Binghamton University
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EECE 301
Signals & Systems
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Discussion #9

- Illustrating the Errors in DFT Processing
- DFT for Sonar Processing

Example #1

Illustrating The Errors in DFT Processing

Illustrating the Errors in DFT processing

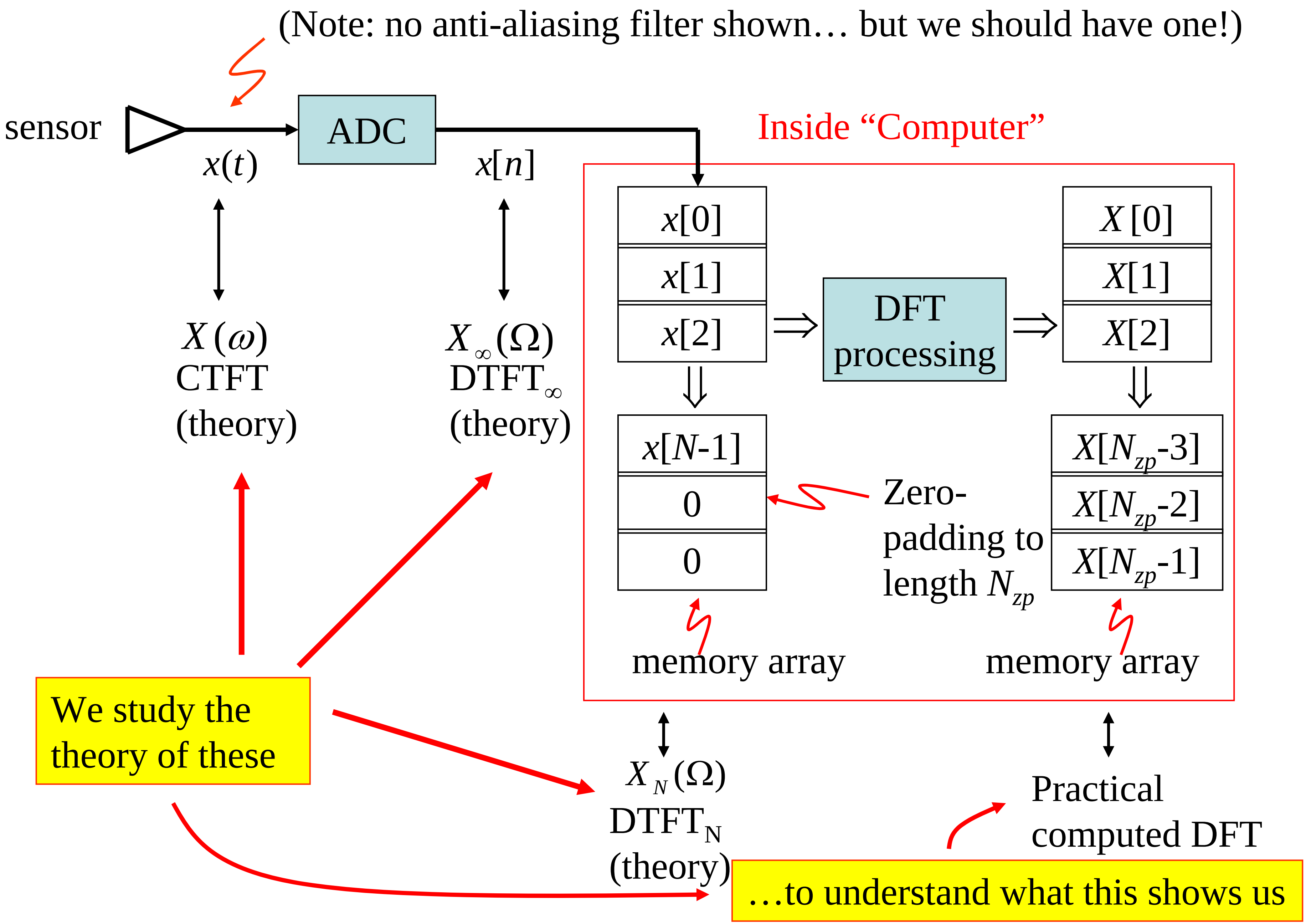
This example does a nice job of showing the relationships between:

- the CTFT,
- the DTFT of the infinite-duration signal,
- the DTFT of the finite-duration collected samples,
- and the DFT computed from those samples.

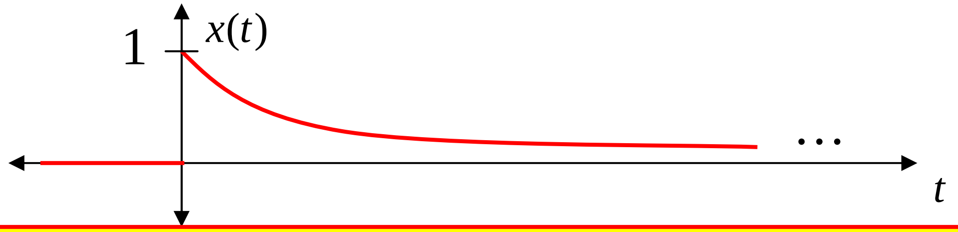
However, it lacks any real illustration of why we do DFT processing in practice.

