

Discussion #7 Example Problem

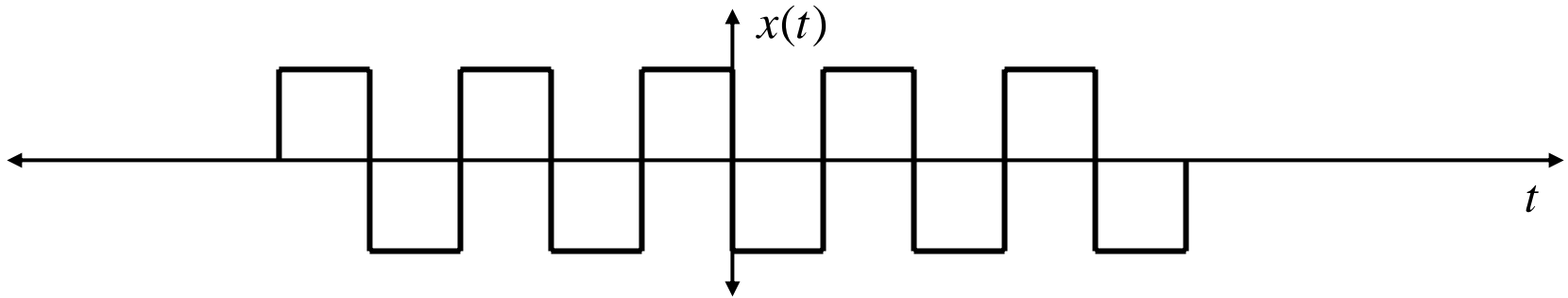
This problem illustrates how Fourier series are helpful tools for analyzing electronic circuits.

Often in electronic circuits we need sinusoids of various frequencies...

But we may already have circuitry in the system that generates a clock signal...

Can we avoid having to build a sinusoidal oscillator by exploiting the clock signal?

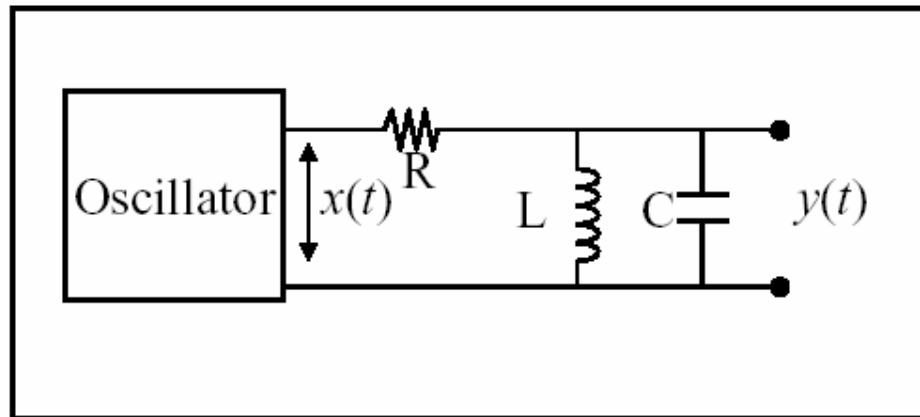
Let's imagine that our job is to design circuitry to convert a square wave into a sinusoid of 10000 Hz. Suppose we have a square wave oscillator that generates the following signal with a period of $100\mu\text{s}$:



The box labeled “oscillator” in the figure below produces this signal.

Assume that the output impedance of the oscillator is very low so that the rest of the circuit doesn't load it (recall our previous discussions of loading).

Assume that the resistor is 800Ω , the capacitor is $1\mu\text{F}$, and the inductor is $250\mu\text{H}$



What we need to do for this design:

- (a) Find the Fourier Series for the square wave signal $x(t)$ that gets applied to the input of the RLC circuit.
- (b) Find the frequency response $H(\omega)$ of the RLC circuit and use a computer to plot $|H(f)|$ vs. $f \geq 0$ (with f in Hz)
- (c) Find the FS spectrum at the output of the circuit and verify that it represents approximately a 10000 Hz sinusoid
- (d) Find the total harmonic distortion of our resultant sinusoid.

