BINGHAMTON UNIVERSITY
State University of New York

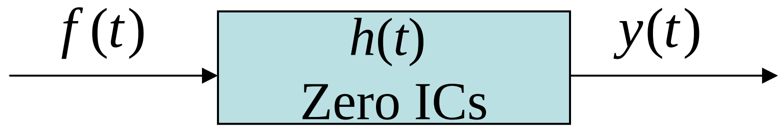
EECE 301
Signals & Systems
Prof. Mark Fowler

<u>**Discussion #4**</u>

- C-T Convolution Examples

C-T Convolution Examples

Example 1:



$$\text{Given}: \ f(t) = e^{t}u(-t) \quad h(t) = -\delta(t) + 2e^{-t}u(t)$$

$$\text{Find}: \ Zero - state\ response: \ y(t) = f(t) * h(t)$$

Solution:

$$y(t) = f(t) * h(t) = f(t) * \left[-\delta(t) + 2e^{-t}u(t)\right]$$

First we'll use properties of convolution to break this down into sub-problems.

By the distributive property we have:

call this $h_1(t)$

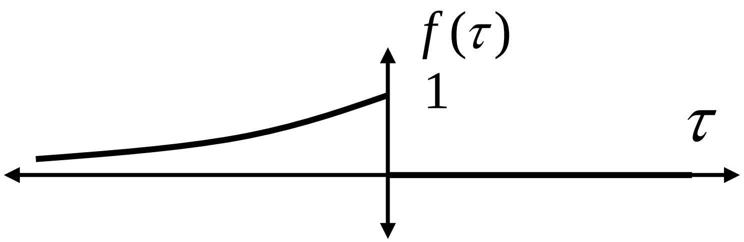
$$y(t) = -f(t) * \delta(t) + f(t) * 2e^{-t}u(t)$$

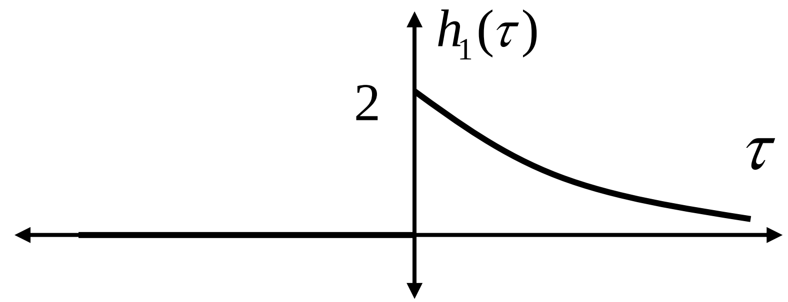
Use Property of Convolution with impulses

Call this convolution $y_1(t)$

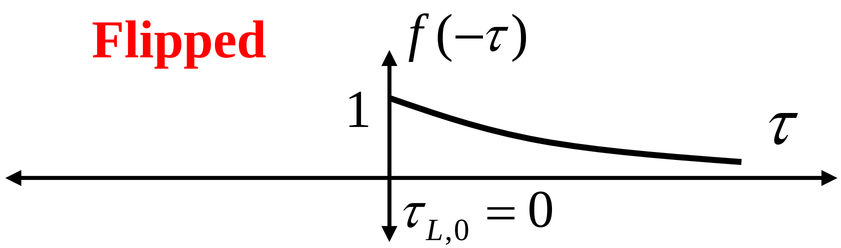
$$y(t) = -f(t) + y_1(t)$$

Write as functions of τ:

