BINGHAMTON UNIVERSITY
State University of New York

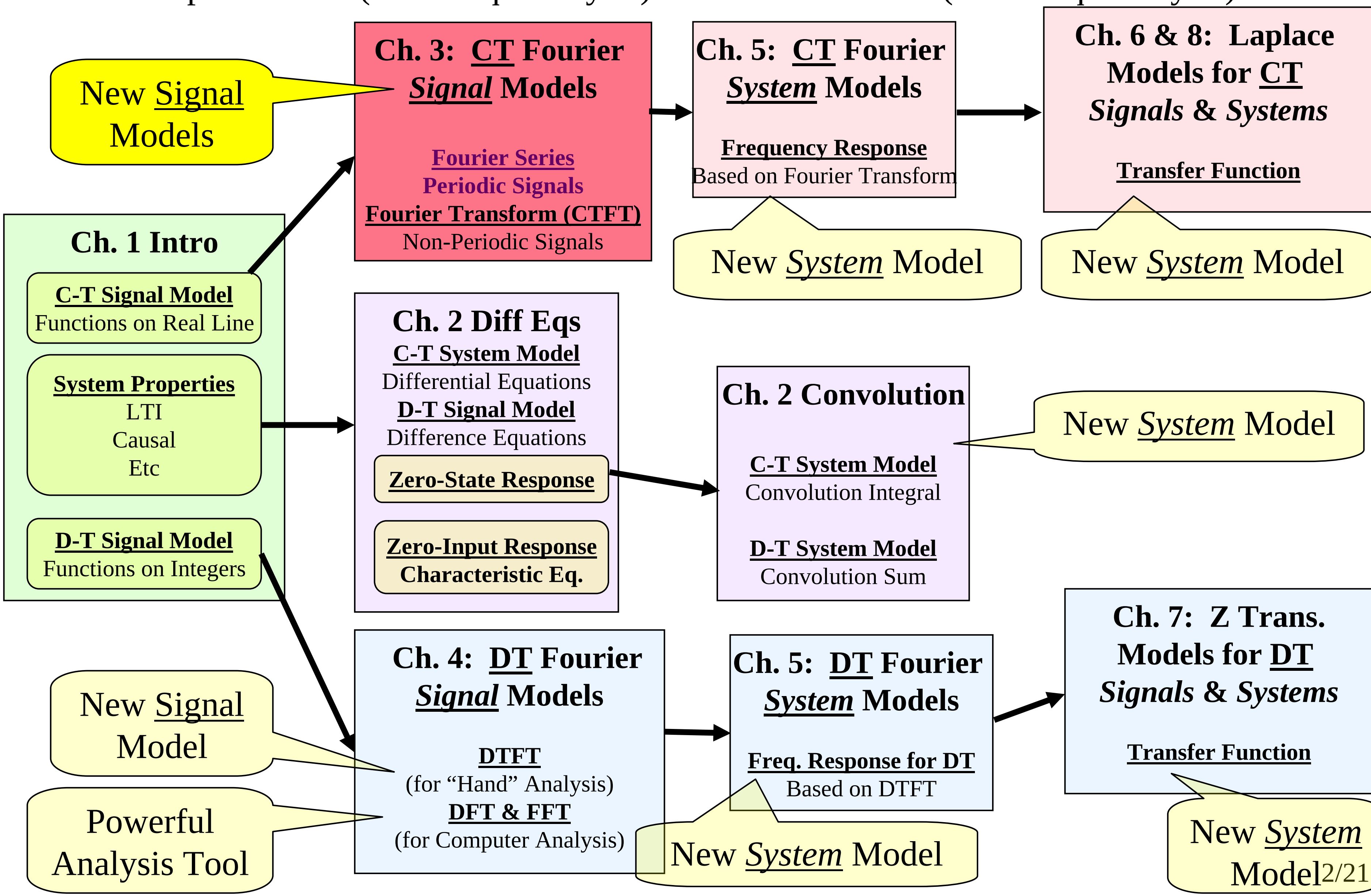
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #13

- C-T Signals: Fourier Series (for Periodic Signals)
- Reading Assignment: Section 3.2 & 3.3 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



3.2 & 3.3 Fourier Series

In the last set of notes we looked at building signals using:

N = finite integer

$$x(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t} \qquad x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(k\omega_0 t + \theta_k)$$

We saw that these build periodic signals.