Total Harmonic Distortion (THD) is used in practice to measure the quality of a sinewave generator (a perfect one has 0% THD).

THD is also used measure the quality of a linear system (e.g., an audio amplifier): you put a sinusoid in and should get a sinusoid of the same frequency out... but if the system is nonlinear you'll get power at other frequencies... THD measures how much power is at these other frequencies relative to how much is at the input frequency.

(a)

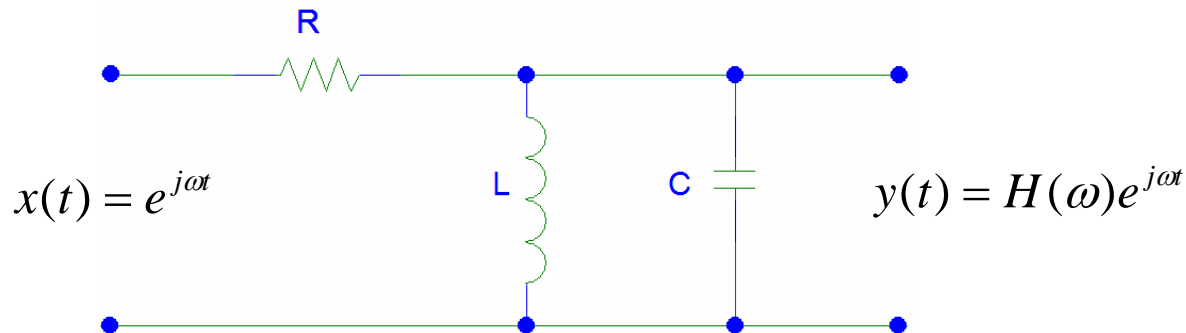
$$\text{For our case } T_0 = 100\mu\text{s} \quad \Rightarrow \omega_0 = \frac{2\pi}{T_0} \quad \Rightarrow f_0 = \omega_0 / 2\pi = \frac{1}{T_0} = 10\text{kHz}$$

So... as we have found before... the FS of this signal is:

$$x(t) = -\frac{4A}{\pi} \sin(2\pi \times 10,000 t) - \frac{4A}{3\pi} \sin(2\pi \times 30,000 t) - \frac{4A}{5\pi} \sin(2\pi \times 50,000 t) \dots$$

(b) Now we find the frequency response $H(\omega)$ of the circuit by asking:

How $e^{j\omega t}$ goes through the system (where we imagine that ω is arbitrary)?



This is just phasor-based analysis of the circuit!!!!

Recall: Phasor-Analysis Steps:

1. **Convert the Circuit into an Equivalent “Complex DC Circuit”**
 - a) Write Sinusoidal Source as a Phasor (phasor = “complex DC source”)
 - b) Write Ls/Cs as impedances (impedance = complex resistance)
 - i. Capacitor: $Z_C(\omega) = 1/j\omega C$ Inductor: $Z_L(\omega) = j\omega L$
2. **Solve the resulting “Complex-Valued DC Circuit” using appropriate combinations of DC analysis methods:**
 - a) Fundamental Rules: KVL & KCL
 - i. Systematic Schemes based on KVL & KCL: Loop, Nodal, Mesh
 - ii. Simple Resulting “Tricks”: Voltage Divider & Current Divider
 - b) Equivalent Circuit Rules:
 - a) Thevenin/Norton,
 - b) Series/Parallel Combinations
3. **Convert output phasor back into real-valued sinusoid**
 - a) Or... for frequency response analysis... just identify $H(\omega)$

