BINGHAMTON UNIVERSITY
State University of New York

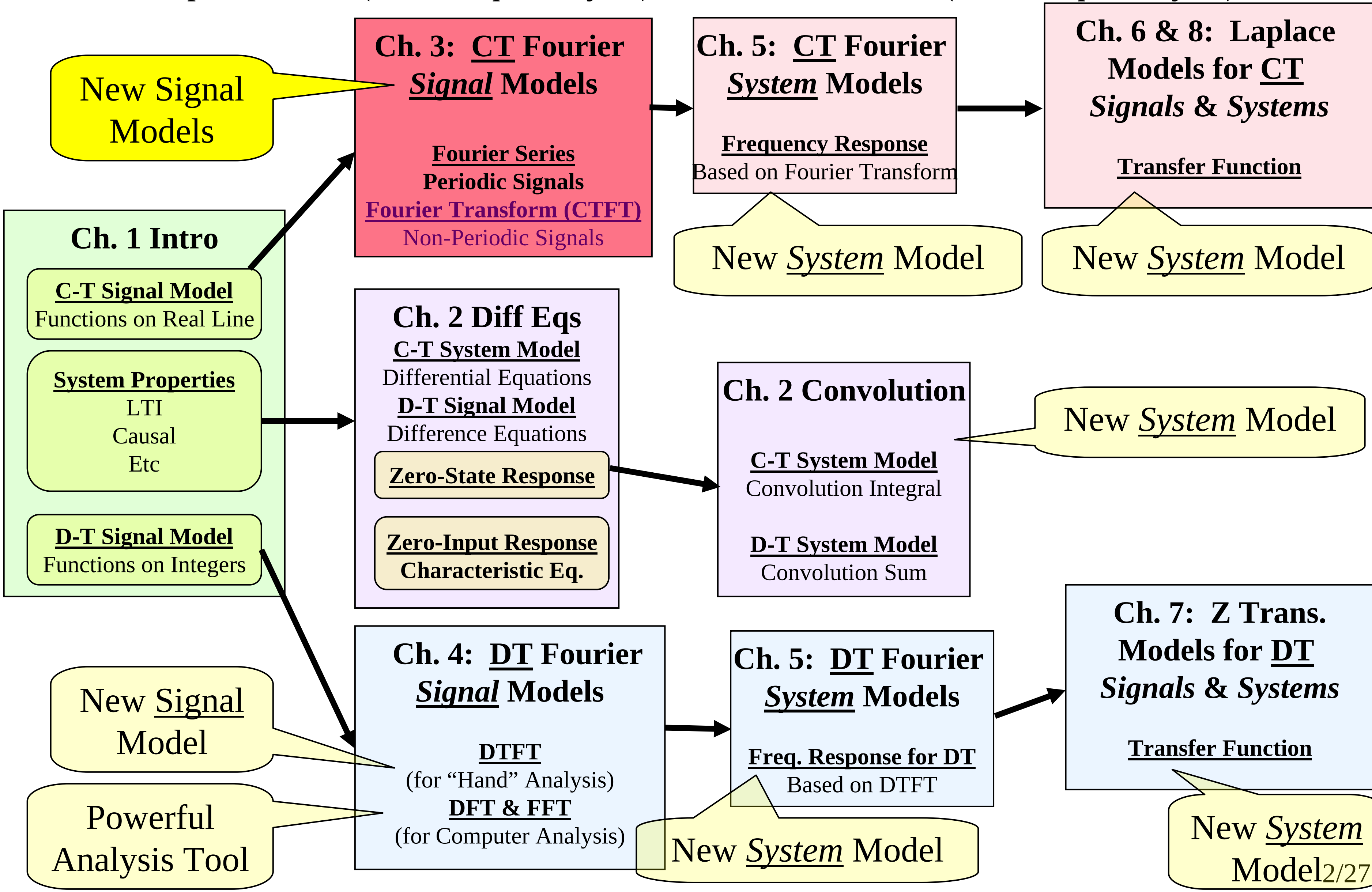
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #14

- C-T Signals: Fourier Transform (for Non-Periodic Signals)
- Reading Assignment: Section 3.4 & 3.5 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



4.3 Fourier Transform

Recall: Fourier Series represents a periodic signal as a sum of sinusoids

or complex sinusoids $e^{jk\omega_0 t}$

Note: Because the FS uses "harmonically related" frequencies $k\omega_0$, it can only create periodic signals

Q: Can we modify the FS idea to handle non-periodic signals?

A: Yes!!

What about $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_k t}$?

With *arbitrary* discrete frequencies…
NOT harmonically related

That will give some non-periodic signals but not some that are important!!

The problem with $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_k t}$ is that it cannot include all possible frequencies!

How about: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Yes… this will work for any *practical* non-periodic signal!!

Called the "Fourier Integral" also, more commonly, called the "**Inverse Fourier Transform**"

Plays the role of c_k

Plays the role of $e^{jk\omega_0 t}$

Integral replaces sum because it can "add up over the continuum of frequencies"!

Okay… given $x(t)$ how do we get $X(\omega)$?

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Called the "**Fourier Transform**" of $x(t)$

Note: $X(\omega)$ is complex-valued function of $\omega \in (-\infty, \infty)$

$|X(\omega)|$ $\qquad$ $\angle X(\omega)$

Need to use two plots to show it

Comparison of FT and FS

Fourier Series: Used for periodic signals

Fourier Transform: Used for non-periodic signals (although we will see later that it can also be used for periodic signals)

	Synthesis	**Analysis**
Fourier Series	$x(t) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ **Fourier Series**	$c_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$ **Fourier Coefficients**
Fourier Transform	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ **Inverse Fourier Transform**	$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ **Fourier Transform**

FS coefficients c_k are a complex-valued function of integer k

FT $X(\omega)$ is a complex-valued function of the variable $\omega \in (-\infty, \infty)$

Synthesis Viewpoints:

FS: $$x(t) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$|c_k|$ shows how much there is of the signal at frequency $k\omega_0$

$\angle c_k$ shows how much phase shift is needed at frequency $k\omega_0$

We need two plots to show these

FT: $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$|X(\omega)|$ shows how much there is in the signal at frequency ω

