BINGHAMTON UNIVERSITY
State University of New York

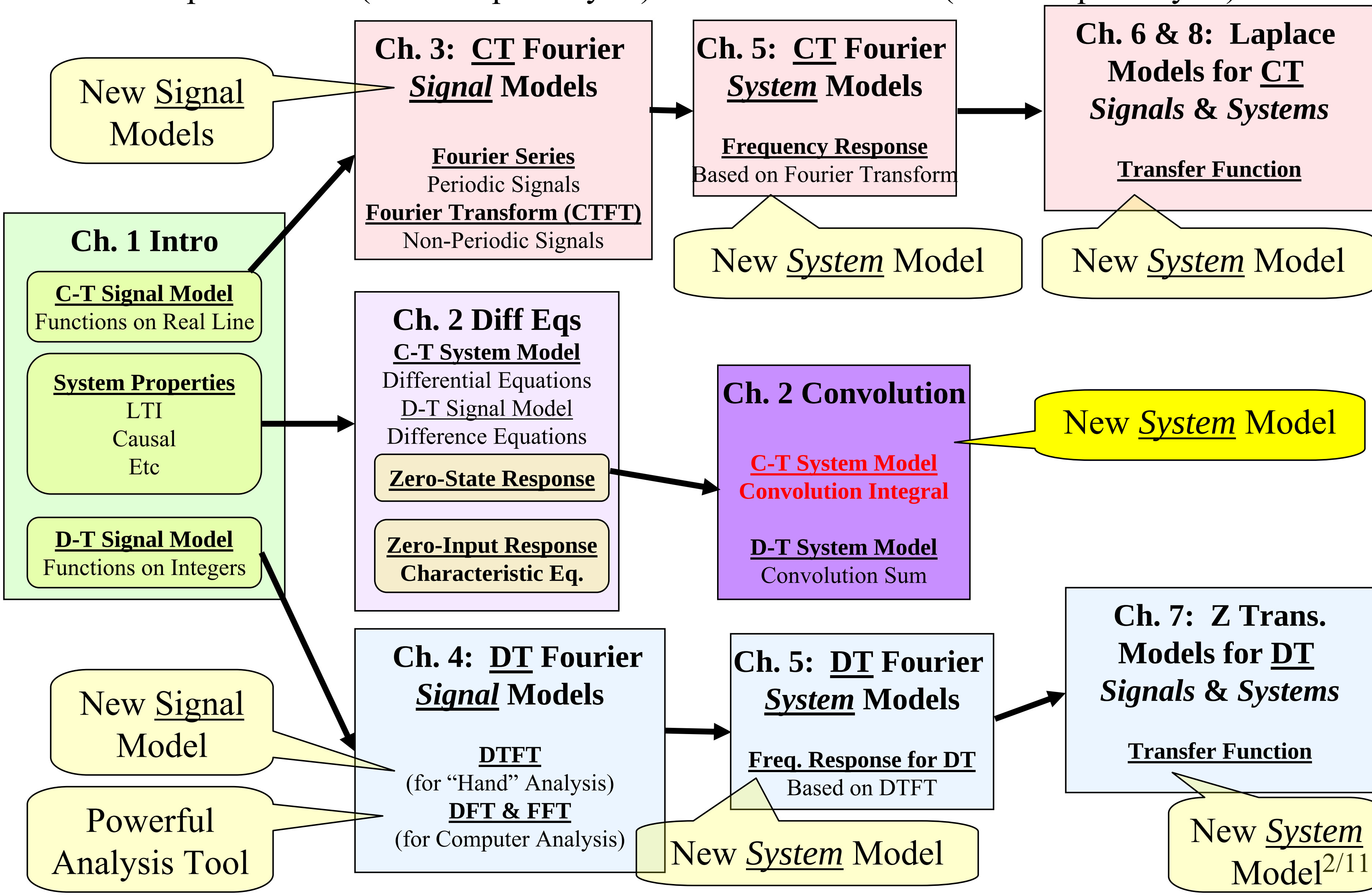
EECE 301
Signals & Systems
Prof. Mark Fowler

<u>Note Set #10</u>

- C-T Systems: Convolution Representation
- Reading Assignment: Section 2.6 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



