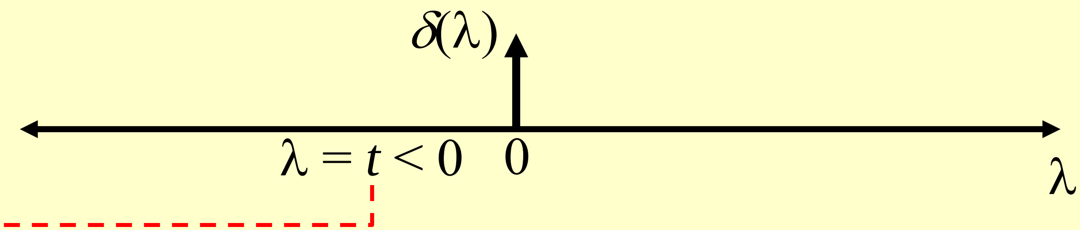
$$(1/3)\sin(-4/3\omega)(-4/3 - 3)^2 = 6.26\sin(-4/3\omega)$$

$$\boxed{\int_{-4}^{7} \sin(\omega t)(t-3)^2 \delta(3t+4)dt = 6.26\sin\left(-4/3\omega\right)}$$

One Relationship Between $\delta(t)$ & $u(t)$

$$\boxed{\int_{-\infty}^{t} \delta(\lambda)\,d\lambda = u(t)}$$

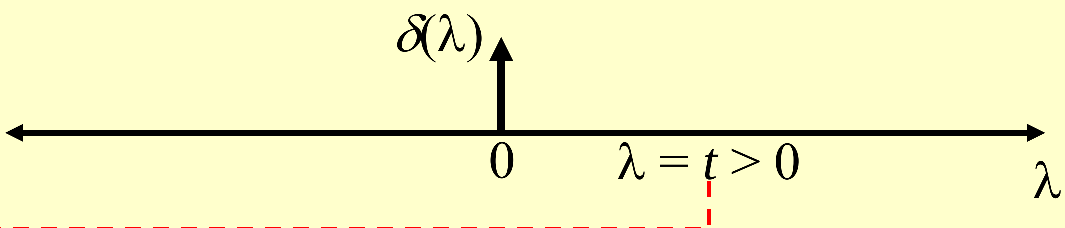
For $t < 0$: the integrand = 0 $\Rightarrow$ **integral = 0 for $t < 0$**



Range of Integration

Defines the unit step function

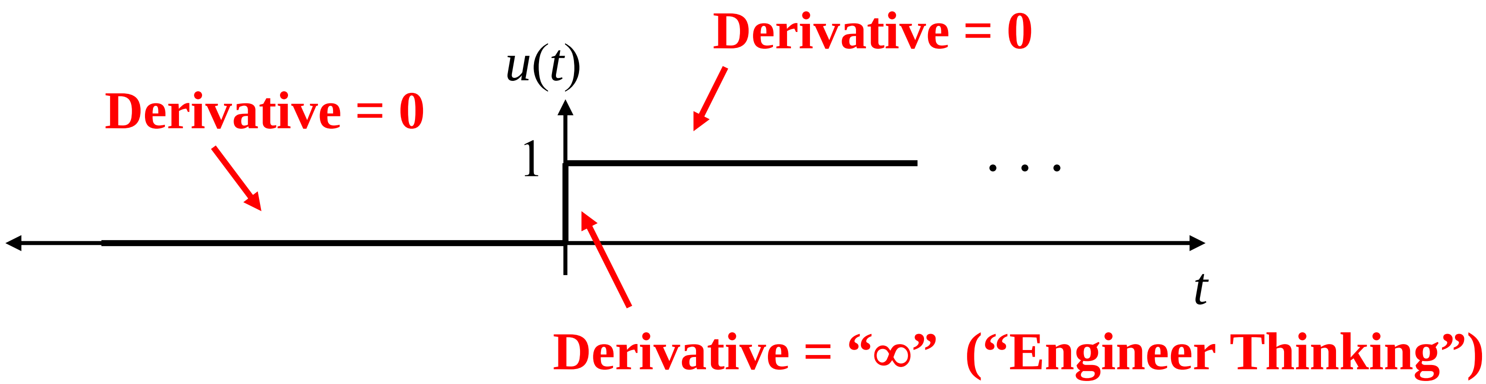
For $t > 0$: we "integrate over" the delta $\Rightarrow$ **integral = 1 for $t > 0$**



Range of Integration

Another Relationship Between $\delta(t)$ & $u(t)$

$$\boxed{\delta(t) = \frac{d}{dt}u(t)}$$



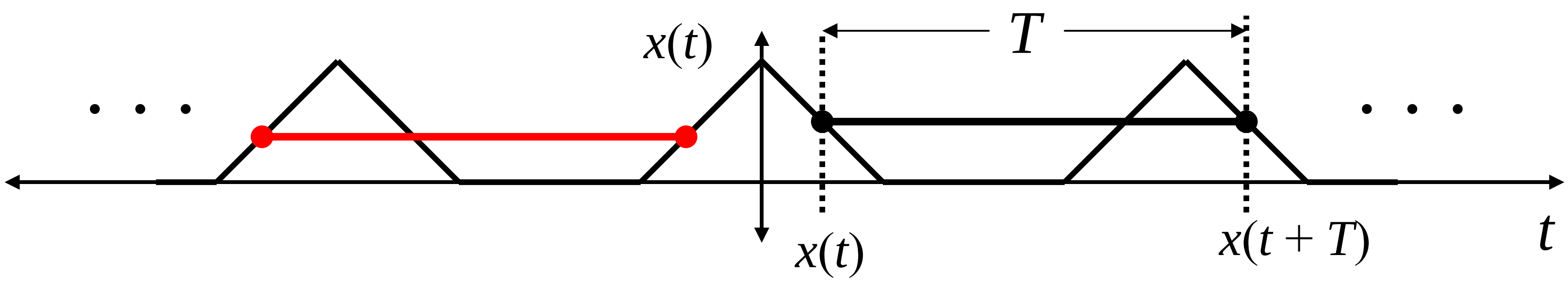
Our view of the delta function having infinite height but zero width matches this interpretation of the values of the derivative of the unit step function!!