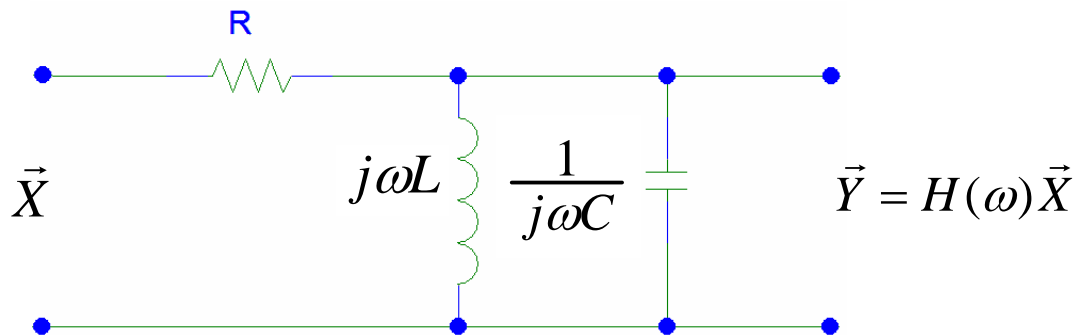
Step 1: In our case we are already part way to a phasor because we are asking about $e^{j\omega t}$, so technically our phasor is just 1 .

But let's just keep the input notated as an arbitrary phasor: \vec{X}

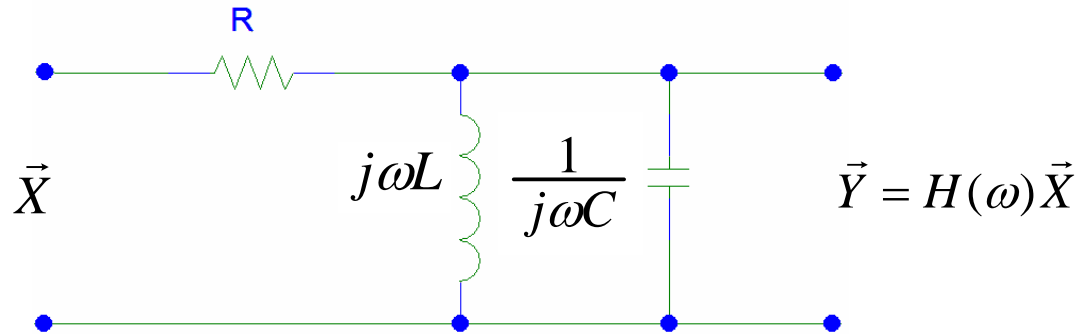
Now convert the capacitor and inductor into impedances at the arbitrary frequency ω :

$$\text{The impedances are: } Z_L(\omega) = j\omega L \quad \& \quad Z_C(\omega) = \frac{1}{j\omega C}$$

The “Complex-DC” circuit to analyze is this:

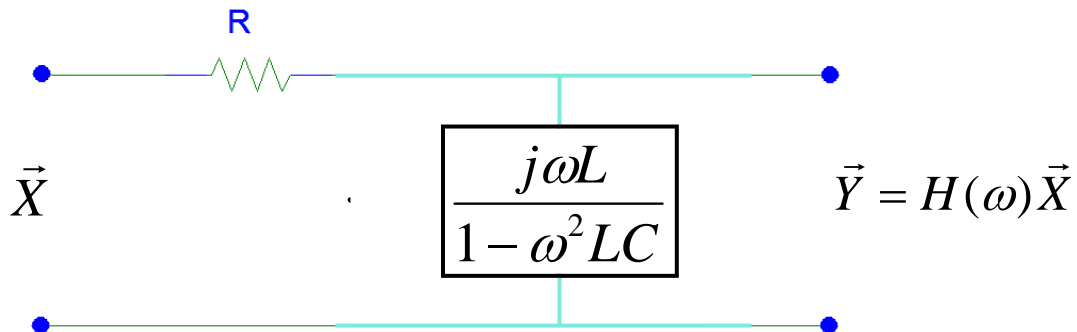


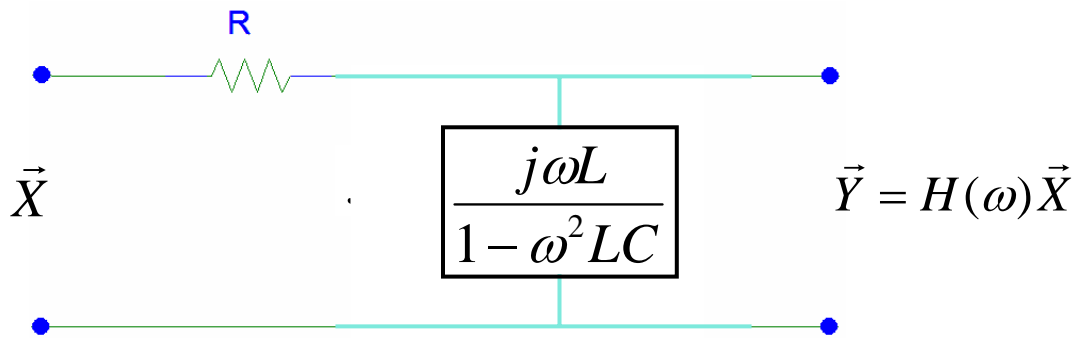
Step 2: Now we solve this Complex DC Circuit:



For *this* circuit we don't need the powerful "Systematic Schemes"...
 We can use "Parallel Combo" together with "Voltage Divider"

Parallel combo:
$$Z_{\parallel} = \frac{Z_L Z_C}{Z_L + Z_C} = \frac{(j\omega L) \left(\frac{1}{j\omega C} \right)}{j\omega L + \frac{1}{j\omega C}} = \frac{j\omega L}{1 - \omega^2 LC}$$





Now **Voltage Divider** gives:

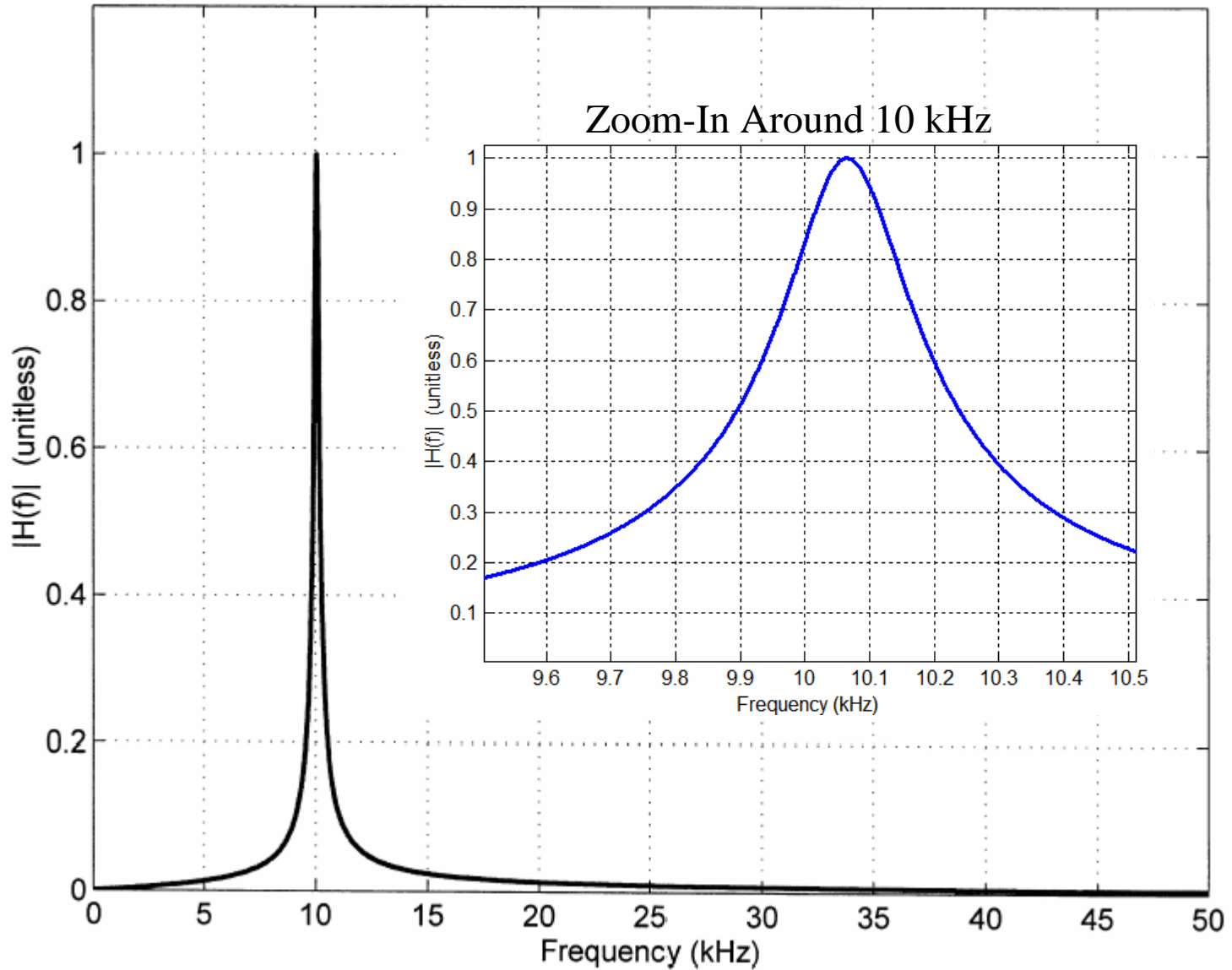
$$\vec{Y} = \left[\frac{Z_{\parallel}}{R + Z_{\parallel}} \right] \vec{X} = \left[\frac{j\omega L / (1 - \omega^2 LC)}{R + j\omega L / (1 - \omega^2 LC)} \right] \vec{X}$$