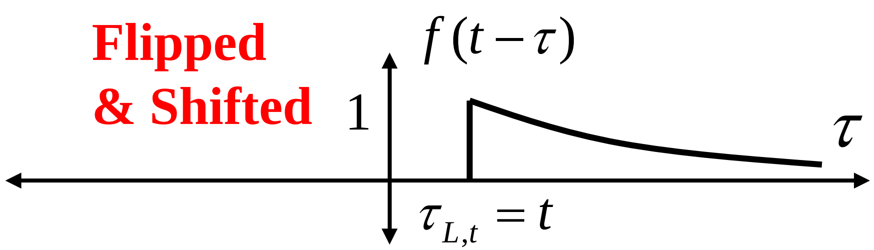




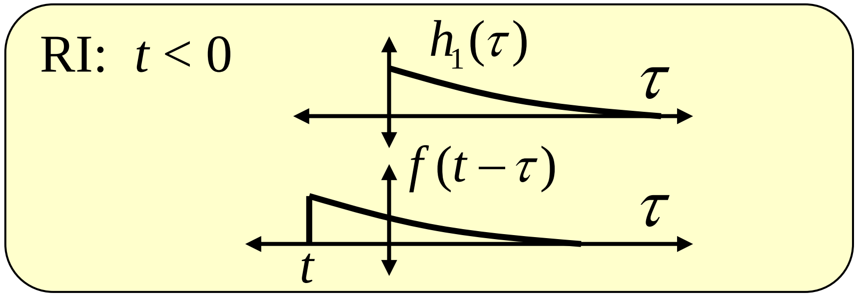
Flipped



Flipped & Shifted



Only two cases for product $h_1(\tau) f(t-\tau)$:

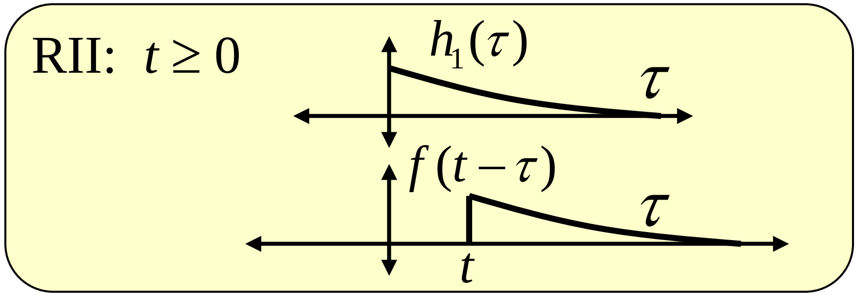


$$\Rightarrow y_1(t) = \int_0^{\infty} \left[e^{t-\tau} \right] 2e^{-\tau} d\tau$$

$$= 2e^t \int_b^{\infty} e^{-2\tau} d\tau$$

$$= 2e^t \left[\frac{-1}{2} e^{-2\tau} \right]_0^{\infty} = -e^t [0-1]$$

$$\boxed{y_1(t) = e^t \quad for \quad t < 0}$$



$$y_1(t) = \int_t^{\infty} \left[e^{t-\tau} \right] 2e^{-\tau} d\tau$$

$$= -e^t \left[e^{-2\tau} \right]_t^{\infty} = -e^t \left[0 - e^{-2t} \right]$$

$$= e^{-t}$$

$$\boxed{y_1(t) = e^{-t} \quad for \quad t \geq 0}$$

Now… assemble these results together:

$$y(t) = -f(t) + y_1(t)$$

So:

$$y(t) = \begin{cases} \underbrace{-e^{t}}_{-f(t)} + \underbrace{e^{t}}_{y_1(t)} = 0, & for \;\; t < 0 \\ 0 + e^{-t} = e^{-t}, & for \; t \geq 0 \end{cases}$$

$-f(t)$ $\quad$ $y_1(t)$

$$\boxed{y(t) = \begin{cases} 0, & for \;\; t < 0 \\ e^{-t}, & for \; t \geq 0 \end{cases}}$$