$\angle X(\omega)$ shows how much phase shift is needed at frequency ω

We need two plots to show these

Some FT Notation:

If $X(\omega)$ is the Fourier transform of $x(t)$…

then we can write this in several ways:

1. $x(t) \leftrightarrow X(\omega)$

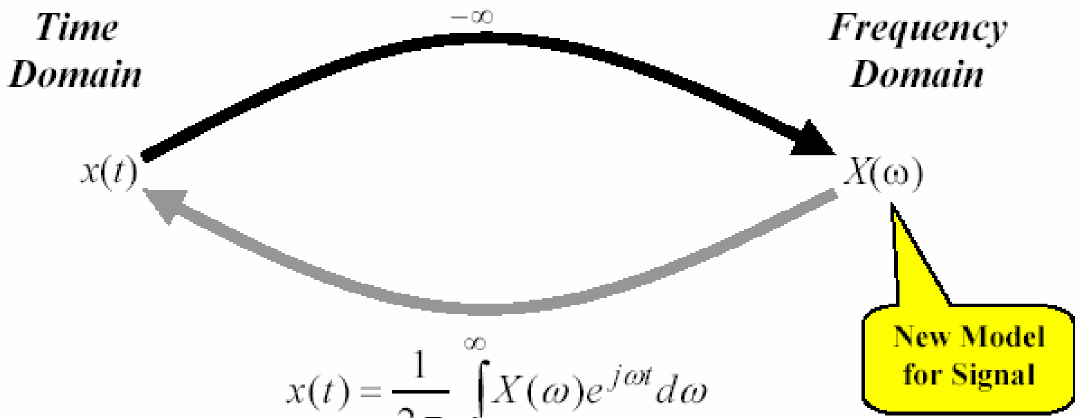
2. $X(\omega) = \mathfrak{F}\{x(t)\}$ $\Rightarrow$ $\mathfrak{F}\{\ \}$ is an “operator” that operates on $x(t)$ to give $X(\omega)$

3. $x(t) = \mathfrak{F}^{-1}\{X(\omega)\}$ $\Rightarrow$ $\mathfrak{F}^{-1}\{\ \}$ is an “operator” that operates on $X(\omega)$ to give $x(t)$

Fourier Transform Viewpoint

View FT as a transformation into a new "domain"

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$x(t)$ is the "time domain" description of the signal

$X(\omega)$ is the "frequency domain" description of the signal

Analogy: Looking at $X(\omega)$ is "like" looking at an x-ray of the signal- in the sense that an x-ray lets you see what is inside the object… shows what stuff it is made from.

In this sense: $X(\omega)$ shows what is "inside" the signal – it shows how much of each complex sinusoid is "inside" the signal

Note: $x(t)$ completely determines $X(\omega)$

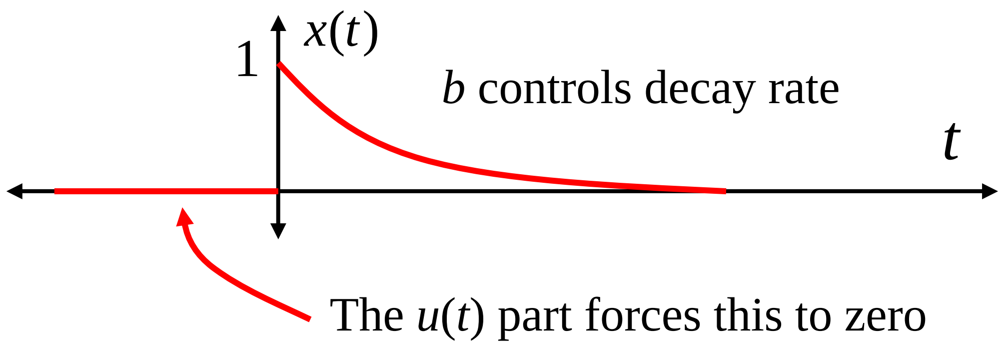
$X(\omega)$ completely determines $x(t)$

There are some advanced mathematical issues that can be hurled at these comments… we'll not worry about them

FT Example: Decaying Exponential

Given a signal $x(t) = e^{-bt}u(t)$ find $X(\omega)$ if $b > 0$

Solution: First see what $x(t)$ looks like:



What does this look like if $b < 0$???

Now…apply the definition of the Fourier transform. Recall the general form:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Now plug in for our signal:

$$X(\omega) = \int_{-\infty}^{\infty} \underbrace{e^{-bt}u(t)e^{-j\omega t}}_{\text{integrand = 0 for } t<0 \text{ due to the } u(t)} dt = \int_{0}^{\infty} e^{-bt}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(b+j\omega)t}dt$$

integrand = 0 for *t* < 0 due to the *u*(*t*)

Set lower limit to 0 and then $u(t) = 1$ over integration range

Easy integral!

$$= \left[\frac{-1}{b+j\omega}e^{-(b+j\omega)t}\right]_{t=0}^{t=\infty} = \frac{-1}{b+j\omega}\left[e^{-(b+j\omega)\infty} - e^{-(b+j\omega)0}\right]$$

$$= \frac{-1}{b+j\omega}\left[\underbrace{e^{-b\infty}}_{=0}\underbrace{e^{-j\omega\infty}}_{mag=1} - \underbrace{e^{0}}_{=1}\right] = \frac{-1}{b+j\omega}[0-1]$$

$$= \frac{1}{b+j\omega}$$

Only if $b>0$… what happens if $b<0$

Summary of FT Result for Decaying Exponential

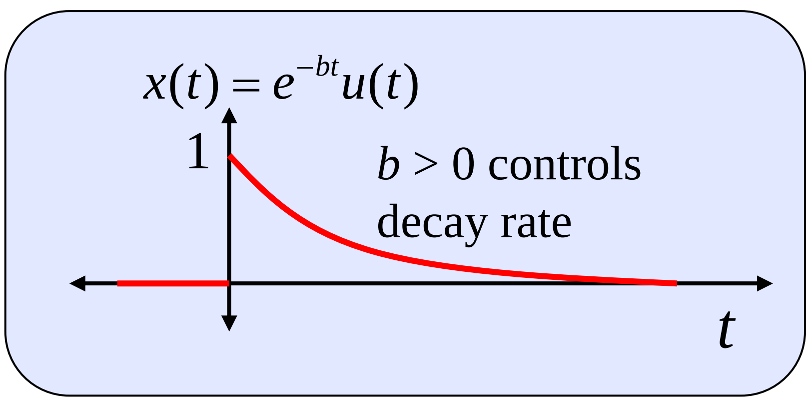
$$x(t) = e^{-bt}u(t) \quad \longleftrightarrow \quad X(\omega) = \frac{1}{b + j\omega}$$