There are many practical applications of the DFT and we'll look at one in the next example.

Recall the processing setup:



Let's imagine we have the following CT Signal: $x(t) = e^{-bt}u(t) \quad for \;\; b > 0$



Now… <u>analyze</u> what we <u>will</u> get from the DFT processing for this signal…

From our FT Table we find the FT of $x(t)$ is:

A

$$X(\omega) = \frac{1}{j\omega + b} \quad \Rightarrow \quad X(f) = \frac{1}{j2\pi f + b}$$

CTFT Result…(Theory)

If we sample $x(t)$ at the rate of F_s samples/second – That is, sample every $T = 1/F_s$ sec – we get the <u>DT Signal</u> coming out of the ADC is:

$$x[n] = x(t)\big|_{t=nT} = x(nT)$$

For this example we get:

$$x[n] = \left[e^{-bt}u(t)\right]_{t=nT} = e^{-bTn}u[n]$$

$$= \left(e^{-bT}\right)^n u[n] \triangleq a^n u[n]$$

Note: $|a| < 1$

Now imagine that <u>in theory</u> we have <u>all</u> of the samples $x[n]$ $-\infty < n < \infty$ at the ADC output.

Then, in theory the DTFT_∞ of this signal is found using the DTFT table to be:

B

$$X_\infty(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

DTFT_∞ Result…(Theory)

For $|a| < 1$ which we have because:

$$a = e^{-bT} \;\; \& \; b > 0, \; T > 0$$

Now, in reality we can "collect" only $N < \infty$ samples in our computer:

$$x_n[n] = a^n, \; 0 \le n \le (N-1)$$

$$(\text{"Assume"} \;\; x_n[n] = 0 \;\; \text{elsewhere})$$

<u>Necessary</u> to connect the DFT result to the theoretical results we'd like to see.

The DTFT of this collected finite-duration is easily found "by hand":

C

$$X_N(\Omega) = \frac{1 - \left(ae^{-j\Omega}\right)^N}{1 - ae^{-j\Omega}}$$

Note that we think of this as follows:

$$x_N[n] = x[n]w_N[n] \qquad w_N[n] = \begin{cases} 1, & 0, 1, 2, \ldots, N-1 \\ 0, & otherwise \end{cases}$$

..and DTFT theory tells us that

$$X_N(\Omega) = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} X_\infty(\Omega-\lambda)W_N(\lambda)d\lambda}_{\text{A form of convolution}} = X_\infty(\Omega) * W_N(\Omega)$$

A form of convolution (DT Freq. Domain Convolution)

…and this convolution has a "smearing" effect.

Finally, the DFT of the zero-padded collected samples is…

$$x_{zp}[n] = \begin{bmatrix} x[0] \\ x[1] \\ \ldots \\ x[N-1] \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

Total of N_{zp} "points"

$$X_{zp}[k] = \sum_{n=0}^{N_{zp}-1} x_{zp}[n] e^{-j\frac{2\pi kn}{N_{zp}}}$$

D

(The only part of this example we'd <u>really</u> "do")

Our theory tells us that the zero-padded DFT is nothing more than "points" on DTFT_N:

$$X_{zp}[k] = X_N(\Omega_k)$$

where $\Omega_k = \dfrac{2\pi k}{N_{zp}}$ $\quad k = 0, 1, 2, \ldots, N-1$

Spacing between DFT "points" is $2\pi/N_{zp}$

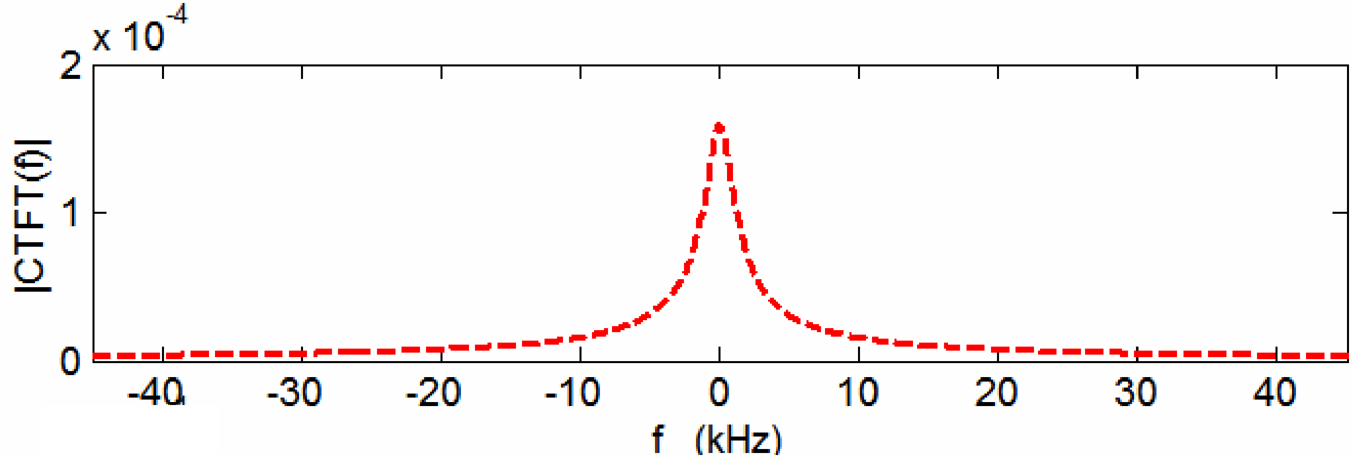
$\Rightarrow$ Increasing the amount of zero-padding gives closer spacing