Q: Can we get ***any*** periodic signal this way?

A: No! There are some periodic signals that need an ***infinite*** number of terms:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Fourier Series
(Complex Exp. Form)

k are integers

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Fourier Series
(Trig. Form)

k are integers

Sect. 3.3

These are two different forms of the same tool!!

There is a 3rd form that we'll see later.

There is a related Form in Sect. 3.2

Q: Does *this* now let us get any periodic signal?

A: No! Although Fourier thought so!

See top of p. 155

Dirichlet showed that there are some that can't be written in terms of a FS!

But… those will never show up in practice!

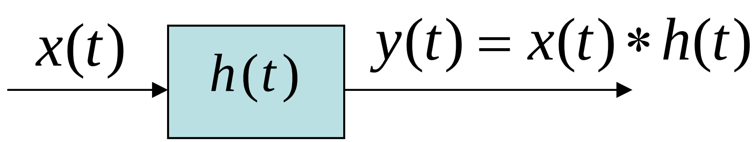
So we can write any practical periodic signal as a FS with infinite # of terms!

So what??!! Here is what!!

We can now break virtually any periodic signal into a sum of simple things…

and we already understand how these simple things travel through an LTI system!

So, instead of:



We break $x(t)$ into its FS components and find how each component goes through. (See chapter 5)

To do this kind of convolution-evading analysis we need to be able to solve the following:

"Time-domain" model		"Frequency-domain model"
Given time-domain signal model $x(t)$	$\longrightarrow$	Find the FS coefficients $\{c_k\}$

Converting "time-domain" signal model into a "frequency-domain" signal model

Q: How do we find the (Exp. Form) Fourier Series Coefficients?

A: Use this formula (it can be proved but we won't do that!)

$$\boxed{c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}\,dt}$$

Integrate over any complete period

Slightly different than book…
It uses $t_0 = 0$.

where: T = fundamental period of $x(t)$ (in seconds)

ω_0 = fundamental frequency of $x(t)$ (in rad/second)

$= 2\pi/T$

t_0 = any time point (you pick t_0 to ease calculations)

$k \in$ all integers

Comment: Note that for $k = 0$ this gives

$$\boxed{c_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt}$$

c_0 is the "DC offset", which is the time-average over one period

Summarizing rules for converting between the Time-Domain Model & the Exponential Form FS Model

<u>"Analysis"</u>

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}$$

Use signal to figure out the FS Coefficients

"Eat food and figure out recipe"

<u>"Synthesis"</u>

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Use FS Coefficients to "Build" the Signal

"Read recipe and cook food"

Time-Domain Model:	**The Periodic Signal Itself**
Frequency-Domain Model:	**The FS Coefficients**

There are similar equations for finding the FS coefficients for the other equivalent forms… But we won't worry about them because once you have the c_k you can get all the others easily…

Three (*Equivalent*) Forms of FS and Their Relationships

Trig Form: Sine-Cosine

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right]$$

$$A_0 = a_0$$

$$A_k = \sqrt{a_k^2 + b_k^2}$$

$$\theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right)$$

$$a_0 = c_0$$

$$a_k = A_k \cos(\theta_k)$$

$$b_k = -A_k \sin(\theta_k)$$

$$a_0 = c_0$$

$$a_k = 2\,\mathrm{Re}\{c_k\}, \quad k = 1, 2, 3, \ldots$$

$$b_k = -2\,\mathrm{Im}\{c_k\}, \quad k = 1, 2, 3, \ldots$$

$$c_0 = a_0$$

$$\left.\begin{aligned} c_k &= \tfrac{1}{2}(a_k - jb_k) \\ c_{-k} &= \tfrac{1}{2}(a_k + jb_k) \end{aligned}\right\} k = 1, 2, 3, \ldots$$

Exponential Form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_0 = A_0$$

$$\left.\begin{aligned} c_k &= \tfrac{1}{2} A_k e^{j\theta_k} \\ c_{-k} &= \tfrac{1}{2} A_k e^{-j\theta_k} \end{aligned}\right\} k = 1, 2, 3, \ldots$$

$$A_0 = c_0$$

$$\left.\begin{aligned} A_k &= 2|c_k| \\ \theta_k &= \angle c_k \end{aligned}\right\} k = 1, 2, 3, \ldots$$

