EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #10

• C-T Systems: Convolution Representation
• Reading Assignment: Section 2.6 of Kamen and Heck
Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).
Convolution for C-T systems

We saw for D-T systems:

- Definition of Impulse Response $h[n]$

- How TI & Linearity allow us to use $h[n]$ to write an equation that gives the output due to input $x[n]$ (That equation is Convolution)

The same ideas arise for C-T systems!

(And the arguments to get there are very similar… so we won’t go into as much detail!!)

**Impulse Response**: $h(t)$ is what “comes out” when $\delta(t)$ “goes in”

If system is **causal**, $h(t) = 0$ for $t < 0$
Note: In D-T systems, $\delta[n]$ has a height of 1

In C-T systems, $\delta(t)$ has a “height of infinity” and a “width of zero”

-So, in practice we can actually make $\delta[n]$

-But we cannot actually make $\delta(t)$!!

How do we know or get the impulse response $h(t)$?

1. It is given to us by the designer of the C-T system.

2. It is measured experimentally

   -But, we cannot just “put in $\delta(t)$”

   -There are other ways to get $h(t)$ but we need chapter 3 and 5 information first

3. Mathematically analyze the C-T system

   -Easiest using ideas in Ch. 3, 5, 6, & 8
In what form will we know $h(t)$?

1. $h(t)$ known analytically as a function
   - e.g. $h(t) = e^{-2t}u(t)$
2. We may only have experimentally obtained samples:
   - $h(nT)$ at $n = 0, 1, 2, 3, \ldots, N-1$

Now we can…

Use $h(t)$ to find the zero-state response of the system for an input

Following similar ideas to the DT case we get that:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)\,d\lambda$$

Notation: $y(t) = x(t) * h(t)$
Example 2.14  Output of RC Circuit with Unit Step Input

We have seen that this circuit is modeled by the following Differential Equation:

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

Problem: Find the zero-state response of this circuit to a unit step input… i.e., let \(x(t) = u(t)\) and find \(y(t)\) for the case of the ICs set to zero (for this case that means \(y(0^-) = 0\)).

We have seen that this circuit is modeled by the following Differential Equation:

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

So… we need to solve this Diff. Eq. for the case of \(x(t) = u(t)\).

The previous slides told us that we can use convolution…
But… to do that we need to know the impulse response $h(t)$ for this system (i.e., for this differential equation)!!

In Chapter 6 we will learn how to find the impulse response by applying the Laplace Transform to the differential equation. The result is:

$$h(t) = \begin{cases} \frac{1}{RC} e^{-(1/RC)t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

or

$$h(t) = \frac{1}{RC} e^{-(1/RC)t} u(t)$$

![Graph of h(t) for RC=1]
For our step input:

\[ x(t) = u(t) \]

This is the convolution we need to do…

\[ y(t) = x(t) * h(t) \]
\[
y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\lambda)x(t - \lambda)d\lambda
\]

\[
= \int_{-\infty}^{\infty} \left[ \frac{1}{RC} e^{-(1/RC)\lambda} u(\lambda) \right]u(t - \lambda)d\lambda
\]

This is the general form for convolution.

Plug in given forms for \(h(t)\) and \(x(t)\).

This makes the integrand = 0 whenever \(\lambda < 0\). And... it is 1 otherwise.

(\text{Note that if } t < 0 \text{ then the integrand is 0 for all } \lambda \text{.)}

So exploiting these facts we see that the only thing the unit steps do here is to limit the range of integration...

\[
y(t) = \begin{cases} 
\frac{1}{RC} \int_{0}^{t} e^{-(1/RC)\lambda} d\lambda, & t > 0 \\
0, & t \leq 0
\end{cases}
\]

So... to find the output for this problem all we have to do is evaluate this integral to get a function of \(t\).
This integral is the easiest one you learned in Calc I!!!

\[
\frac{1}{RC} \int_{0}^{t} e^{-(1/RC)\lambda} d\lambda = \frac{1}{RC} \left[ -RC e^{-(1/RC)\lambda} \right]_{0}^{t} = \left[ - e^{-(1/RC)\lambda} \right]_{0}^{t} = \left[ - e^{-(1/RC)t} \right] - \left[ - e^{0} \right]
\]

\[
= 1 - e^{-(1/RC)t}
\]

\[
y(t) = \begin{cases} 
1 - e^{-(1/RC)t}, & t > 0 \\
0, & t \leq 0 
\end{cases}
\]

Once we can compute this kind of integral in general we can find out what the output looks like for any given input!

Recall Time-Constant Rules:

- 63% after 1 TC
- \( \approx 100\% \) after 5 TCs
Big Picture

For a LTI C-T system in zero state we no longer need the differential equation model…

-Instead we need the impulse response $h(t)$ & convolution

New alternative model!

Equivalent Models (for zero state)