Now run the m-file called DFT_Relations.m for different F_s , N, & N_{ap} values

Results from DFT_Relations.m

Plot #1: shows CTFT computed using: $X(f) = \dfrac{1}{j2\pi f + b}$ **A**

Notice that this is not ideally bandlimited, but is essentially bandlimited.



Plot #2: shows DTFT_∞ computed using:

(For plotting) $$X_\infty(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$ B

Our theory says that:

(For analysis) $$X_\infty(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\left(\frac{\Omega + k2\pi}{2\pi}\right)F_s\right)$$

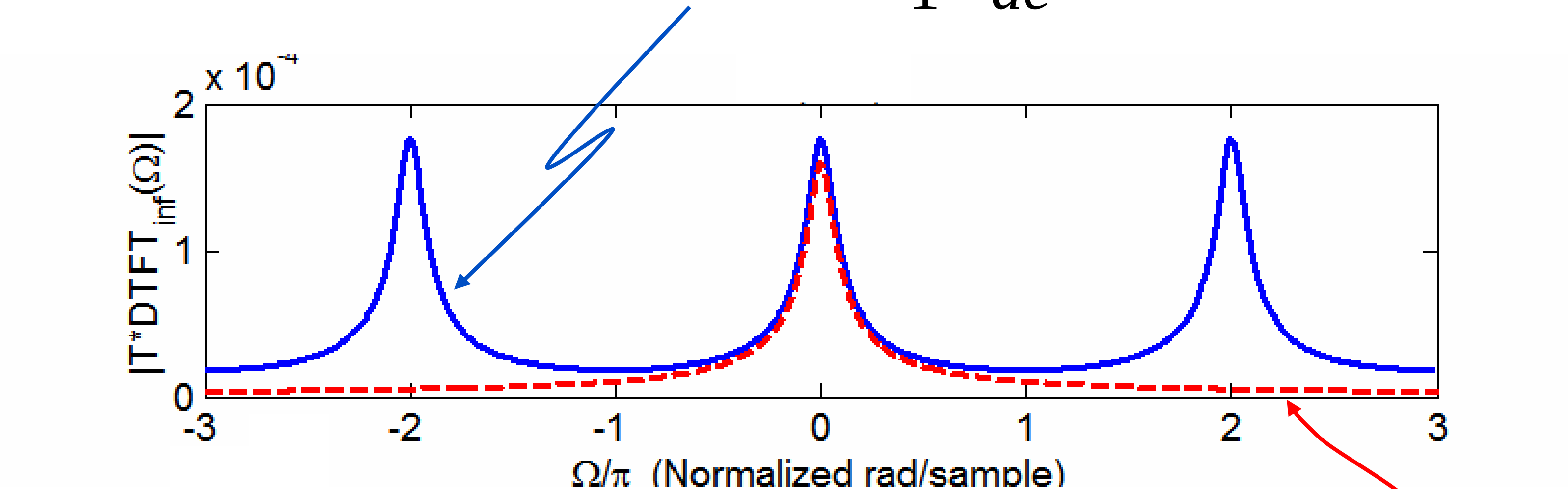
(CTFT rescaled to Ω and then shifted by multiples of 2π)

So we should see "replicas" in $X_\infty(\Omega)$ and we do!

We plot $TX_\infty(\Omega)$ to undo the 1/T here

Plot #2:

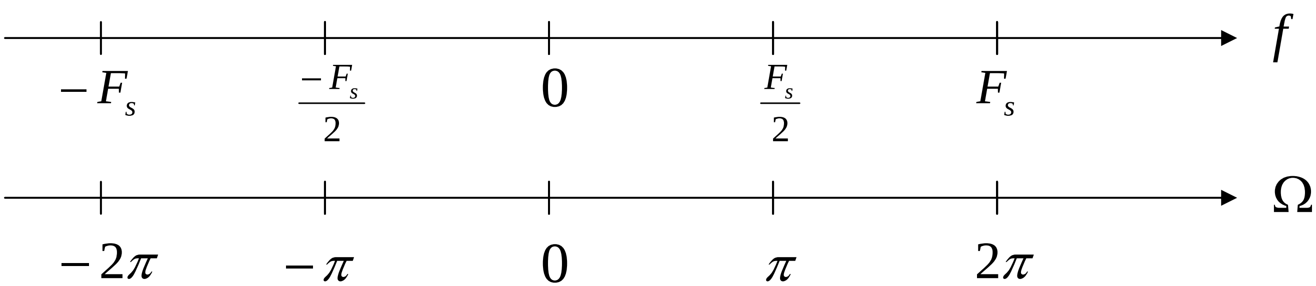
$$\text{DTFT}_\infty:\ X_\infty(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$