Convolution for C-T systems

We saw for D-T systems:

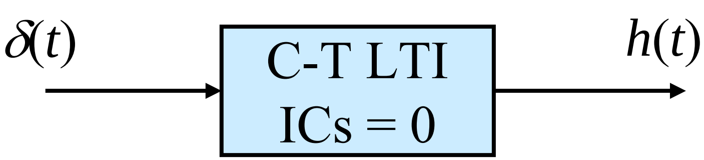
-Definition of Impulse Response $h[n]$

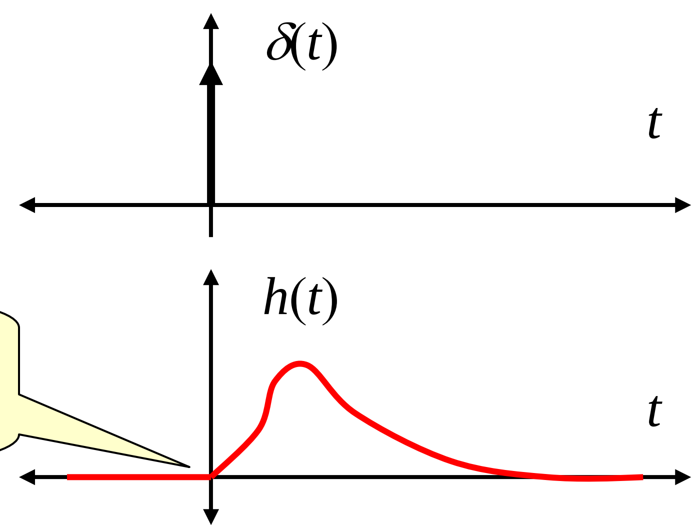
-How TI & Linearity allow us to use $h[n]$ to write an equation that gives the output due to input $x[n]$ (That equation is Convolution)

The same ideas arise for C-T systems!

(And the arguments to get there are very similar… so we won't go into as much detail!!)

Impulse Response: $h(t)$ is what "comes out" when $\delta(t)$ "goes in"





If system is causal,
$h(t) = 0$ for $t < 0$

Note: In D-T systems, $\delta[n]$ has a height of 1

In C-T systems, $\delta(t)$ has a "height of infinity" and a "width of zero"

-So, in practice we can actually make $\delta[n]$

-But we cannot actually make $\delta(t)$!!

How do we know or get the impulse response $h(t)$?

1. It is given to us by the designer of the C-T system.

2. It is measured experimentally

   -But, we cannot just "put in $\delta(t)$"

   -There are other ways to get $h(t)$ but we need chapter 3 and 5 information first

3. Mathematically analyze the C-T system

   -Easiest using ideas in Ch. 3, 5, 6, & 8

In what form will we know *h*(*t*)?

Our focus is here

1. $h(t)$ known analytically as a function

   -e.g. $h(t) = e^{-2t}u(t)$

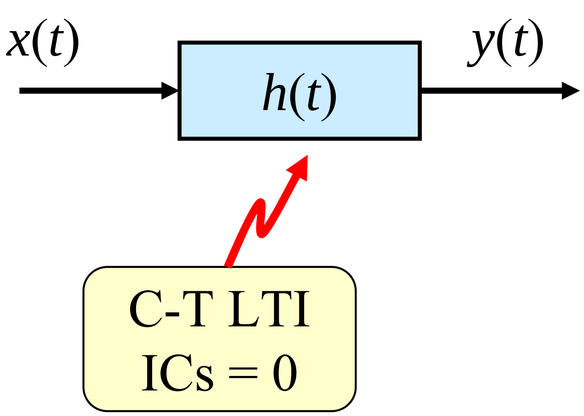
2. We may only have experimentally obtained samples:

   - $h(nT)$ at $n = 0, 1, 2, 3, \ldots, N-1$

Now we can…

Use $h(t)$ to find the zero-state response of the system for an input

Following similar ideas to the DT case we get that:

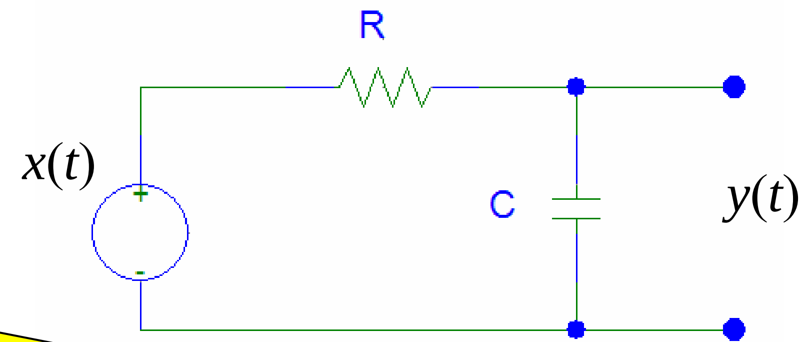


CONVOLUTION

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

Notation: $y(t)=x(t)*h(t)$

Example 2.14 Output of RC Circuit with Unit Step Input



The book considers a different case… $x(t)$ is a pulse

Problem: Find the zero-state response of this circuit to a unit step input… i.e., let $x(t) = u(t)$ and find $y(t)$ for the case of the ICs set to zero (for this case that means $y(0^-) = 0$).

We have seen that this circuit is modeled by the following Differential Equation:

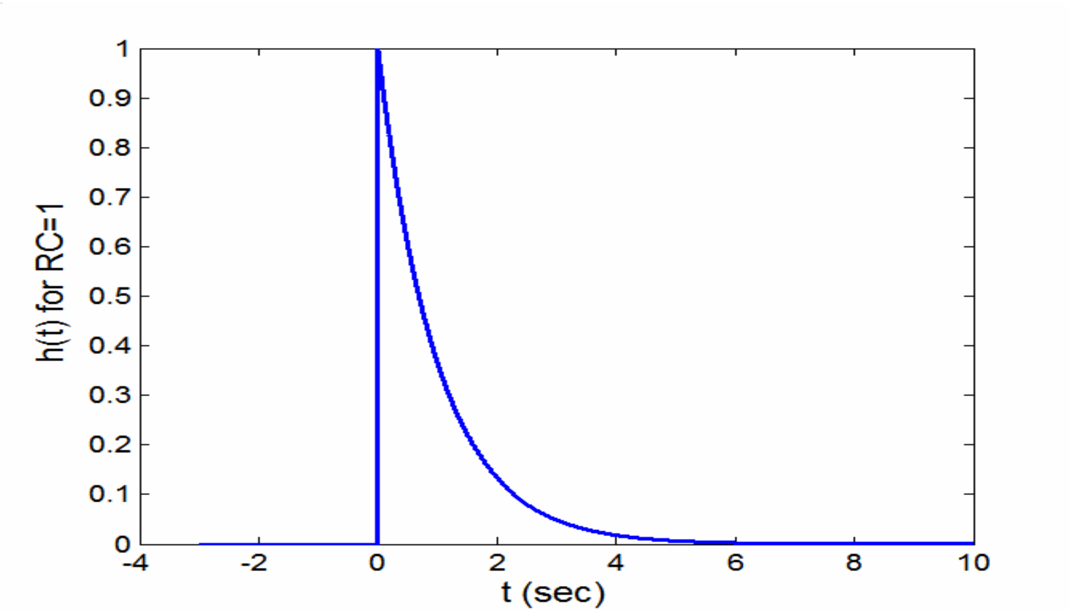
$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

So… we need to solve this Diff. Eq. for the case of $x(t) = u(t)$. The previous slides told us that we can use convolution…

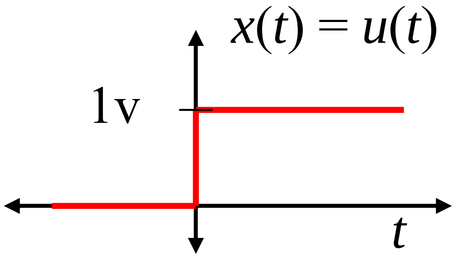
But… to do that we need to know the impulse response $h(t)$ for this system (i.e., for this differential equation)!!!