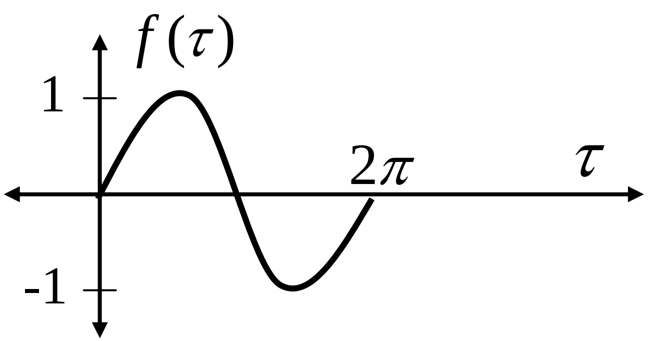
Example 2:

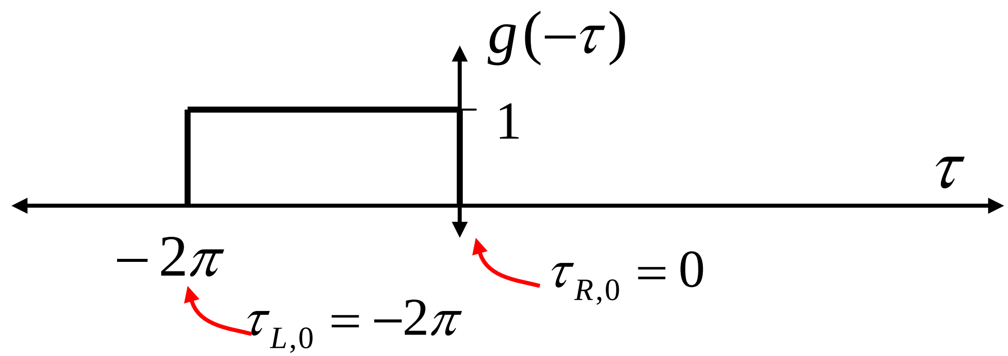
Given: $f(t) = \sin(t)\left[u(t) - u(t-2\pi)\right]$

$g(t) = \left[u(t) - u(t-2\pi)\right]$

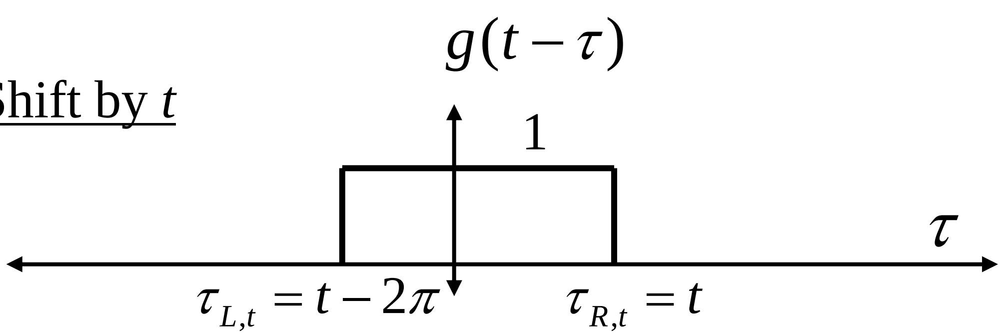
Find: $c(t) = f(t) * g(t)$

First write "as τ" and flip one:



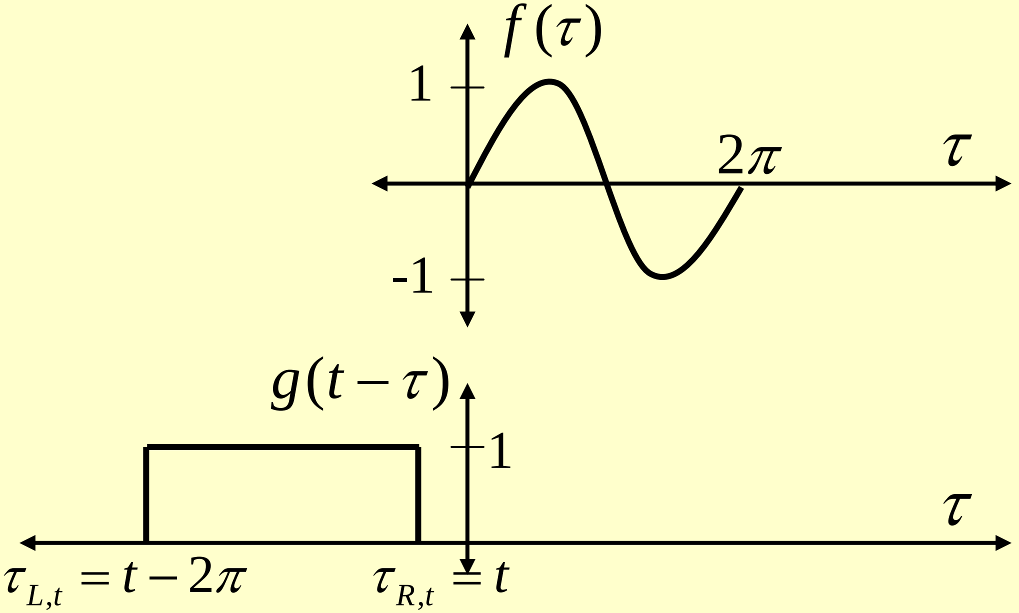


Shift by t



Now find the Regions of overlap… there are Four Regions

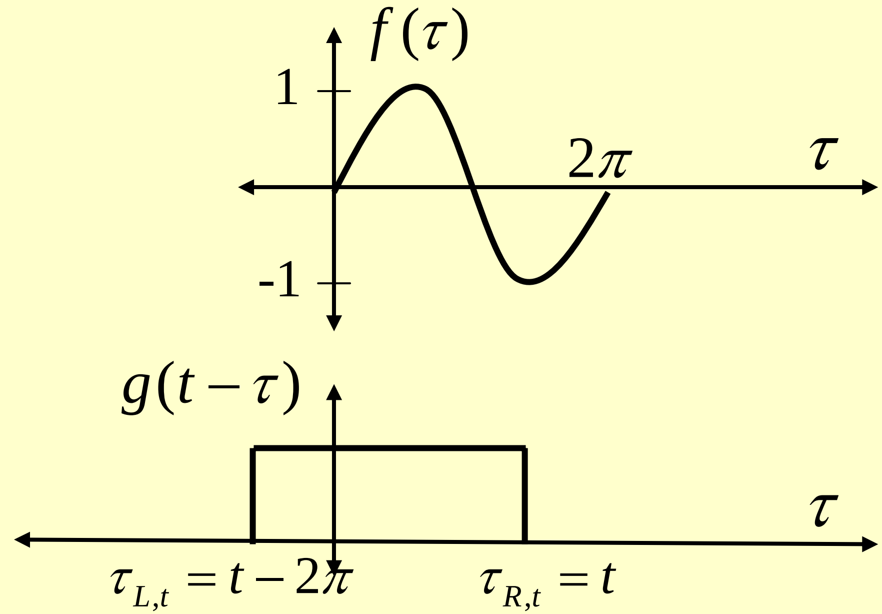
$RI:$ $t \le 0$



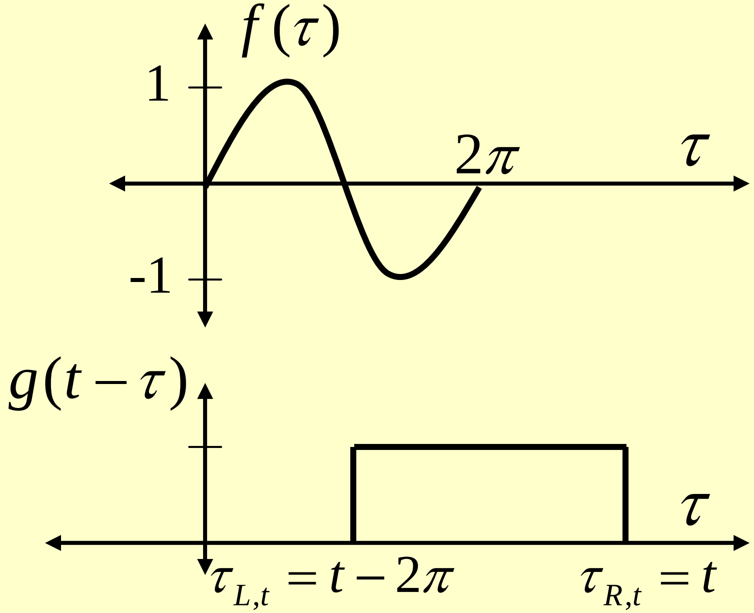
$RII:$

$$t > 0$$
$$t - 2\pi \le 0$$