We also plot the CTFT against $f \times \left(\frac{2\pi}{F_s}\right) = \Omega$

$$X(f) = \frac{1}{j2\pi f + b} \Rightarrow X(\Omega) = \frac{1}{j2\pi(\Omega \times F_s / 2\pi) + b}$$

Recall the two equivalent axes:

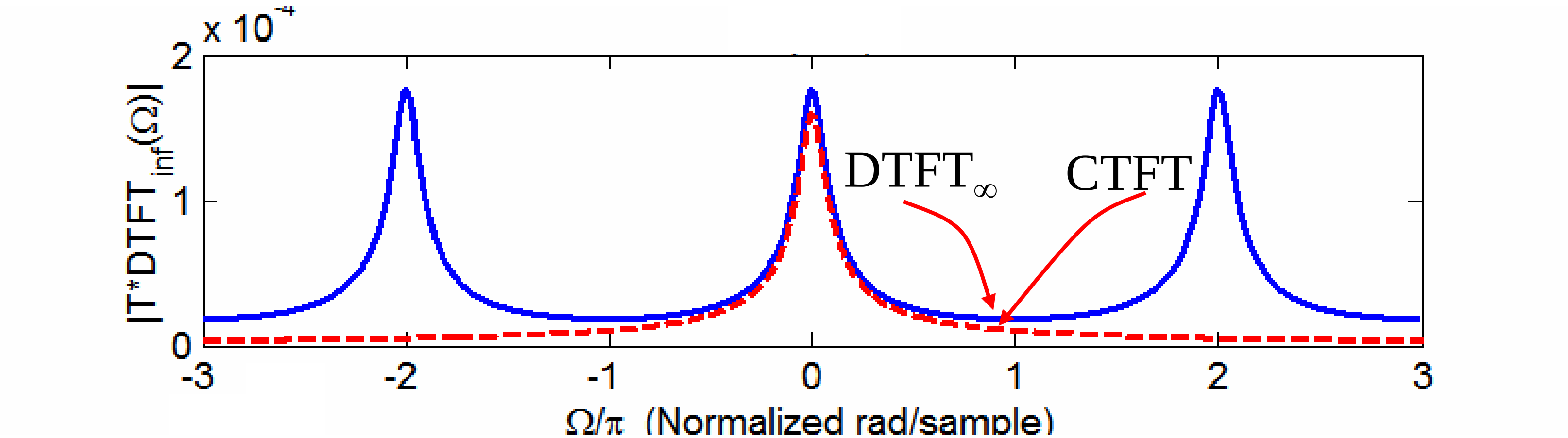


The theory in

$$X_\infty(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\left(\frac{\Omega + k2\pi}{2\pi}\right)F_s\right)$$

says we'll see significant aliasing in $X_\infty(\Omega)$ unless F_s is high enough

The first error – visible in plot #2

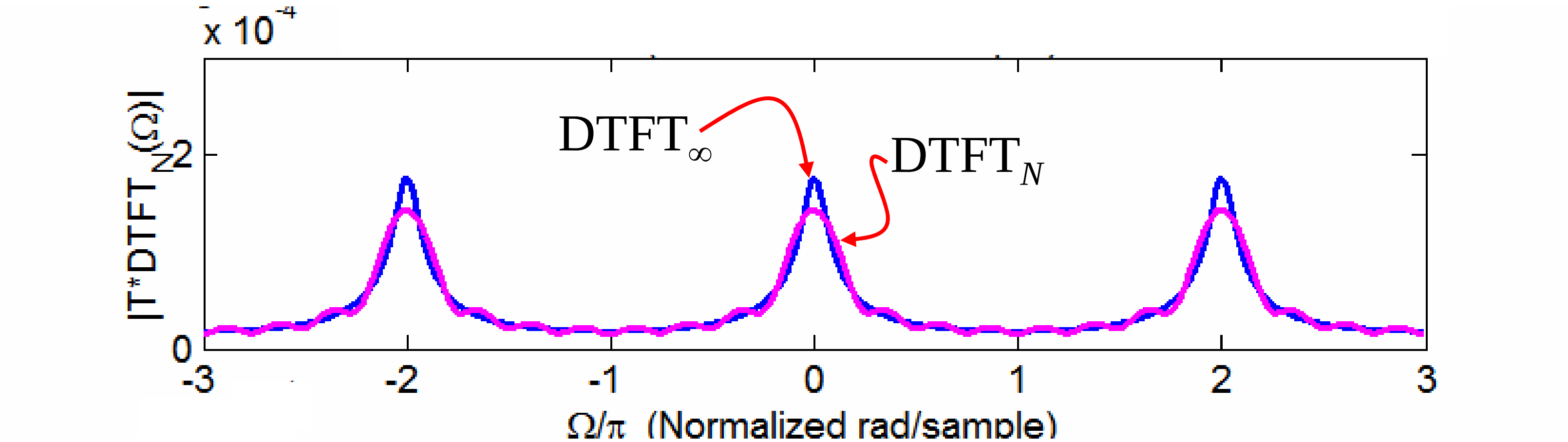


Plot #3 shows DTFT_N computed using $X_N(\Omega) = \dfrac{1-(ae^{-j\Omega})^N}{1-ae^{-j\Omega}}$ C