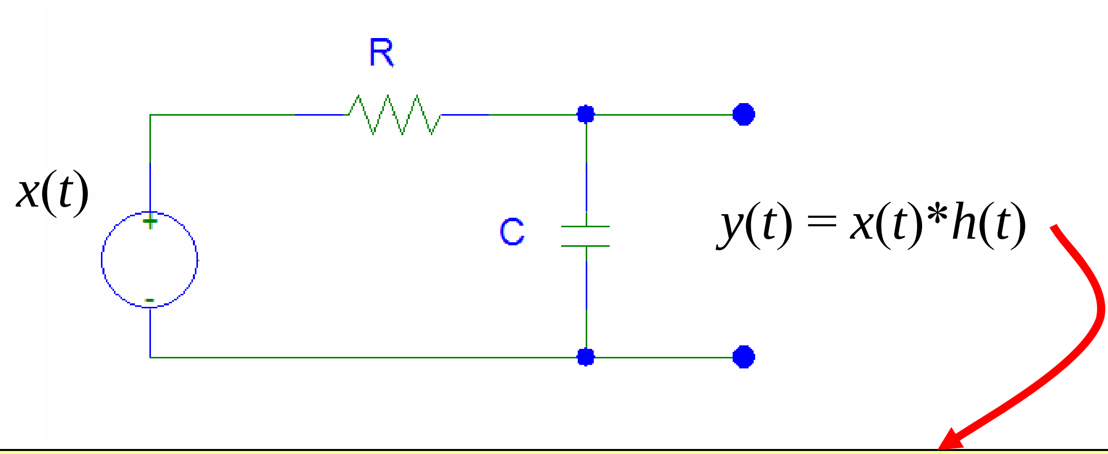
In Chapter 6 we will learn how to find the impulse response by applying the Laplace Transform to the differential equation. The result is:

$$h(t) = \begin{cases} \frac{1}{RC} e^{-(1/RC)t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad or \qquad h(t) = \frac{1}{RC} e^{-(1/RC)t} u(t)$$

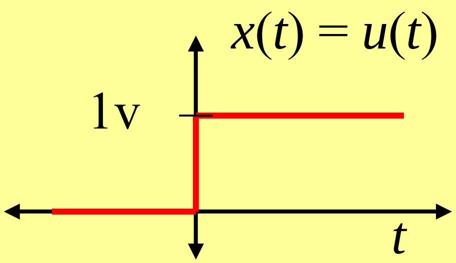


For our step input:





This is the convolution we need to do…



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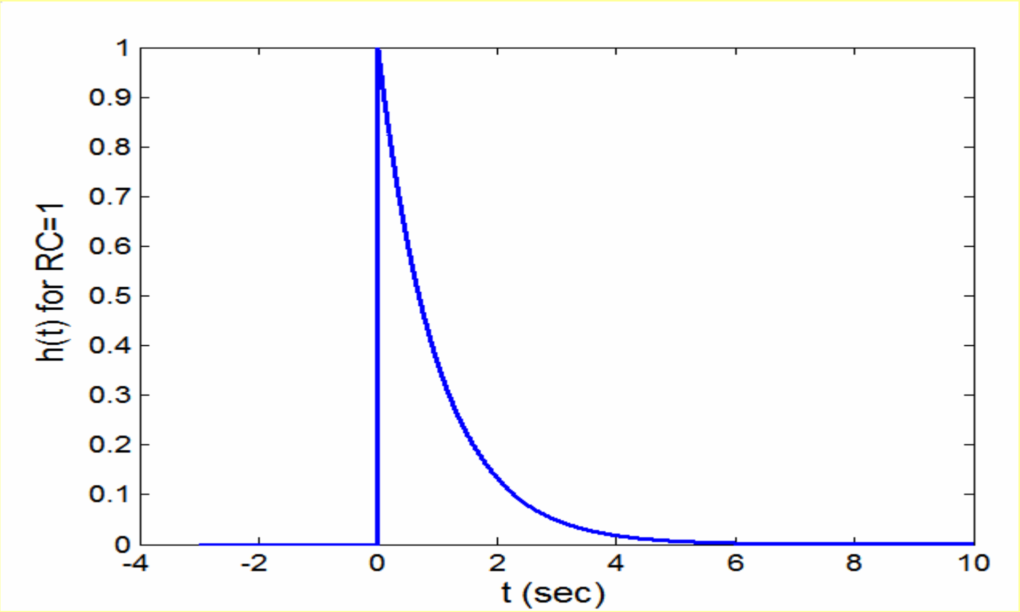