For $b > 0$

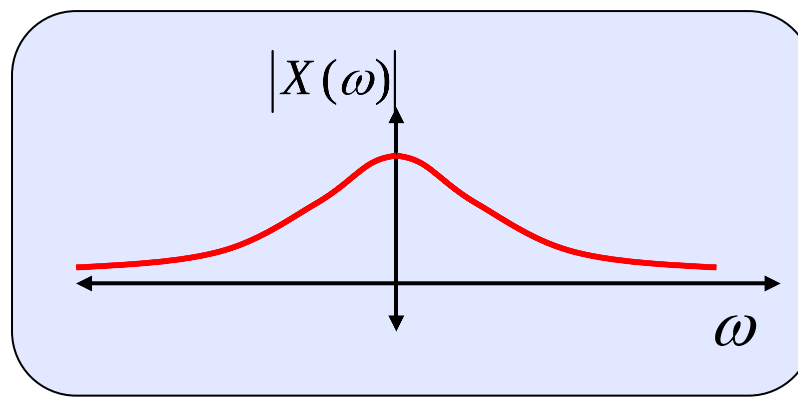
(Complex Valued)

$$|X(\omega)| = \frac{1}{\sqrt{b^2 + \omega^2}}$$ Magnitude

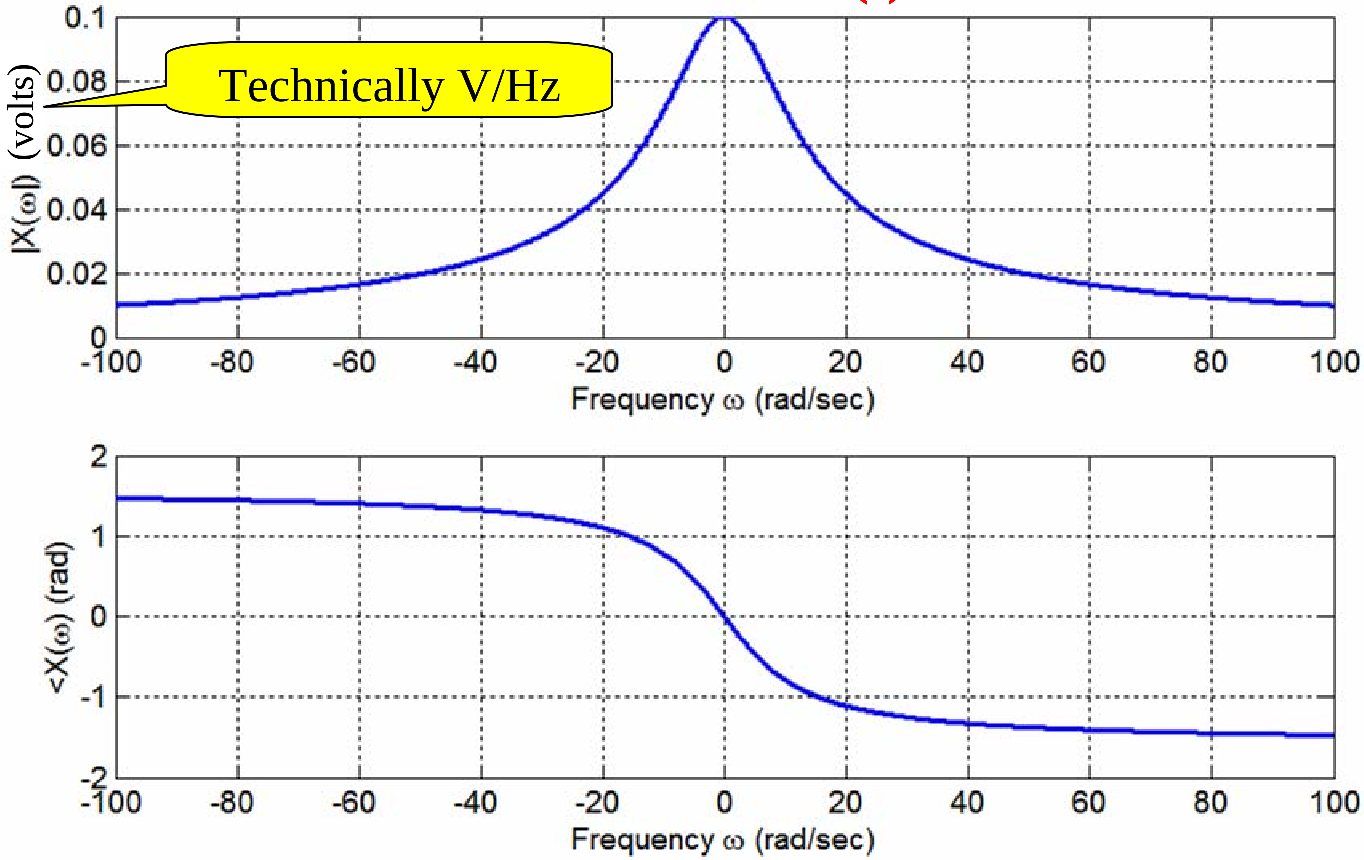
$$\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{b}\right)$$ Phase



$\longleftrightarrow$



Fourier Transform of $e^{-bt}u(t)$ for $b = 10$



Note that magnitude plot has <u>even</u> symmetry

Note that phase plot has <u>odd</u> symmetry

True for <u>every</u> real-valued signal

<u>MATLAB Commands to Compute FT</u>

```
w=-100:0.2:100;
b=10;
X=1./(b+j*w);
```