We see that $X_N(\Omega)$ shows signs of the "smearing" due to: $X_N(\Omega) = X_\infty(\Omega) * W_N(\Omega)$

Also called "leakage" error

The second error – visible in plot #3

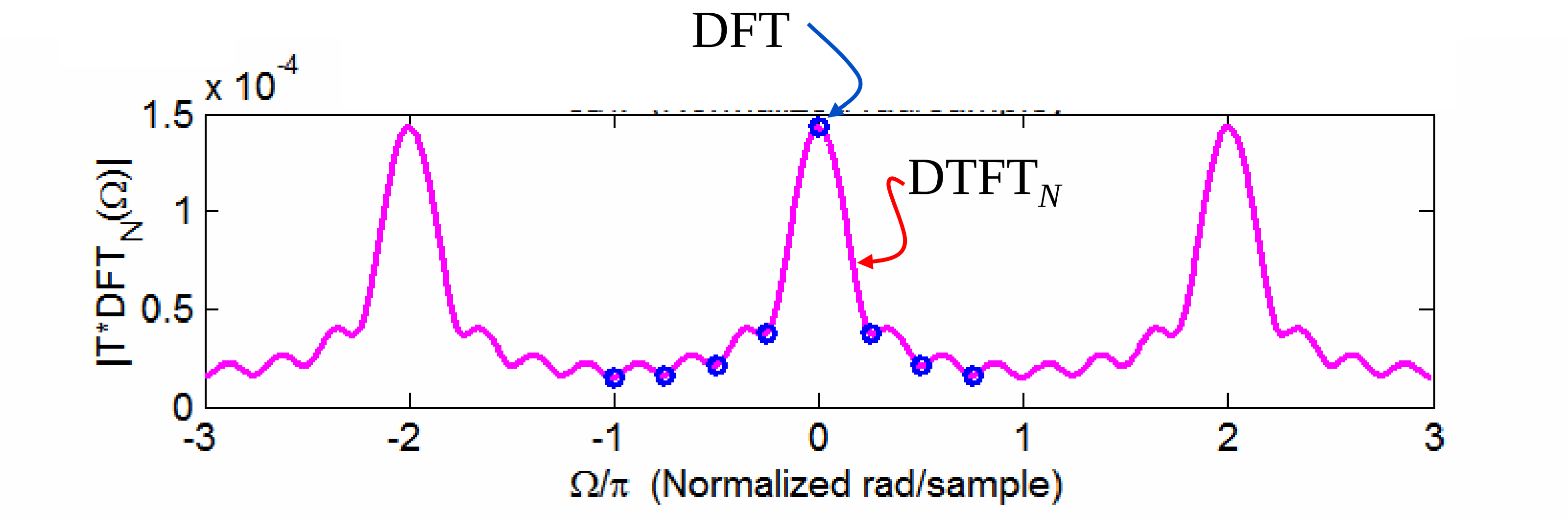


This "leakage" error is less significant as we increase *N*, the number of collected samples

Plot #4 shows DFT computed using: $X_{zp}[k] = \sum_{n=0}^{N_{zp}-1} x_{zp}[n] e^{\frac{-j2\pi kn}{N_{zp}}}$ D

It is plotted vs. $\Omega_k = \frac{2\pi k}{N_{zp}}$ …but with the "right half" moved down to lie between $-\pi$ & 0 rad/sample