$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

This is the general form for convolution

$$= \int_{-\infty}^{\infty} \left[\tfrac{1}{RC} e^{-(1/RC)\lambda} u(\lambda)\right] u(t-\lambda)d\lambda$$

Plug in given forms for $h(t)$ and $x(t)$

This makes the integrand = 0 whenever $\lambda < 0$. And… it is 1 otherwise.

This makes the integrand = 0 whenever t- $\lambda < 0$ or in other words whenever $\lambda > t$. And… it is 1 otherwise.

(Note that if $t < 0$ then the integrand is 0 for all λ)

So exploiting these facts we see that the only thing the unit steps do here is to limit the range of integration…

$$y(t) = \begin{cases} \frac{1}{RC}\int_{0}^{t} e^{-(1/RC)\lambda} d\lambda, & t > 0 \\ 0, & t \le 0 \end{cases}$$

So… to find the output for this problem all we have to do is evaluate this integral to get a function of t

This integral is the easiest one you learned in Calc I!!!

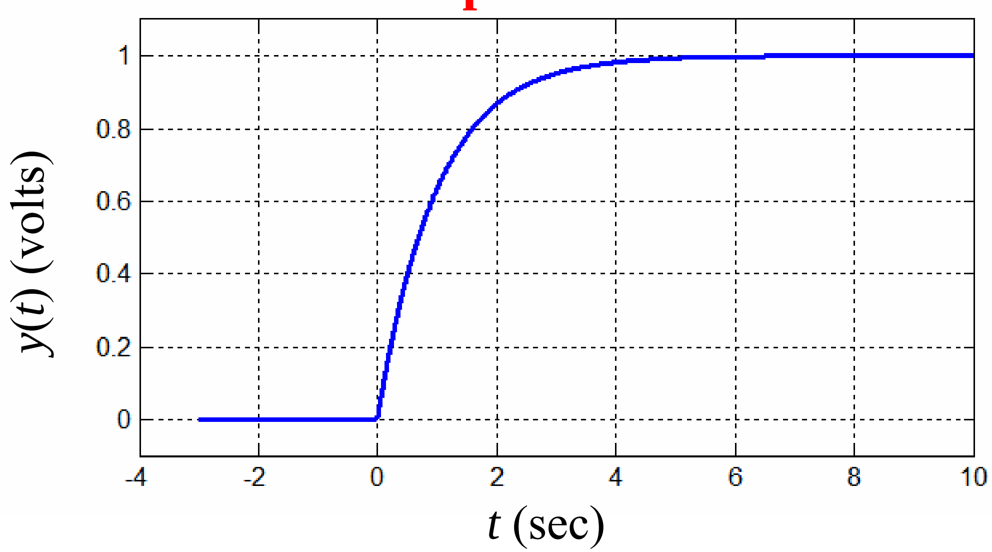
$$\frac{1}{RC}\int_0^t e^{-(1/RC)\lambda}d\lambda = \frac{1}{RC}\left[-RCe^{-(1/RC)\lambda}\right]_0^t = \left[-e^{-(1/RC)\lambda}\right]_0^t = \left[-e^{-(1/RC)t}\right]-\left[-e^{0}\right]$$

$$= 1-e^{-(1/RC)t}$$

$$y(t) = \begin{cases} 1-e^{-(1/RC)t}, & t>0 \\ 0, & t\le 0 \end{cases}$$

Once we can compute this kind of integral in general we can find out what the output looks like for any given input!

Output for *RC* = 1



Recall Time-Constant Rules:

- ▶ 63% after 1 TC
- ▶ ≈100% after 5 TCs

Compare to “Big Picture” for DT Case… Same!

Big Picture

For a LTI C-T system in zero state we no longer need the differential equation model…

-Instead we need the impulse response *h*(*t*) & convolution

New alternative model!