$$0 < t \le 2\pi$$

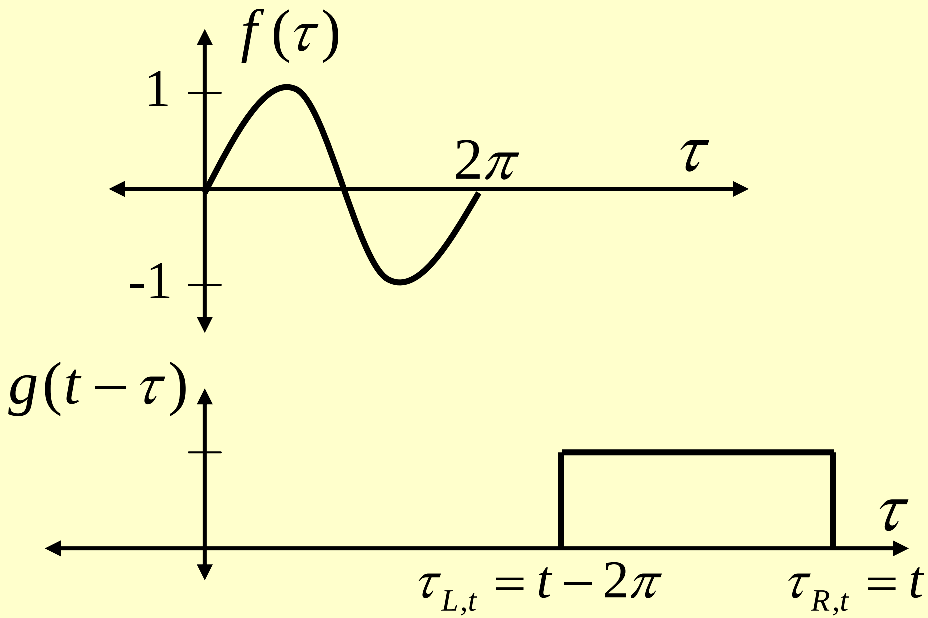
Note: this region includes the single *t* value at which you get complete overlap



$RIII:$ $\left.\begin{array}{l} t-2\pi \leq 2\pi \\ t > 2\pi \end{array}\right\}$ $\boxed{2\pi < t \leq 4\pi}$