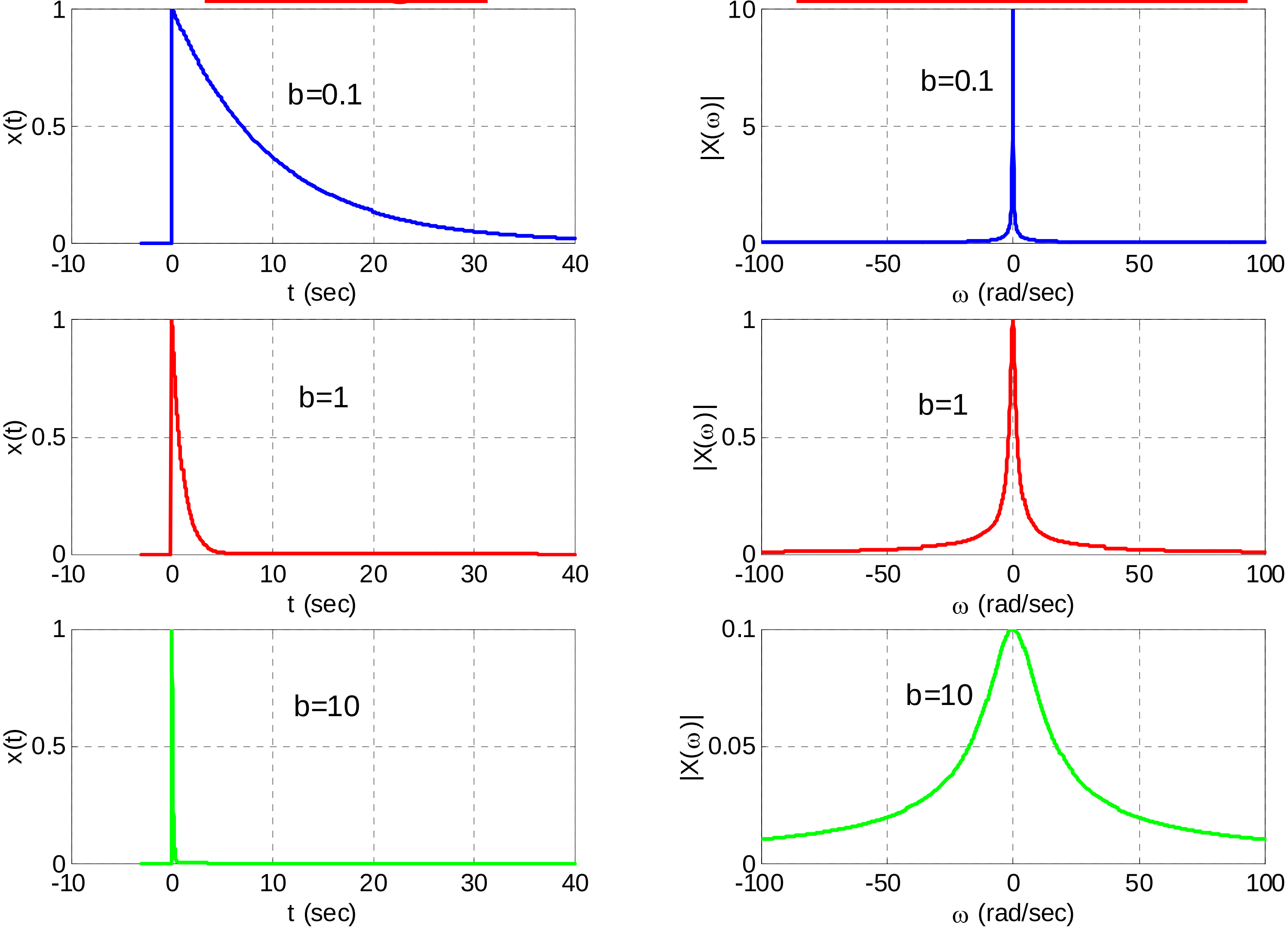
<u>Plotting Commands</u>

```
subplot(2,1,1); plot(w,abs(X))
xlabel('Frequency \omega (rad/sec)')
ylabel('|X(\omega|) (volts)'); grid
subplot(2,1,2); plot(w,angle(X))
xlabel('Frequency \omega (rad/sec)')
ylabel('<X(\omega) (rad)'); grid
```

Note: Book's Fig. 3.12 only shows <u>one</u>-sided spectrum plots

Time Signal

Fourier Transform



Exploring Effect of decay rate *b* on the Fourier Transform's Shape

Short Signals have FTs that spread more into High Frequencies!!!

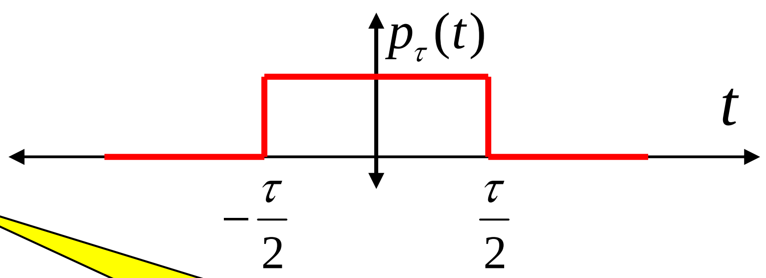
<u>Note</u>: As *b* increases…

1. **Decay rate in time signal increases**
2. **High frequencies in Fourier transform are more prominent.**

Example: FT of a Rectangular pulse

τ = pulse width

Given: a rectangular pulse signal $p_\tau(t)$



Recall: we use this symbol to indicate a rectangular pulse with width τ

Find: $P_\tau(\omega)$… the FT of $p_\tau(t)$

Note the Notational Convention: lower-case for time signal and corresponding upper-case for its FT

Solution:

Note that

$$p_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & otherwise \end{cases}$$

Now apply the definition of the FT:

$$P_\tau(\omega) = \int_{-\infty}^{\infty} p_\tau(t) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

Limit integral to where $p_\tau(t)$ is non-zero… and use the fact that it is 1 over that region

$$= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{\frac{-\tau}{2}}^{\frac{\tau}{2}} = \frac{2}{\omega}\left[\frac{e^{j\frac{\omega\tau}{2}} - e^{-j\frac{\omega\tau}{2}}}{j2}\right]$$