Periodic Signals

Periodic signals are important because many human-made signals are periodic. Most test signals used in testing circuits are periodic signals (e.g., sine waves, square waves, etc.)

A Continuous-Time signal $x(t)$ is periodic with period T

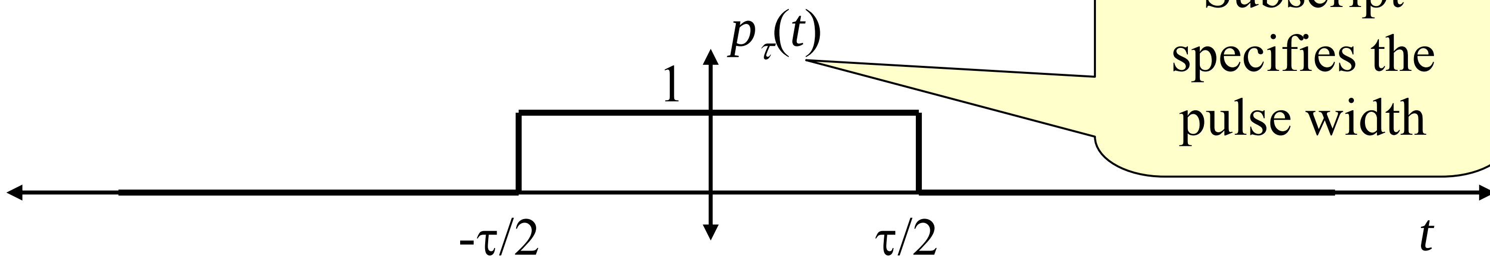
<u>if:</u> $$x(t+T) = x(t) \quad \forall t$$



Fundamental period = <u>smallest</u> such T

When we say "Period" we almost always mean "Fundamental Period"

Rectangular Pulse Function: $p_\tau(t)$

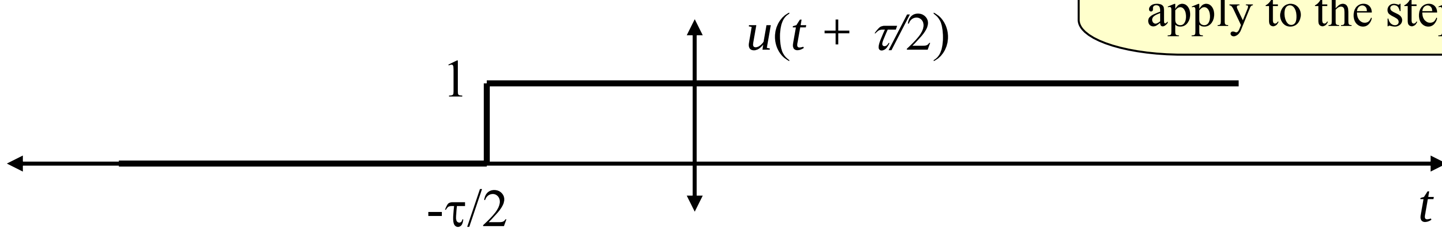


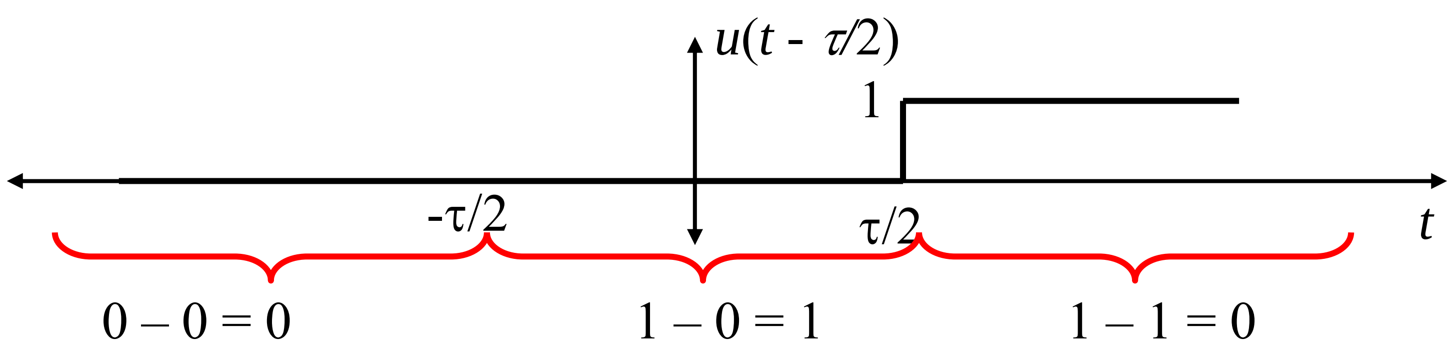
Subscript specifies the pulse width

We can build a Rectangular Pulse from Unit Step Functions:

$$p_\tau(t) = u(t + \tau/2) - u(t - \tau/2)$$

This is helpful because we will have lots of results that apply to the step function





Building Signals with Pulses: shifted pulses are used to "turn other functions on and off". This allows us to mathematically describe complicated functions in terms of simpler functions.