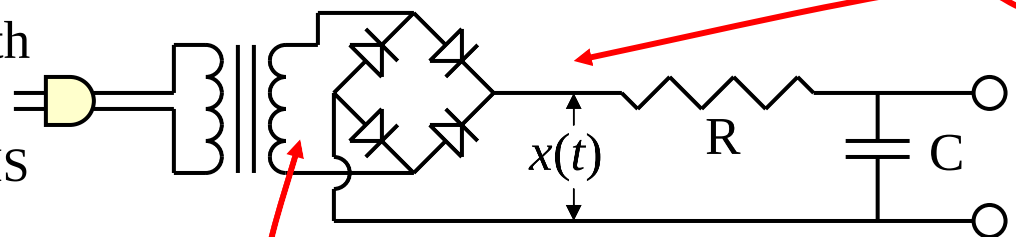
Trig Form: Amplitude & Phase

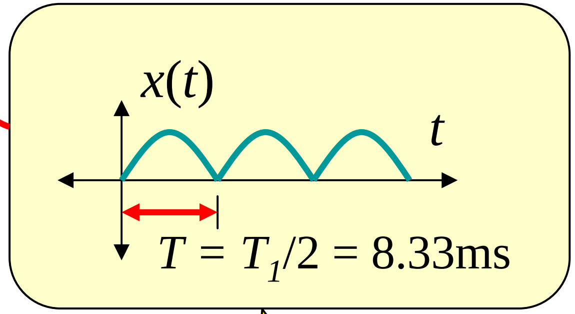
$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Example of Using FS Analysis

In electronics you have seen (or will see) how to use diodes and an RC filter circuit to create a DC power supply:

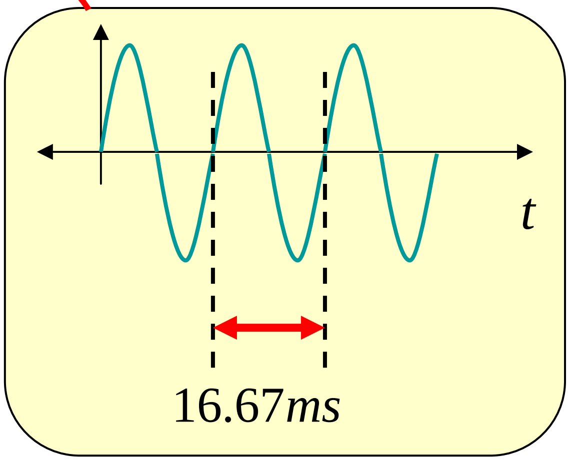
60Hz Sine wave with around 110V RMS



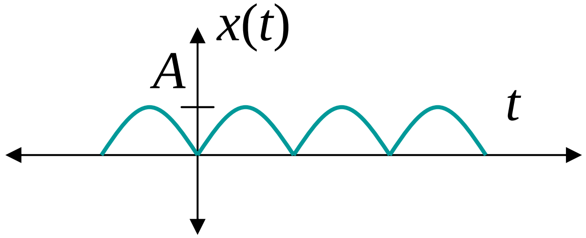


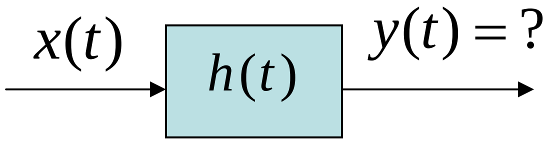
60 Hz

$$T_1 = \frac{1}{60} = 16.67ms$$



1. **This signal goes into the RC filter… what comes out? (Assume zero ICs)**
2. **How do we choose the desired *RC* values?**