Artificially inserted 2 in numerator and denominator

$$= \sin\left(\frac{\omega\tau}{2}\right)$$

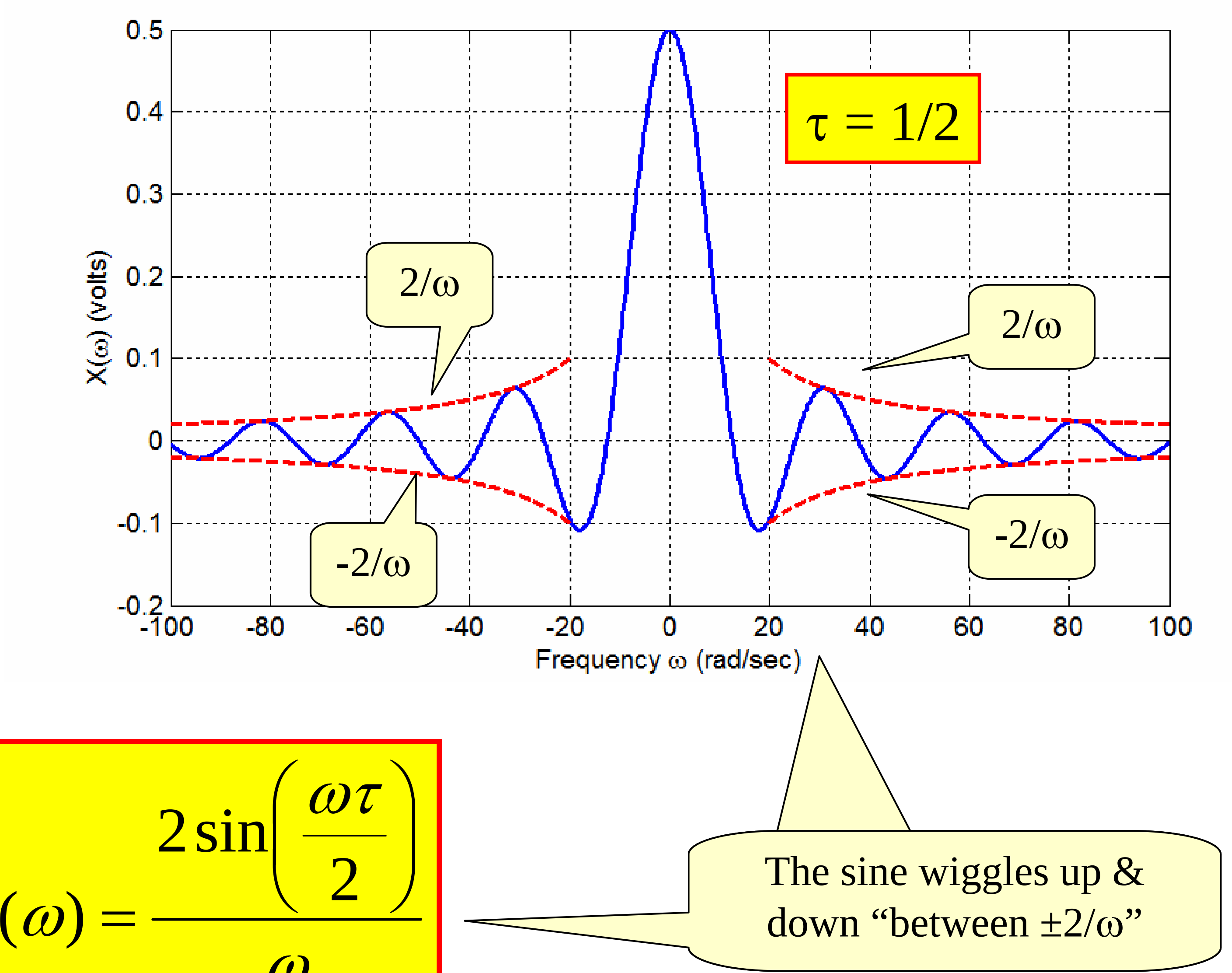
Use Euler's Formula

$$\Rightarrow \quad P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

sin goes up and down between -1 and 1

1/ω decays down as |ω| gets big… this causes the overall function to decay down

For ***this*** case the FT is real valued so we can plot it using a single plot (shown in solid blue here):



$$P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

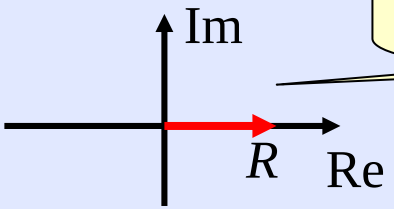
Now… let's think about how to make magnitude/phase plot…

Even though this FT is real-valued we can still plot it using magnitude and phase plots:

We can view any real number as a complex number that has zero as its imaginary part

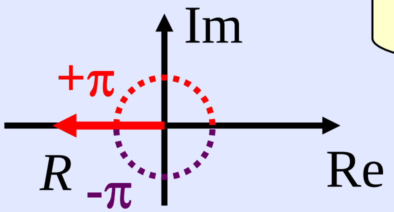
A *positive* real number R will have:

$|R| = R \qquad \angle R = 0$



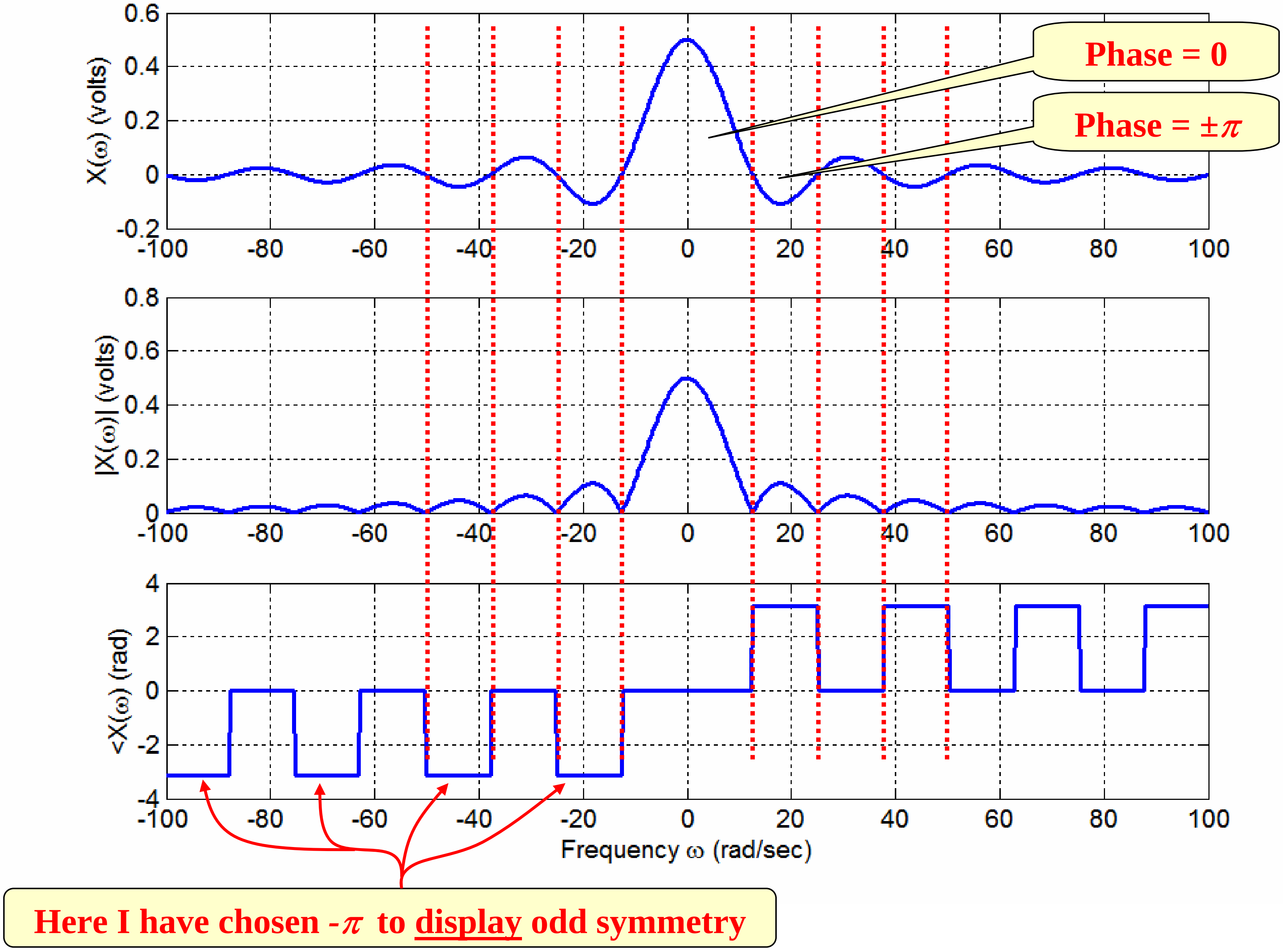
A *negative* real number R will have:

$|R| = -R \qquad \angle R = \pm\pi$

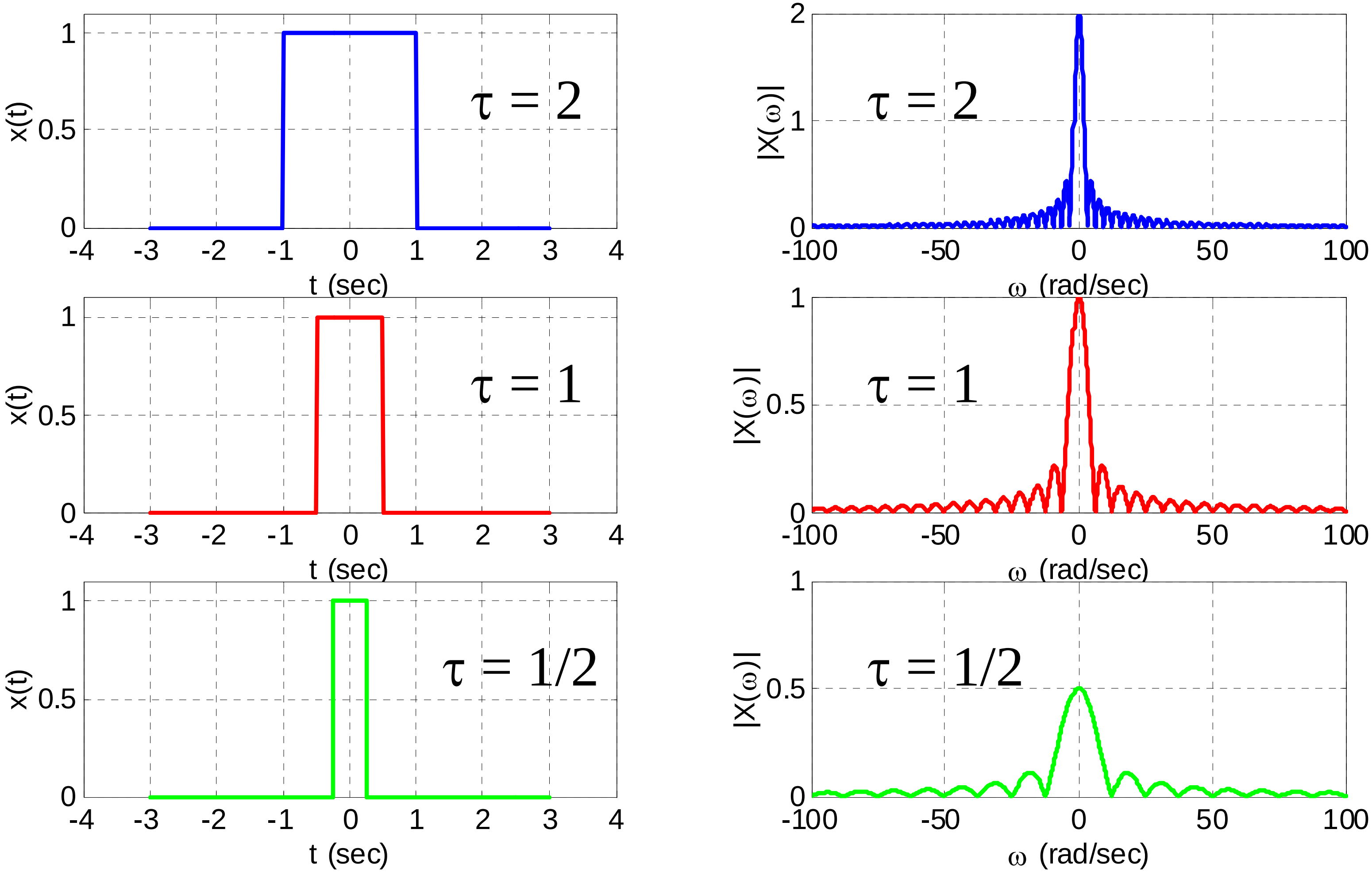


Can use either one!!