For comparison we also plot $X_N(\Omega)$



Note: We show an artificially small number of DFT points here

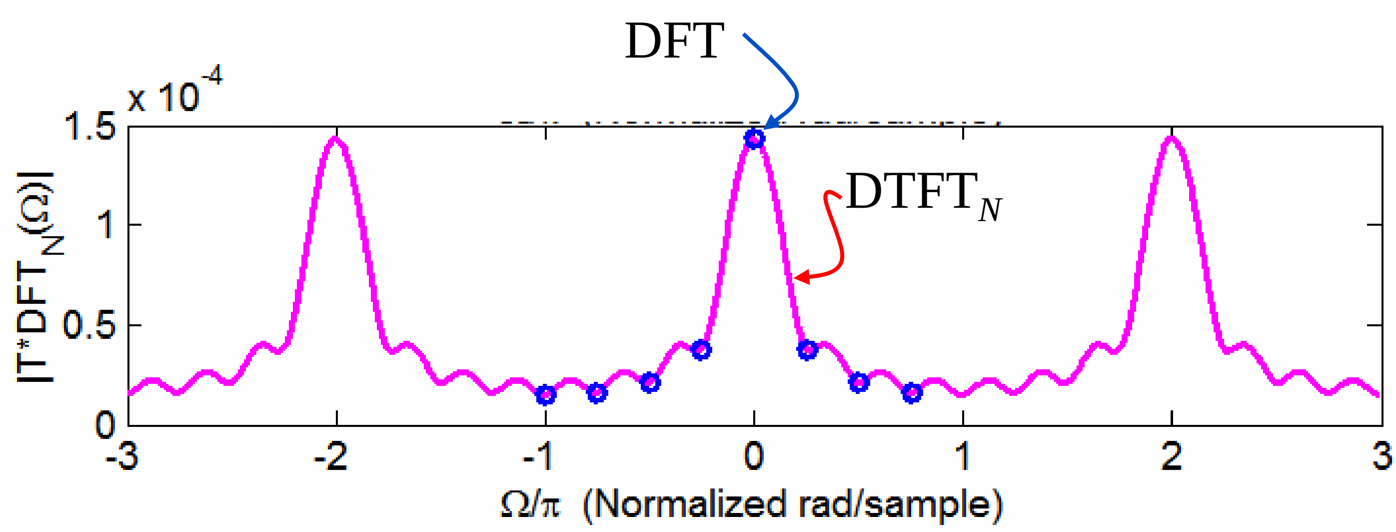
Theory says… $X_{zp}[k]$ points should lie on top of $X_N(\Omega)$… not $X_\infty(\Omega)$!!

We see that this is true

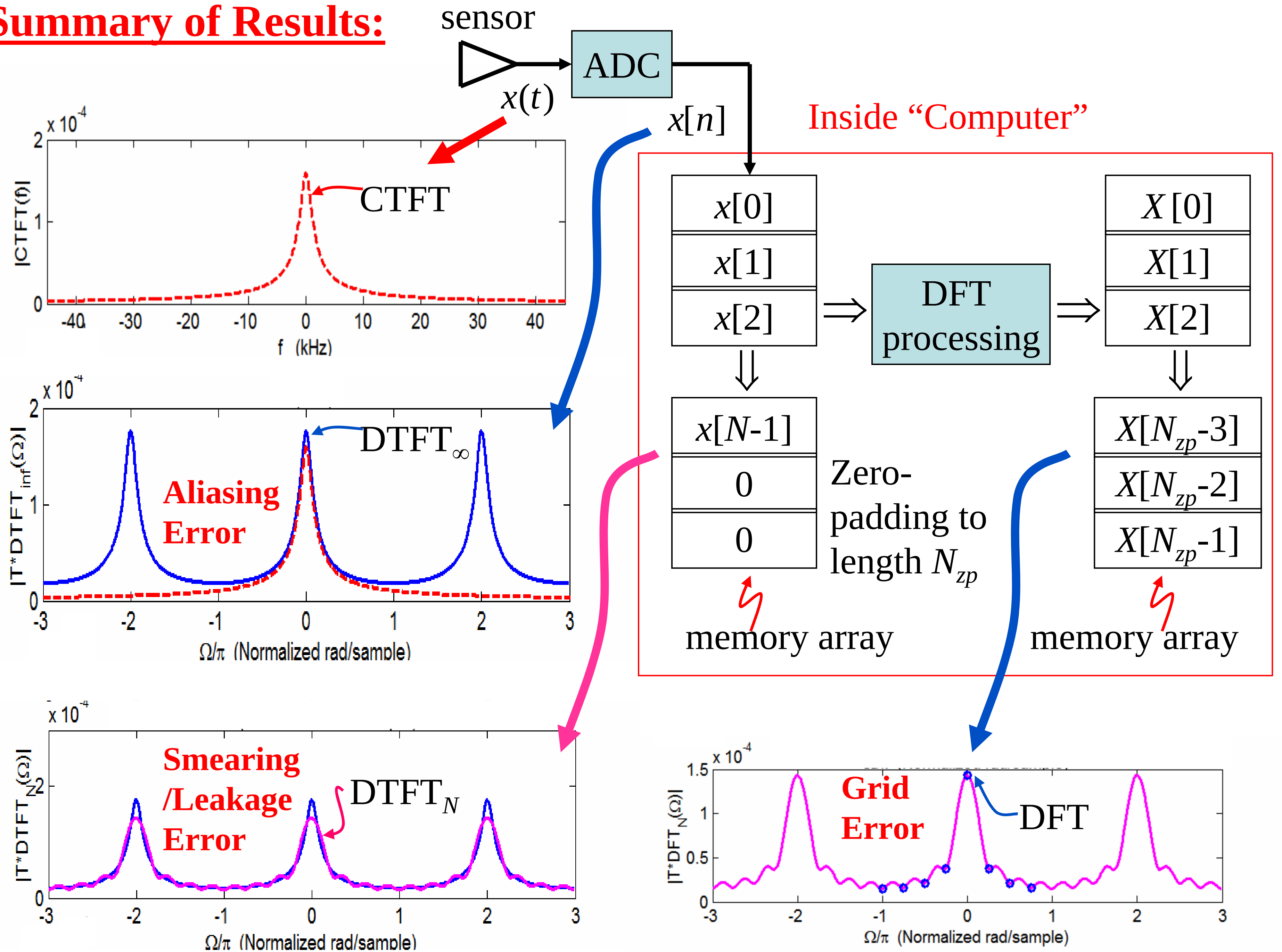
If N_{zp} is too small (i.e. $N_{zp} = N$) then there aren't enough "DFT points" on $X_N(\Omega)$ to allow us to see the real underlying shape of $X_N(\Omega)$

This is "Grid Error" and it is less significant when N_{zp} is large.