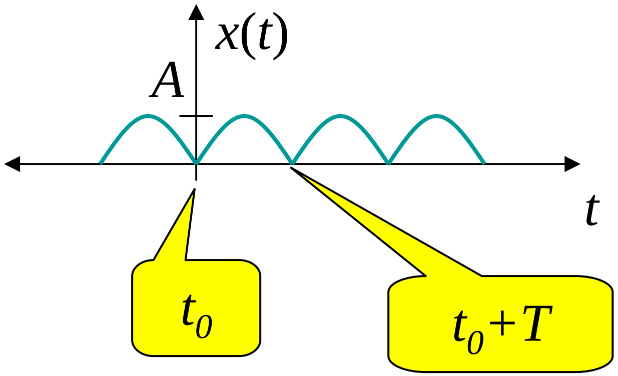


For now we will just find the FS of *x*(*t*)…

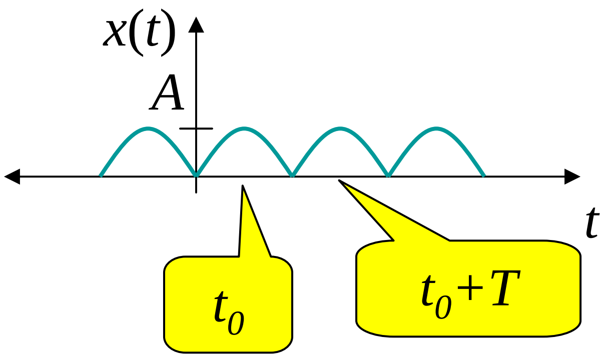
<u>Later</u> (Ch. 5) we will use it to analyze what *y*(*t*) looks like

The equation for the FS coefficients is: $c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt \qquad \omega_0 = \frac{2\pi}{T}$

Choose $t_0 = 0$ here to make things easier:



This kind of choice would make things harder:

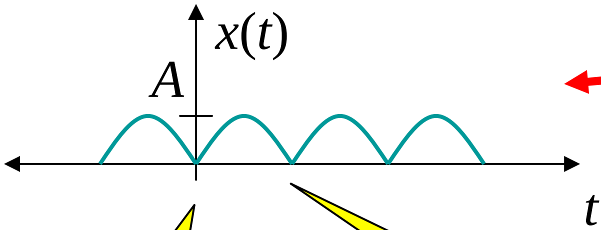


Now what is the equation for $x(t)$ over $t \in [0,T]$?

i.e., over the range of integration

$$\Rightarrow x(t) = A\sin\left(\frac{\pi}{T}t\right) \quad 0 \le t \le T$$

Determined by looking at the plot



t_0

t_0+T

So using this we get:

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_0^T A\sin\left(\frac{\pi}{T}t\right)e^{-jk\left(\frac{2\pi}{T}\right)t}dt$$

$$\omega_0 = \frac{2\pi}{T}$$

So… now we "just" have to evaluate this integral as a function of k…

To evaluate the integral: $c_k = \frac{1}{T}\int_0^T A\sin\left(\frac{\pi}{T}t\right)e^{-jk\left(\frac{2\pi}{T}\right)t}dt$

…we do a <u>Change of Variables</u>. There are three steps:

1. Identify the new variable and sub it into the integrand
2. Determine its impact on the differential
3. Determine its impact on the limits of integration

<u>Step 1</u>: $\boxed{\tau = \frac{\pi}{T}t}$ $\boxed{\sin(\tau)e^{-jk2\tau}}$

<u>Step 2</u>: $\boxed{d\tau = \frac{\pi}{T}dt}$ $\boxed{\Rightarrow dt = \frac{T}{\pi}d\tau}$

<u>Step 3</u>: when $t = 0 \Rightarrow$ $\boxed{\tau = \frac{\pi}{T}0 = 0}$

when $t = T \Rightarrow$ $\boxed{\tau = \frac{\pi}{T}T = \pi}$

$$\boxed{\begin{aligned} c_k &= \frac{1}{T}\int_0^T A\sin\left(\frac{\pi}{T}t\right)e^{-jk\left(\frac{2\pi}{T}\right)t}dt \\ &= \frac{1}{T}\int_0^\pi A\sin(\tau)e^{-jk2\tau}\left(\frac{T}{\pi}d\tau\right) \\ &= \frac{A}{\pi}\int_0^\pi \sin(\tau)e^{-jk2\tau}d\tau \end{aligned}}$$

So… to evaluate the integral given by: $\boxed{c_k = \frac{A}{\pi}\int_0^{\pi} \sin(\tau) e^{-jk2\tau} d\tau}$

… use your favorite <u>Table of Integrals</u> (a short one is available on the course web site):

$$\int e^{ax} \sin(bx)dx = \frac{e^{ax}[a\sin(bx) - b\cos(bx)]}{a^2 + b^2}$$

A general entry from an integral table

We get our case with: $a = -j2k \quad b = 1$

So…

$$c_k = \frac{A}{\pi}\left[\frac{e^{-j2k\tau}[-j2k\sin(\tau) - \cos(\tau)]}{1-4k^2}\right]_0^{\pi}$$

<u>Recall</u>: $\sin(0) = \sin(\pi) = 0$

So the sin term above goes away
(Finesse the problem… don't use brute force!)