Applying these Ideas to the Real-valued FT $P_\tau(\omega)$



Effect of Pulse Width on the FT $P_\tau(\omega)$



Note: As width decreases, FT is more widely spread

➔ Narrow pulses "take up more frequency range"

Definition of "Sinc" Function

The result we just found had this mathematical form: $P_\tau(\omega) = \dfrac{2\sin\left(\dfrac{\omega\tau}{2}\right)}{\omega}$

This kind of structure shows up frequently enough that we define a special function to capture it:

Define: $$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Note that sinc(0) = 0/0.
So… Why is sinc(0) = 1?

It follows from L'Hopital's Rule

With a little manipulation we can re-write the FT result for a pulse in terms of the sinc function:

Recall:

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Need π times something…

$$P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \frac{2\sin\left(\frac{\pi}{\pi}\frac{\omega\tau}{2}\right)}{\omega} = \frac{2\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\omega}$$

Now we need the same thing down here as inside the sine…

$$= \pi\frac{\tau}{2\pi}\frac{2\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\pi\frac{\tau}{2\pi}\omega} = \tau\frac{\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\pi\frac{\omega\tau}{2\pi}} = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

$$P_\tau(\omega) = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

Table of Common Fourier Transform Results

We have just found the FT for two common signals…

$$x(t) = e^{-bt}u(t) \longleftrightarrow X(\omega) = \frac{1}{b + j\omega}$$

$$p_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & otherwise \end{cases} \longleftrightarrow P_\tau(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

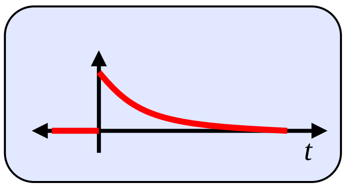
There are tables in the book but I recommend that you use the Tables I provide on the Website

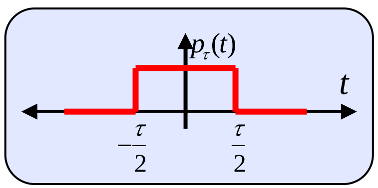
See FT Table on the Course Website for a list of these and many other FT.

You should study this table…

- If you encounter a time signal or FT that is on this table you should recognize that it is on the table without being told that it is there.
- You should be able to recognize entries in graphical form as well as in equation form (so… it would be a good idea to make plots of each function in the table to learn what they look like! See next slide!!!)
- You should be able to use multiple entries together with the FT properties we'll learn in the next set of notes (and there will be another Table!)

For your FT Table you should spend time making sketches of the entries … like this:





$\delta(t - c) \leftrightarrow e^{-j\omega c}$, c any real number

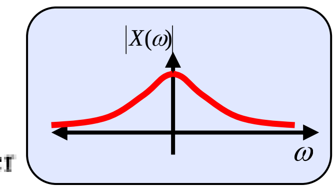
$e^{-bt}u(t) \leftrightarrow \dfrac{1}{j\omega + b}$, $b > 0$

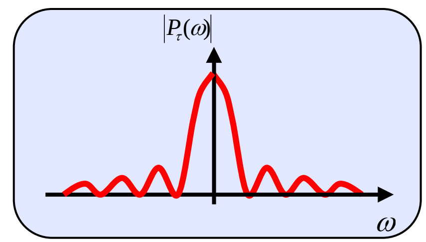
$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$, ω_0 any real number

$p_\tau(t) \leftrightarrow \tau \operatorname{sinc} \dfrac{\tau\omega}{2\pi}$

$\tau \operatorname{sinc} \dfrac{\tau t}{2\pi} \leftrightarrow 2\pi p_\tau(\omega)$

$\left(1 - \dfrac{2|t|}{\tau}\right) p_\tau(t) \leftrightarrow \dfrac{\tau}{2} \operatorname{sinc}^2\left(\dfrac{\tau\omega}{4\pi}\right)$



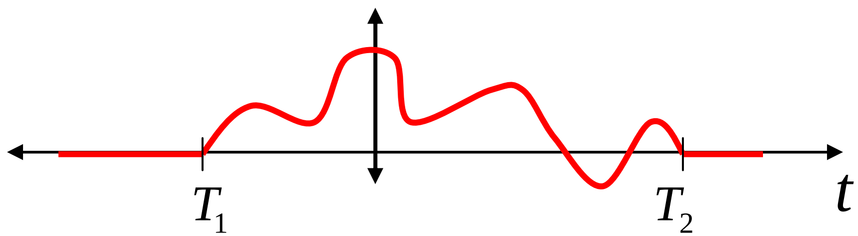


Bandlimited and Timelimited Signals

Now that we have the FT as a tool to analyze signals, we can use it to identify certain characteristics that many practical signals have.

A signal $x(t)$ is **timelimited** (or of finite duration) if there are 2 numbers T_1 & T_2 such that:

$$x(t) = 0 \quad \forall t \notin [T_1, T_2]$$

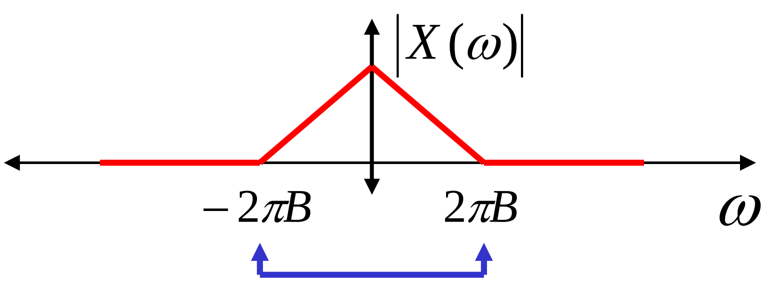


A (real-valued) signal $x(t)$ is **bandlimited** if there is a number B such that

$$|X(\omega)| = 0 \quad \forall \omega > 2\pi B$$

2πB is in rad/sec

B is in Hz



Recall: If $x(t)$ is real-valued then $|X(\omega)|$ has "even symmetry"