$RIV:$ $t-2\pi > 2\pi$ $\boxed{t > 4\pi}$

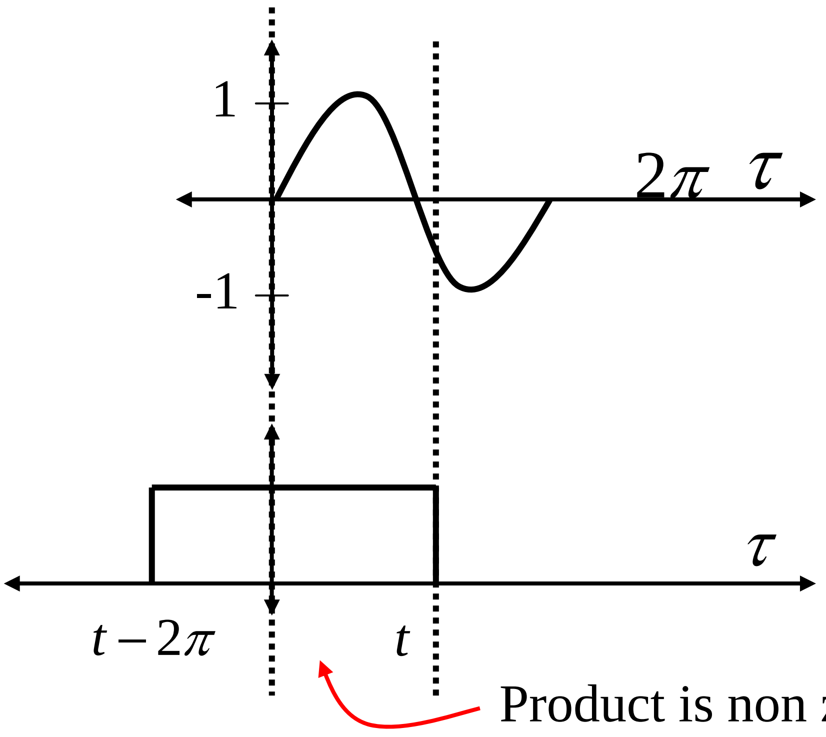


RI and RIV Integrals

No Overlap $\Rightarrow$ Integrand = 0 $\Rightarrow$

$c(t) = 0$ for $t \le 0$
and $t > 4\pi$

RII Integral



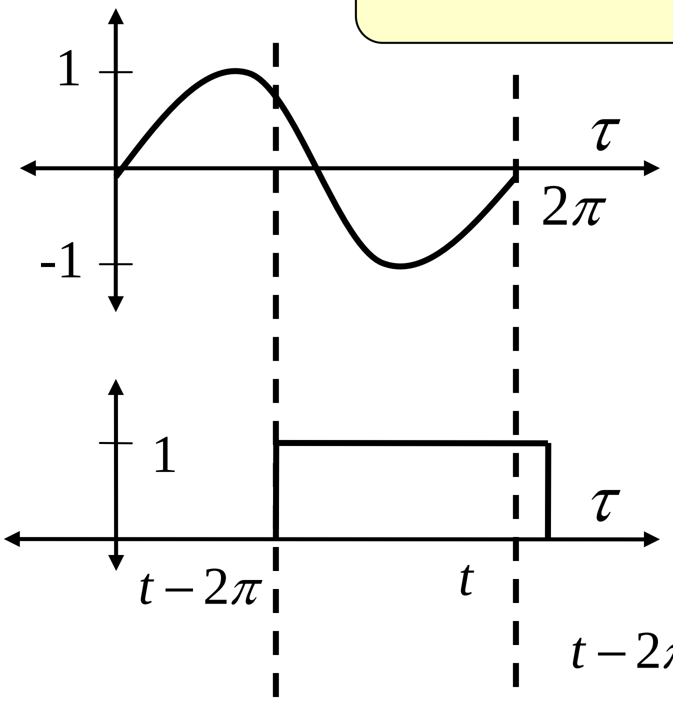
Product is non zero only here
$\Rightarrow$ integrate 0 to t

$$\int_0^t 1\sin(\tau)d\tau = \left[-\cos(\tau)\right]_0^t = -\left[\cos(t)-1\right]$$

$$= 1-\cos(t) \quad for \;\; 0 < t \le 2\pi$$

RIII Integral

Comment: note that you do not get a sinusoid out, because you only put part of a sinusoid in!