$$= \underbrace{\left[\frac{j\omega L}{R(1 - \omega^2 LC) + j\omega L} \right]}_{H(\omega)} \vec{X}$$

$$\Rightarrow H(\omega) = \frac{j\omega L}{R(1 - \omega^2 LC) + j\omega L}$$

The plot of $H(\omega)$ on the next page (for the given RLC values) shows that this circuit is a narrow “bandpass” filter – it passes a narrow band of frequencies (in this case centered at 10kHz, the sinusoidal oscillator frequency we want!)

This Circuit's Frequency Response



(c) Each term in the FS of the input looks like this:

$$b_n \sin(n\omega_0 t) \quad n = 1, 3, 5, \dots$$

or

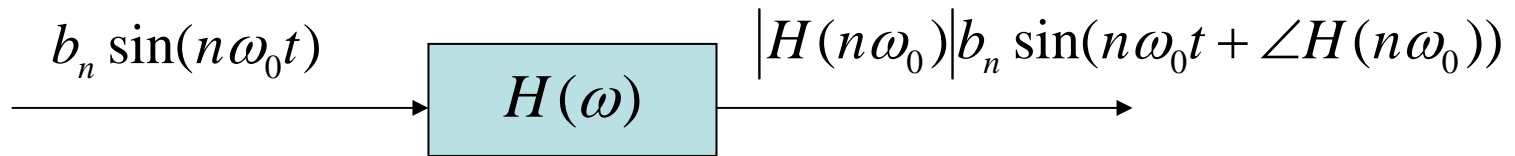
$$b_n \sin(n \times 2\pi \times 10,000t)$$

So we get the following frequencies: 10kHz

30kHz

50kHz etc.

Each of these sine waves goes through the circuit according to our general result to give:

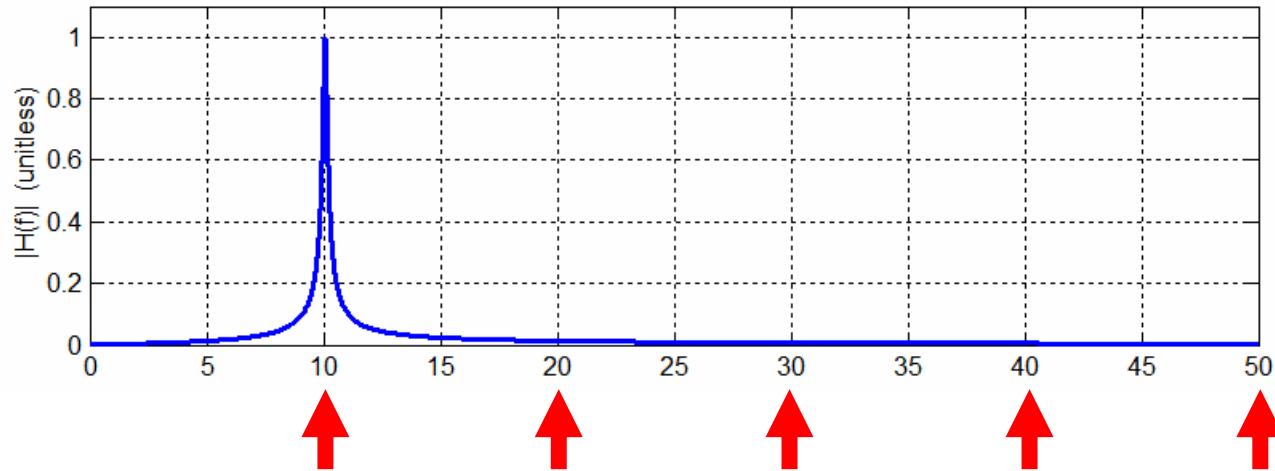


So the output is: $|H(n\omega_0)| b_n \sin(n\omega_0 t + \angle H(n\omega_0))$

Now from the plot of $|H(\omega)|$ we see that the sinusoidal term at 10kHz in $x(t)$ will get multiplied by:

$$|H(2\pi \times 10,000)| \approx 1$$

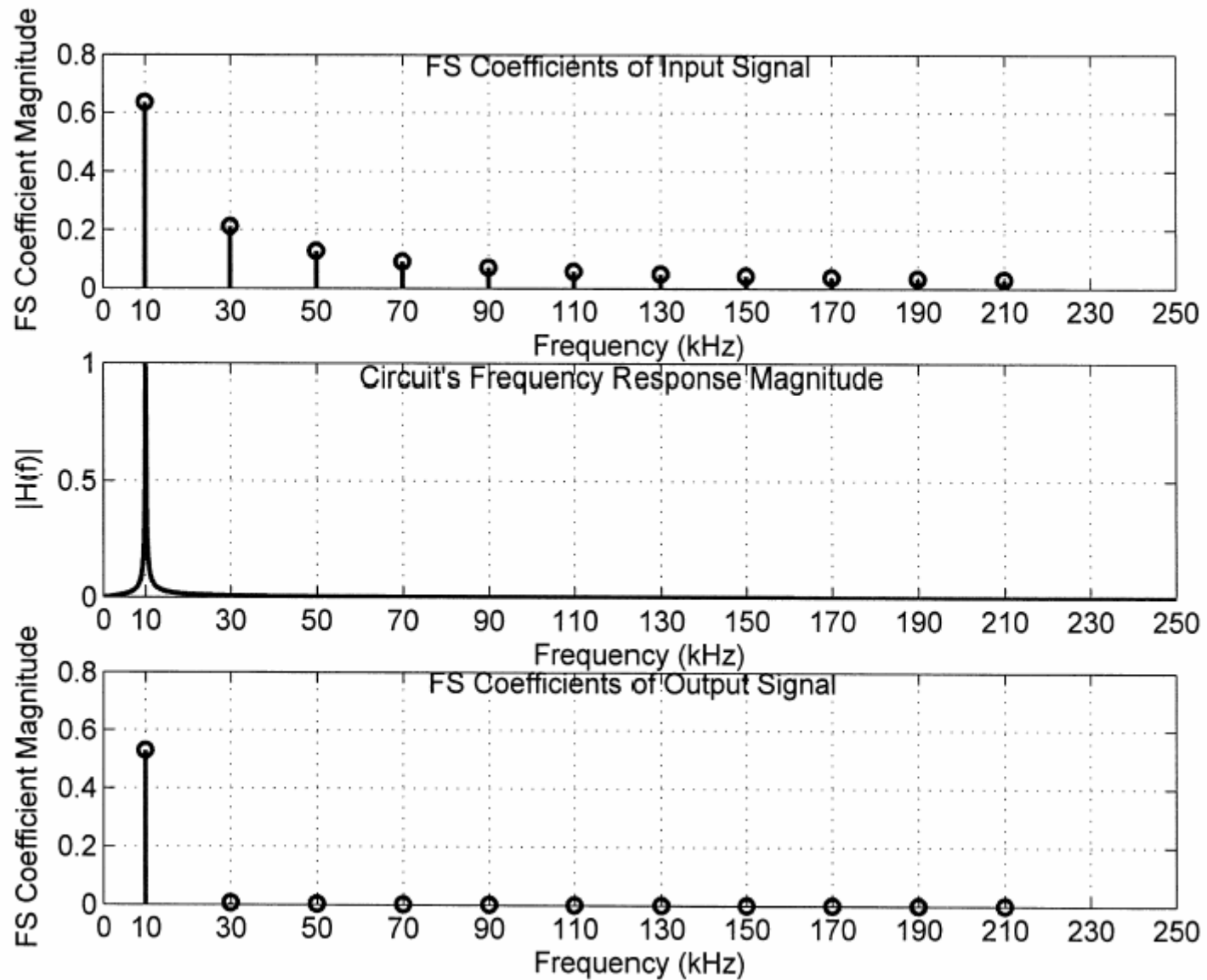
But all the other frequencies (20 kHz, 30 kHz, 40 kHz, etc.) get multiplied by very small numbers.