The third error – visible in plot #4



Summary of Results:



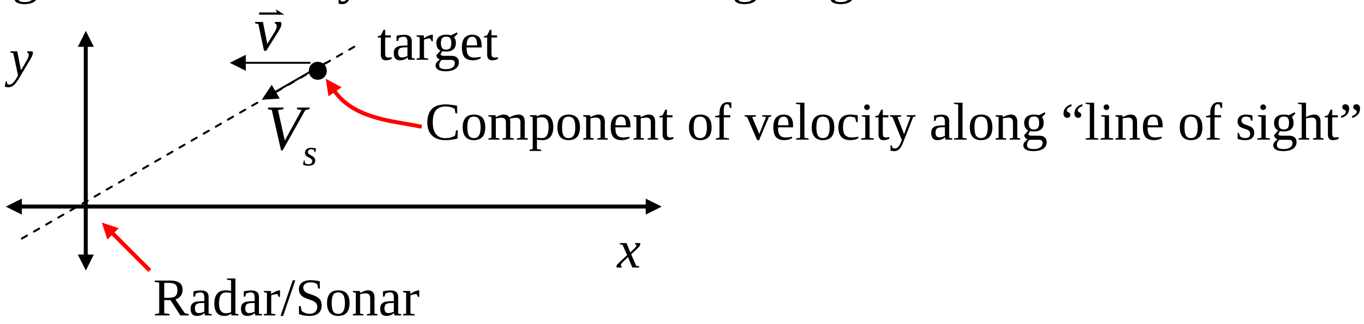
Example #2

Sonar Processing

using the DFT

Radar/Sonar Processing using the DFT

Imagine a stationary sonar and moving target



Say we transmit a sinusoidal pulse:

$$x_{TX}(t) = \begin{cases} A\cos(2\pi f_o t), & 0 \le t \le T_o \\ 0, & else \end{cases}$$

"Tx" = Transmit

"Rx" = Receive

Physics tells us (Doppler effect) that the reflected signal received will be:

$$x_{RX}(t) = \begin{cases} \alpha A\cos\left(2\pi\left(f_o + \underbrace{\frac{f_o V_s}{c}}\right)t + \phi\right), & 0 \le t \le T_o \\ 0, & else \end{cases}$$

Doppler shift in Hz

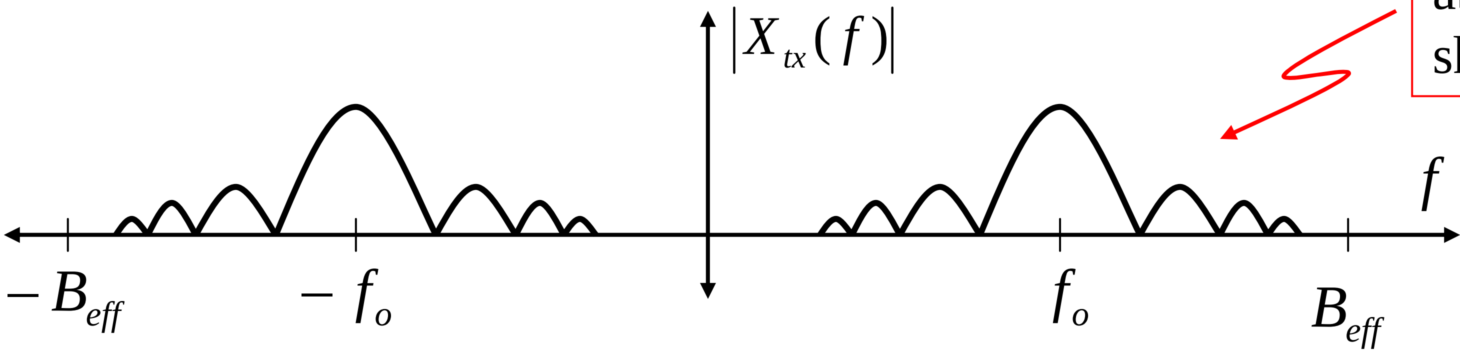
(c – speed of propagation ≈ 331m/s for sound in air)

(for radars, this is generally in the kHz range)

(for sonar, this is in the 100's of Hz range)

