EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #18

• C-T Systems: Frequency-Domain Analysis of Systems
• Reading Assignment: Section 5.2 of Kamen and Heck
Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).
5.2 Response to Periodic Inputs

Since \( x(t) \) is periodic, write it as FS:

\[
x(t) = \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t}
\]

Sum these to get output

\[
y(t) = \sum_{k=-\infty}^{\infty} [H(\omega_0 k) c_k^x] e^{jk\omega_0 t}
\]

\( H(\omega_0 k) \) is the FS coefficient of \( y(t) \)

\( \text{Output} = \text{Sum of Individual Responses} \)

Linear System: So… Output = Sum of Individual Responses

But each individual response is to a complex sinusoid input ⇒ EASY!
General Insights from this Analysis

1. periodic in, periodic out

2. The system’s frequency response $H(\omega)$ works to modify the input FS coefficients to create the output FS coefficients:

$$c_k^y = H(k\omega_0) c_k^x$$
Example (Ex. 5.4 with Some Injected Reality)

Problem: suppose you have a circuit board that has a digital clock circuit on it. It makes the rectangular pulse train shown below:

(Of course most digital clock circuits would run much faster)

Suppose you need to connect this clock signal to a circuit on another circuit board using a twisted pair of wires:

Q: What effect does the cable have on the clock signal at the 2nd board???

Pair of wires can be modeled as an RC circuit:

Assume: The circuit “driving” the cable has an infinitesimally small output impedance (that is good!):
Assume: The circuit being “driven” by the cable has infinite input impedance (that is good!) i.e. No loading of the RC circuit

So…

\[ x(t) \rightarrow y(t) \rightarrow \text{(goes to driven circuit having infinite input impedance)} \]

**Goal:** Perform an analysis to enable you to recommend an acceptable value of cable RC time constant \((\text{Analysis Drives Design!})\)

**Step 1:** Analytically find FS of input and compute truncated FS sum:

From Ex. 3.4 we get:

\[ x(t) \approx \sum_{k=-N}^{N} c_k^x e^{jk\omega t} \]

Then plot vs. time \(t\)

\[ c_k^x = \begin{cases} 
\frac{1}{k\pi}, & k = \pm 1, \pm 5, \pm 9, \ldots \\
-\frac{1}{k\pi}, & k = \pm 3, \pm 7, \pm 11, \ldots \\
0, & k = \pm 2, \pm 4, \pm 6, \ldots \\
\frac{1}{2}, & k = 0 
\end{cases} \]

**Step 2:** Find cable’s frequency response as a function of RC:

\[ H(\omega) = \frac{1}{1 + j\omega RC} \]

(See Ex. in section 5.1)
**Step 3 (optional)**  (But it really helps you see what is going on!)

Look at frequency domain plots of Input and System (for various RC values)

- "stem" plot of FS coefficients’ magnitude $|c_k^x|
- "continuous" plot of Magnitude of system’s Frequency Resp. $|H(\omega)|$

**Step 4 (optional)**  (This also really helps you see what is going on)

Compute output FS coefficients: $c_k^y = H(k\omega_0)c_k^x$

Look at the result $\rightarrow$ “stem” plot of $|c_k^y|$

**Step 5:** Compute truncated FS sum to see output signal

$$y(t) \approx \sum_{k=-N}^{N} c_k^y e^{j\omega_0 t}$$

Plot vs. time $t$

See plots on next 3 pages for three RC time constant values:

- $RC = 0.01\ s$
- $RC = 0.1\ s$
- $RC = 1\ s$

*Note: Short RC time constant passes high frequencies better than long RC time constant*
RC Circuit Analysis w/ Square Wave Input

Input Signal

**Artifact from summing only a finite # of terms**

\[
RC = 0.01 \text{ s}
\]

Output Signal

**Does Decent Job of “Passing” Most of the Significant Frequencies of the Input**

Input Signal

Cable’s Freq. Resp.

Output Signal

Input Signal’s FS Coefficients

Cable’s Freq. Resp.

Output Signal’s FS Coefficients
RC Circuit Analysis w/ Square Wave Input

**Input Signal**

![Input Signal Graph](image1)

**Output Signal**

![Output Signal Graph](image2)

**Input Signal’s FS Coefficients**

![FS Coefficients Graph](image3)

**Cable’s Freq. Resp.**

![Frequency Response Graph](image4)

**Output Signal’s FS Coefficients**

![FS Coefficients Graph](image5)

RC = 0.1 s

Does Moderate Job of “Passing” Most of the Significant Frequencies of the Input
RC Circuit Analysis w/ Square Wave Input

Input Signal

Output Signal

RC = 1 s

Does Poor Job of “Passing” Most of the Significant Frequencies of the Input

Input Signal’s FS Coefficients

Cable’s Freq. Resp.