The input magnitude spectrum, the frequency response, and the output magnitude spectrum are shown in the plots on the next page.

\Rightarrow The output spectrum shows one significant sinusoid with all the other terms negligible

\Rightarrow Output \approx single sinusoid @ 10kHz



The bottom plot is found by multiplying the top plot by the middle plot

(d) Total Harmonic Distortion is the power at the desired frequency relative to the power at all other frequencies (as %)

$$THD = 100\% \times \frac{P_{@ 20kHz} + P_{@ 30kHz} + P_{@ 40kHz} + \dots}{P_{@ 10kHz}}$$

$$= 100\% \times \left[\frac{\sum_{n=2}^{\infty} (b_n |H(n\omega_0)|)^2 / 2}{(b_1 |H(\omega_0)|)^2 / 2} \right]$$

We'll use the computer...
so we'll only compute this sum for
some large enough # terms

From computer calculation:

$$\text{Power @ 10kHz} = 0.1406 \text{ W}$$

$$\text{Power Elsewhere} = 0.33 \times 10^{-4} \text{ W}$$

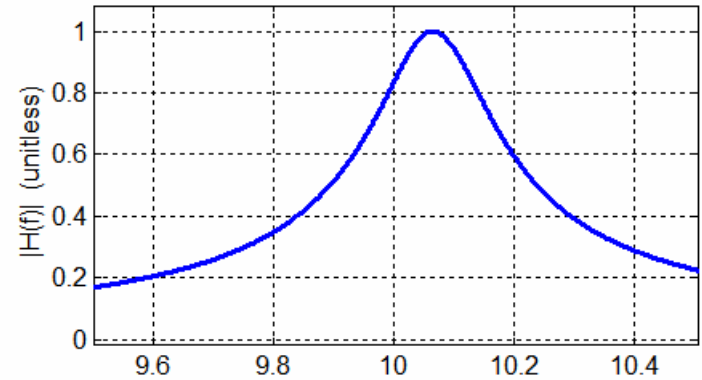
$$THD = 100\% \times \left(\frac{0.33 \times 10^{-4}}{0.1406} \right)$$

$$THD = 0.02\% \quad \text{Very Low!}$$

⇒ Output is a good quality sinusoid

Some practical things to worry about:

Note that our peak doesn't fall exactly at 10 kHz !!!



1. There are only certain typical R , L , C values that are available
 - a. Standard 1% resistors are available having R s have first two digits of
10 11 12 13 15 16 18 20 22 24 27 30 33 36 39 43 47 51
56 62 68 75 82 91
 - b. Capacitors and Inductors are also available in certain standard values
 - c. **So... we may not be able to choose standard values to get a peak exactly at 10 kHz!!**
2. All components have tolerances: accurate to 1%, 5%, 10%, 20%, etc. of the nominal value
 - a. **So... we should choose the tolerance needed to ensure that the peak will not move too much over the possible component value ranges!!!**