Our CTFT theory tells us that the CTFT of the Tx signal will be:

abs. value of shifted sinc

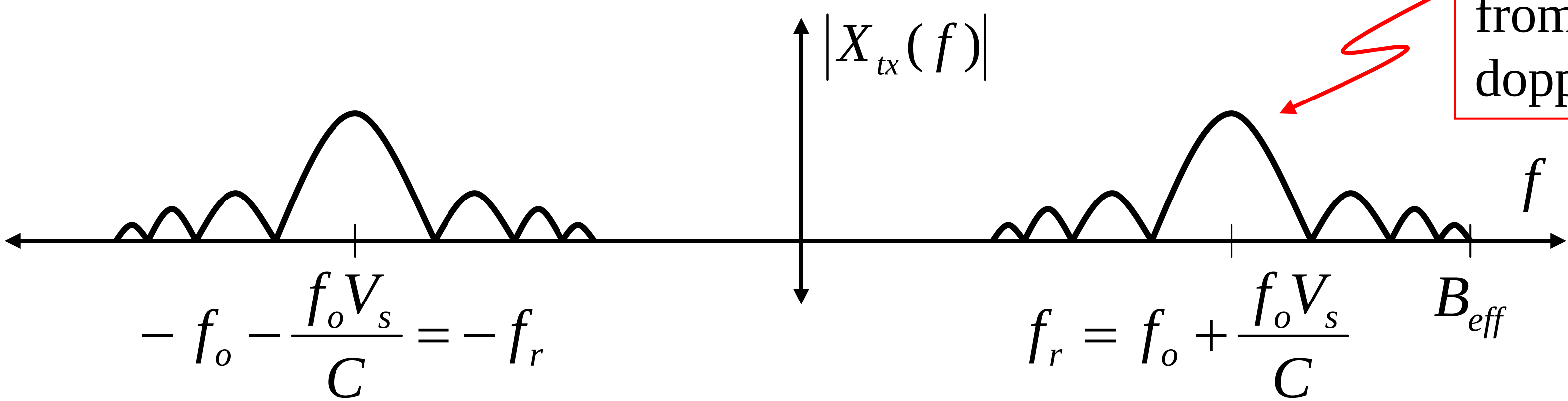


Assume that f_o is large enough that this decays to a negligible level by $f \approx 0$ Hz and by $f \approx \pm B_{eff}$

$\Rightarrow$ "Center" f_o between 0 and B_{eff} $\Rightarrow F_s \approx 2\, B_{eff}$

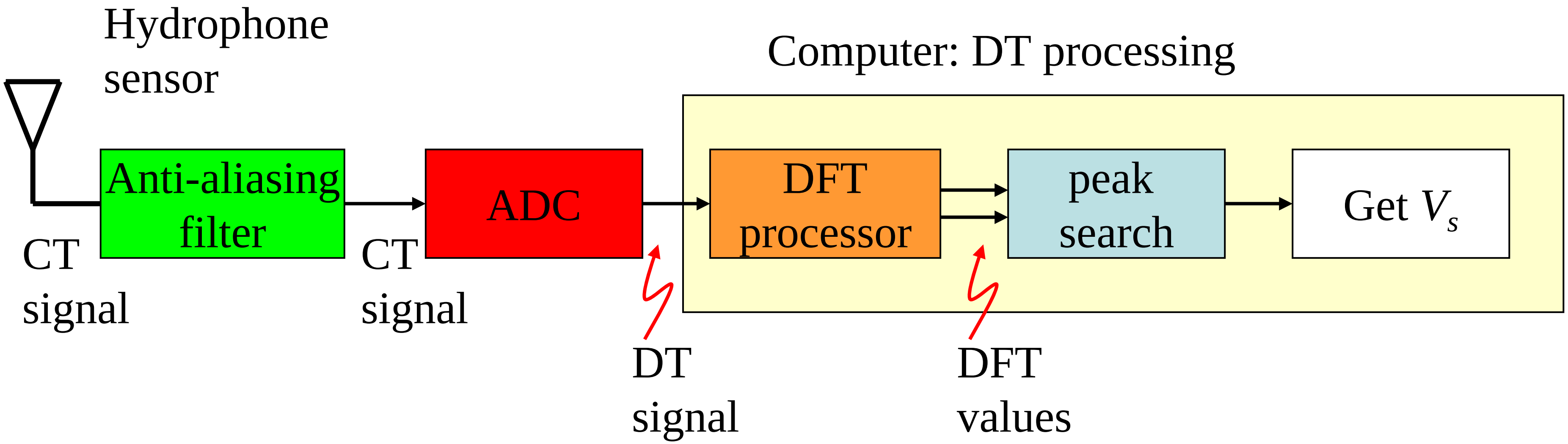
Also CTFT theory tells us that the CTFT of the Rx signal will be:

Peak is shifted from f_o by the doppler shift



If we know f_o and we can find where this peak is… then we can find V_s:

$$V_s = c\left(\frac{f_r}{f_o} - 1\right) \quad (m/s)$$


Hydrophone
sensor
Computer: DT processing
Anti-aliasing
filter
ADC
DFT
processor
peak
search
Get V_s
CT
signal
CT
signal
DT
signal
DFT
values