So…

$$c_k = \frac{-A}{\pi(1-4k^2)}\left[e^{-j2k\tau}\cos(\tau)\right]_0^{\pi}$$

So…

$$c_k = \frac{-A}{\pi(1-4k^2)}\underbrace{\left[\underbrace{e^{-j2\pi k}}_{=1}\underbrace{\cos(\pi)}_{=-1} - \underbrace{e^{-j2k0}}_{=1}\underbrace{\cos(0)}_{=1}\right]}_{=-2}$$

So…

$$\boxed{c_k = \frac{2A}{\pi(1-4k^2)}}$$

FS coefficient for full-wave rectified sine wave of amplitude A

Things you would never know if you can't work *arbitrary* cases

Notes: 1. This does not depend on T

2. c_k is proportional to A

So: 1. If you change the input sine wave's frequency the c_k does not change

2. If you, say, double A… you'll double c_k

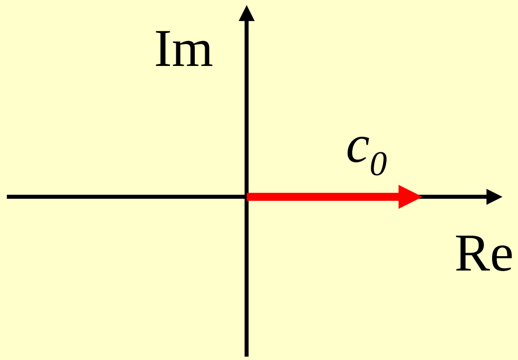
Now find the magnitude and phase of the FS coefficients:

$$\boxed{c_k = \frac{2A}{\pi(1-4k^2)}}$$

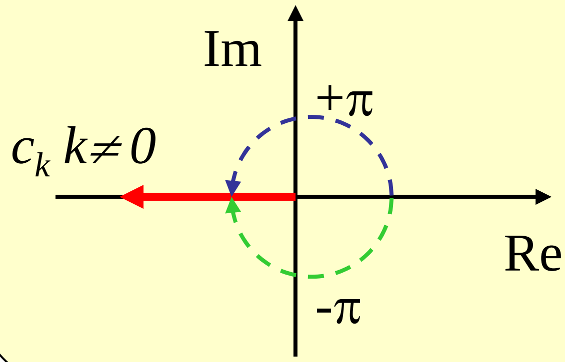
$$|c_0| = \frac{2A}{\pi} \qquad\qquad |c_k| = \frac{2A}{\pi(4k^2-1)} \quad k \neq 0$$