<u>FACT:</u> A signal can not be both timelimited <u>and</u> bandlimited

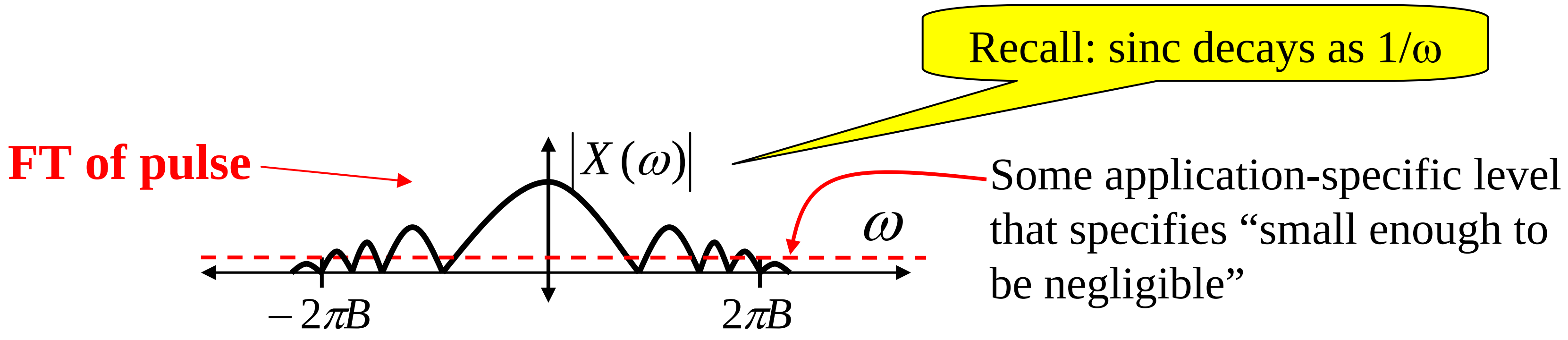
$\Rightarrow$ Any timelimited signal is <u>not</u> bandlimited

$\Rightarrow$ Any bandlimited signal is <u>not</u> timelimited

Note: All practical signals must “start” & “stop”

$\Rightarrow$ timelimited $\Rightarrow$ <u>Practical</u> signals are <u>not</u> bandlimited!

But… engineers say <u>practical</u> signals are <u>effectively bandlimited</u> because for <u>almost all</u> practical signals $|X(\omega)|$ decays to zero as ω gets large

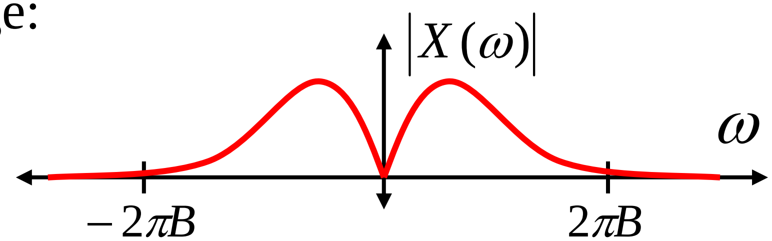


<u>This signal is effectively bandlimited to B Hz</u> because $|X(\omega)|$ falls below (and stays below) the specified level for all ω above $2\pi B$

Bandwidth (Effective Bandwidth)

Abbreviate Bandwidth as "BW"

For a lot of signals – like audio – they fill up the lower frequencies but then decay as ω gets large:



Signals like this are called "lowpass" signals

We say the signal's BW = B in Hz if there is "negligible" content for $|\omega| > 2\pi B$

Must specify what "negligible" means

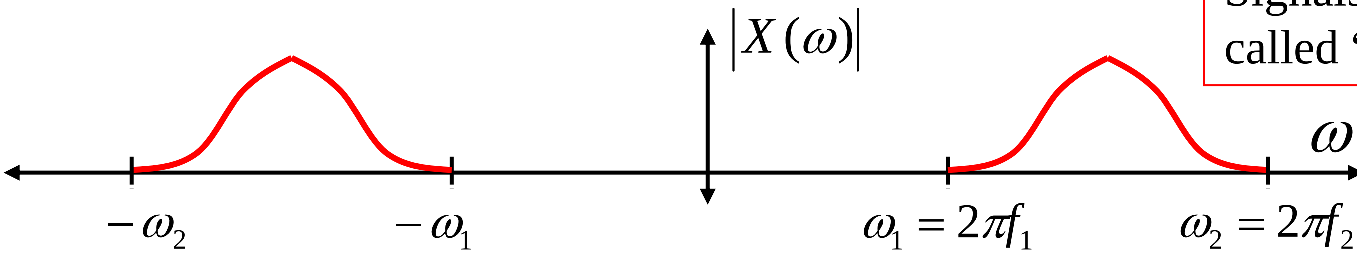
For Example:

1. High-Fidelity Audio signals have an accepted BW of about 20 kHz
2. A speech signal on a phone line has a BW of about 4 kHz

Early telephone engineers determined that limiting speech to a BW of 4kHz still allowed listeners to understand the speech

For other kinds of signals – like "radio frequency (RF)" signals – they are concentrated at high frequencies

Signals like this are called "bandpass" signals



If the signal's FT has negligible content for $|\omega| \notin [\omega_1, \omega_2]$ then we say the signals BW = $f_2 - f_1$ in Hz

<u>For Example</u>:

1. The signal transmitted by an FM station has a BW of 200 kHz = 0.2 MHz
   a. The station at 90.5 MHz on the "FM Dial" must ensure that its signal does not extend outside the range [90.4, 90.6] MHz
   b. Note that: FM stations all have an odd digit after the decimal point. This ensures that adjacent bands don't overlap:
      i. FM90.5 covers [90.4, 90.6]
      ii. FM90.7 covers [90.6, 90.8], etc.
2. The signal transmitted by an AM station has a BW of 20 kHz
   a. A station at 1640 kHz must keep its signal in [1630, 1650] kHz
   b. AM stations have an even digit in the tens place and a zero in the ones