Output Signal’s FS Coefficients
Insight from Example:

- We used a simple model for the cable to make it easy to analyze
  - But... the method would be the same even if we had a more detailed model for the cable
- The input clock signal has nice sharp transitions due to its significant high frequency components
- Cables that significantly suppressed the input’s high frequency components provided a low-quality clock signal to the 2nd board
- We made assumptions about the driver circuit and the driven circuit
  - The driver was assumed to have zero output resistance
    - If that were not true, its output impedance gets added to the resistor and that would further degrade the performance (in fact the driver’s output impedance may be more than the cable resistance in which case it would be the dominant factor)
  - The driven circuit was assumed to have infinite input impedance
    - If that were not true we would have to combine it in parallel with the capacitor’s impedance... this would further degrade the performance
- Typically the RC value of a cable increases with length
  - So performance would decrease with length of cable
function \([x,t]=example_5_4(RC)\)

\(k=1:4:200;\)
\(K=[k;k+1;k+2;k+3];\)
\(C_k=1/(K*pi);\) % fill matrix with this form… keep first row
\(C_k(2,:)=zeros(size(k));\) % replace 2nd row w/ zeros
\(C_k(3,:)=-1*C_k(3,:);\) % replace 3rd row w/ negatives
\(C_k(4,:)=zeros(size(k));\) % replace 4th row w/ zeros
\(c_x_k=C_k(:);\) % turn into col. vector by going down matrix columns

\(t=-3:(6/800):3;\) % create time vector with approp. spacing
\(k=(1:max(max(K)));\) % create FS term index
\([T,K]=meshgrid(t,k);\) % create time matrix and index matrix

\(wo=pi;\)
\(EXP_pos=exp(j*T.*K*wo);\) % Each row is a sinusoid term
\(EXP_neg=exp(-j*T.*K*wo);\)
% Compute the FS summation to get approx. input time signal
\(x=0.5+sum(c_x_k(:,ones(1,length(t))).*EXP_pos)\)
+\(sum(conj(c_x_k(:,ones(1,length(t)))).*EXP_neg);\)
% The above cmdm adds up the rows of EXP weighted by the c_x_k
subplot(2,3,1)
plot(t,x)
xlabel('time (sec)')
ylabel('Input Signal x(t)')

subplot(2,3,4)
\(w_k=k*wo;\) % create vector of FS frequencies
% In the next line we have to attach c 0=0.5 and its freq
\(stem([0 w_k]/(2*pi),[0.5 abs(c_x_k).'])\) % plot vs freq in Hz
xlabel('f_o (Hz)')
ylabel('c^x_k')
axis([-0.5 15 0 0.6])

\(H_k=1./(1+j*w_k*RC);\) % compute Freq Resp @ these Freqs
\(H_0=1./(1+j*0*RC);\)
\(c_y_k=c_x_k.*H_k;\) % compute output FS coeffs
\(stem([0 w_k]/(2*pi),[0.5*H_0 abs(c_y_k).'])\)
axis([-0.5 15 0 0.6])

subplot(2,3,6)
FS summation to get approx. output time signal
\(y=0.5*H_0+sum(c_y_k(:,ones(1,length(t))).*EXP_pos)\)
+\(sum(conj(c_y_k(:,ones(1,length(t)))).*EXP_neg);\)
plot(t,y)
xlabel('time (sec)')
ylabel('Output Signal y(t)')

4 types of indices:
1,5,9,…
2,6,10,…
3,7,11,…
4,8,12,…

Matlab Code for Example’s Plots

\[
\begin{bmatrix}
1 & 1 & 1 & \cdots \\
2 & 2 & 2 & \cdots \\
3 & 3 & 3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\(w=0:0.1:max(w_k);\) % create finely-spaced frequency
\(H=1./(1+j*w*RC);\) % compute Freq Resp @ these Freqs
plot(w/(2*pi),abs(H)) % plot vs. freq in Hz
xlabel('f (Hz)')
ylabel('|H(f)|')
axis([-0.5 15 0 1.1])

\(w_k=k*wo;\) % create vector of FS frequencies
\(H_k=1./(1+j*w_k*RC);\) % compute Freq Resp at FS freqs
\(H_0=1./(1+j*0*RC);\)
\(c_y_k=c_x_k.*H_k;\) % compute output FS coeffs
\(stem([0 w_k]/(2*pi),[0.5*H_0 abs(c_y_k).'])\)
xlabel('f_o (Hz)')
ylabel('c^y_k')
axis([-0.5 15 0 0.6])

\(w_k=k*wo;\) % create vector of FS frequencies
\(\% In the next line we have to attach c 0=0.5 and its freq
\(\% \) plot vs freq in Hz
\(\% \) xaxis([-0.5 15 0 0.6])