$$\angle c_0 = 0 \qquad\qquad \angle c_k = \pm\pi \quad\quad k \neq 0$$

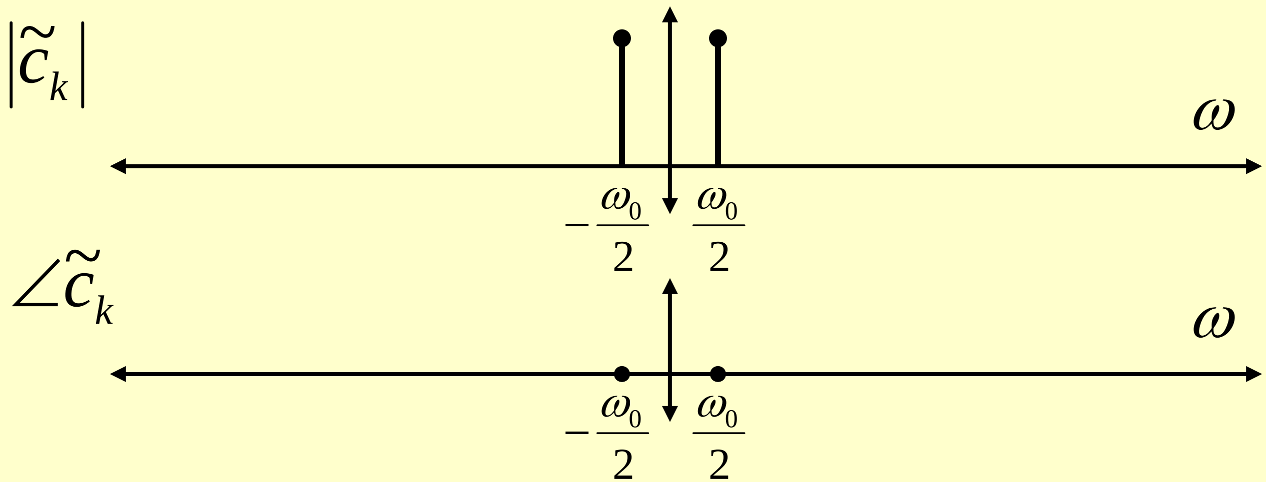
Because c_0 is real and > 0



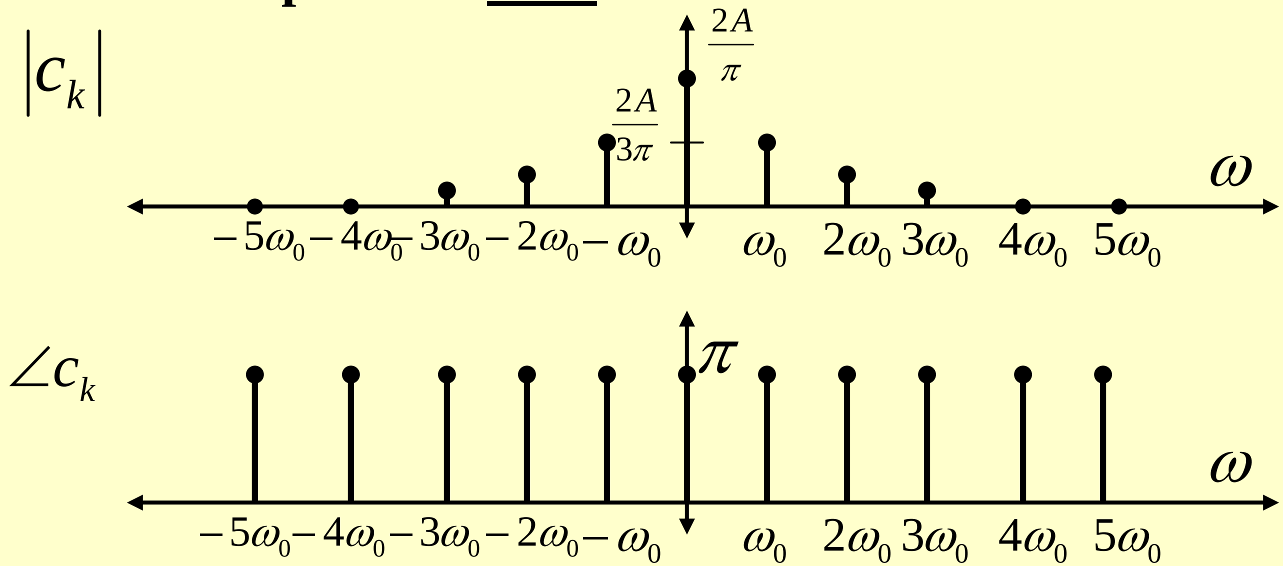
Because c_k is real and < 0



The spectrum before the rectifier (input is a single sinusoid):



So the two-sided spectrum after the rectifier:



Note: The rectifier *creates* frequencies because it is *non-linear*

Soon you should be able to explain why an LTI system cannot create frequencies!

Now you can find the Trigonometric form of FS

Once you have the c_k for the Exp. Form, Euler's formula gives the Trig Form as:

$$\boxed{x(t) = c_0 + \sum_{k=1}^{\infty} 2|c_k| \cos(k\omega_0 t + \angle c_k)}$$

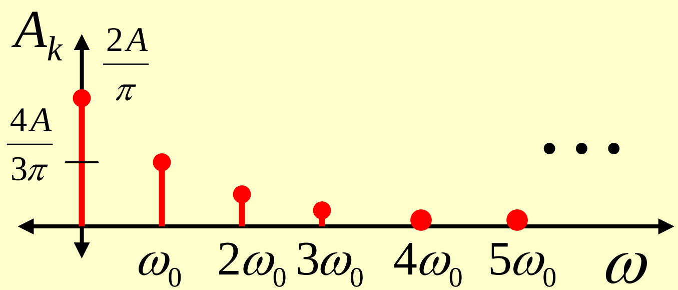
General Result!!

The rectified sine wave has Trig. Form FS:

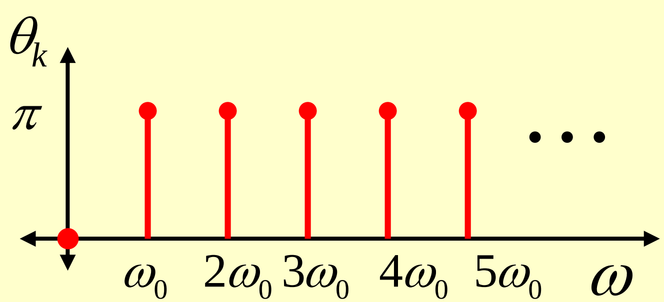
$$x(t) = \underbrace{\frac{2A}{\pi}}_{A_0} + \sum_{k=1}^{\infty} \underbrace{\frac{4A}{\pi(4k^2-1)}}_{A_k} \cos(k\omega_0 t + \underbrace{\pi}_{\theta_k})$$

The one-sided spectrum is:

Magnitude Spectrum



Phase Spectrum



So… the input to the RC circuit consists of a superposition of sinusoids… and you know from circuits class how to:

1. **Handle a superposition of inputs to an RC circuit**
2. **Determine how a single sinusoid of a given frequency goes through an RC circuit**

Preliminary to "Parseval's Theorem" (Not in book)

Imagine that signal $x(t)$ is a voltage.

If $x(t)$ drops across resistance R, the instantaneous power is $p(t) = \dfrac{x^2(t)}{R}$

Sometimes we don't know what R is there so we "normalize" this by ignoring the R value:

$$p_N(t) = x^2(t)$$

Once we have a specific R we can always un-normalize via $p_N(t) \big/ R$

(In "Signals & Systems" we will drop the N subscript)

Recall: power = energy per unit time $\Rightarrow p(t) = \dfrac{dE(t)}{dt} \quad \Rightarrow \underbrace{dE(t) = x^2(t)dt}_{\text{differential increment of energy}}$

(1 W = 1 J/s)

$$\Rightarrow \text{Energy in one period} = \int_{t_0}^{T+t_0} dE(t) = \int_{t_0}^{T+t_0} x^2(t)dt$$

$$\text{The Total Energy} = \int_{-\infty}^{\infty} x^2(t)dt$$

$= \infty$ for a <u>periodic signal</u>

Recall: power = energy per unit time

$$\text{Average power over one period} = \frac{\text{Energy in One Period}}{T}$$

$$\boxed{P = \frac{1}{T}\int_{t_0}^{T+t_0} x^2(t)\,dt}$$

Often just called "Average Power"

For periodic signals we use the average power as measure of the "size" of a signal.

The Average Power of practical periodic signals is finite and non-zero.

(Recall that the total energy of a periodic signal is infinite.)

Parseval's Theorem

We just saw how to compute the average power of a periodic signal if we are given its time-domain model:

$$\boxed{P = \frac{1}{T}\int_{t_0}^{T+t_0} x^2(t)\,dt}$$

Q: Can we compute the average power from the frequency domain model

$$\boxed{\{c_k\}, \quad k = 0, \pm 1, \pm 2, \ldots}$$

A: Parseval's Theorem says… Yes!

Parseval's theorem gives this equation

$$\boxed{P = \sum_{k=-\infty}^{\infty} |c_k|^2}$$

as an alternate way to compute the average power of a periodic signal whose complex exponential FS coefficients are given by c_k

Another way to view Parseval's theorem is this equality:

$$\boxed{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2}$$

Interpreting Parseval's Theorem

$$\underbrace{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,dt}_{\text{"sum" of squares in time-domain model}} = \underbrace{\sum_{k=-\infty}^{\infty} |c_k|^2}_{\text{"sum" of squares in freq.-domain model}}$$

"sum" of squares in time-domain model = "sum" of squares in freq.-domain model

$x^2(t)$ = power at time t (includes effects of all frequencies)

We can find the power in the time domain by "adding up" all the "powers at each time"

$|c_k|^2$ = power at frequency $k\omega_0$ (includes effects of all times)

We can find the power in the frequency domain by adding up all the "powers at each frequency"