$t-2\pi$

$2\pi \Rightarrow$ Integrate (t - 2π) to 2π

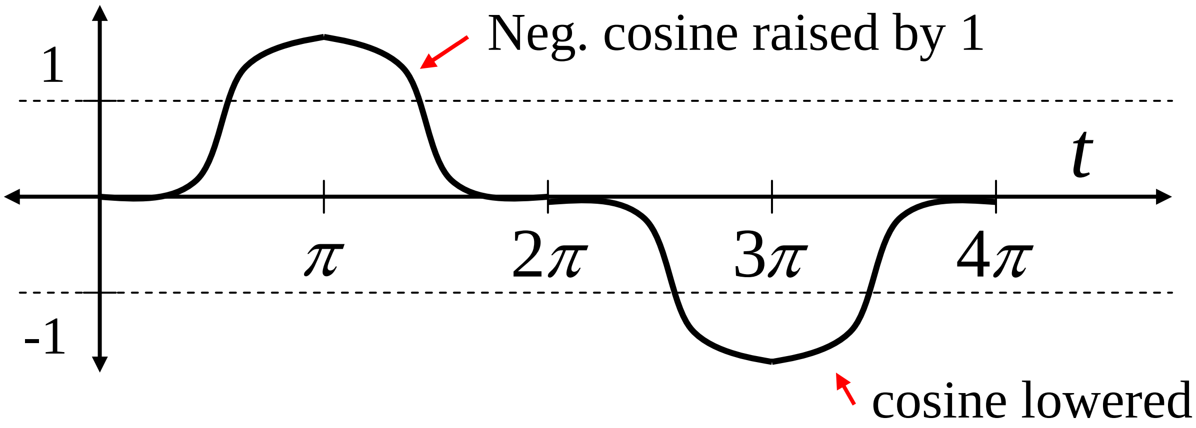
$$\int_{(t-2\pi)}^{2\pi} 1\sin(\tau)d\tau = \left[-\cos(\tau)\right]_{t-2\pi}^{2\pi} = -\left[\cos(2\pi) - \underbrace{\cos(t-2\pi)}\right]$$

$$= \underbrace{\cos(t) - 1} \quad \textit{for} \quad 2\pi < t \le 4\pi$$

cosine lowered by 1

= cos(t) by periodicity

Now… assemble the parts that make up the output:



$$c(t) = \begin{cases} 0, & t \le 0 \\ 1 - \cos t, & 0 < t \le 2\pi \\ \cos t - 1, & 2\pi < t \le 4\pi \\ 0, & t > 4\pi \end{cases}$$

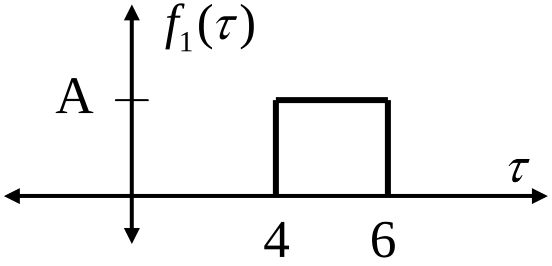
Example 3:

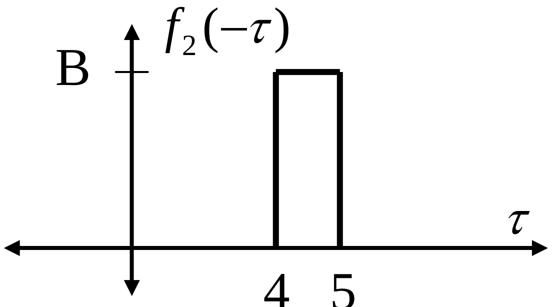
Given: $f_1(t) = A[u(t-4) - u(t-6)]$

$f_2(t) = [u(t+5) - u(t+4)]$

Find: $c(t) = f_1(t) * f_2(t)$

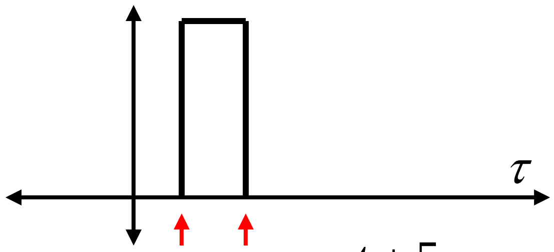
Write "as τ" and flip one:





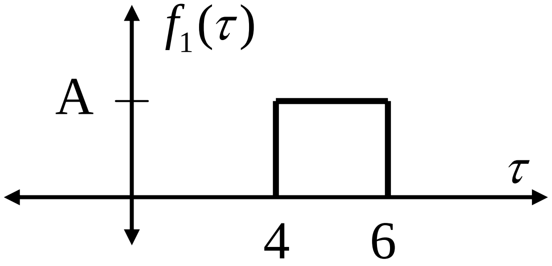
$\tau_{L,0} = 4$ $\tau_{R,0} = 5$

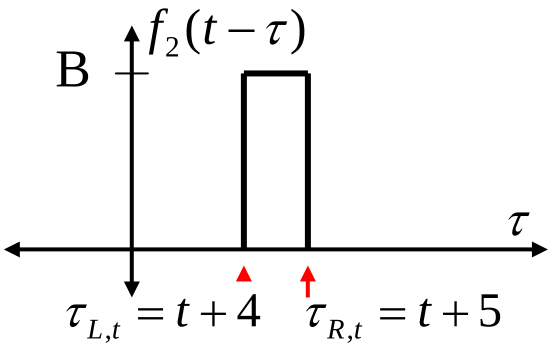
"Shift by t"



$\tau_{L,t} = t+4$ $\tau_{R,t} = t+5$

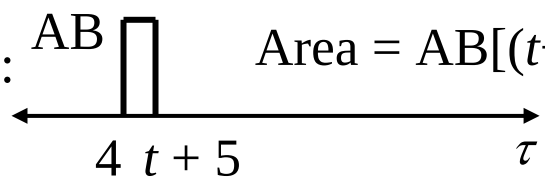
RI: $t + 5 < 4 \Rightarrow t < -1$ prod. $= 0 \Rightarrow$ $c(t) = 0 \quad \forall\, t < -1$





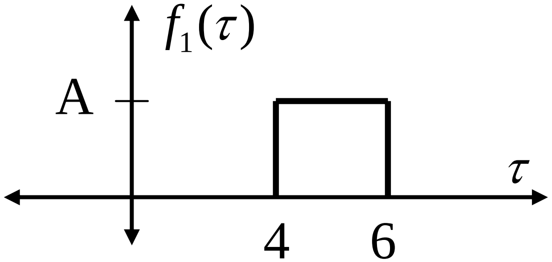
<u>RII:</u> $\left.\begin{array}{l} t+5 \geq 4 \\ t+4 < 4 \end{array}\right\}$ $-1 \leq t < 0$

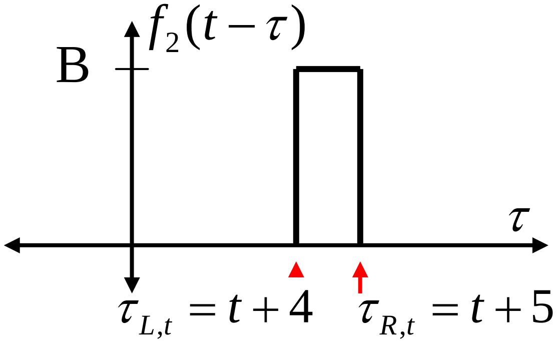
product:



Area = AB[(t+5)-4] = ABt + AB

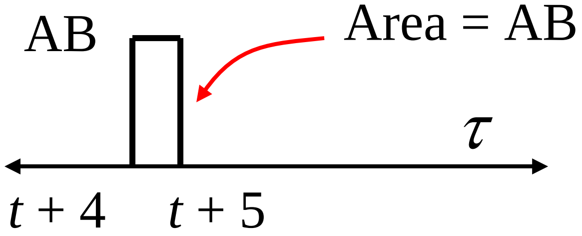
$$c(t) = \int_4^{t+5} AB\,d\tau = \boxed{ABt + AB \quad for \quad -1 \leq t < 0}$$





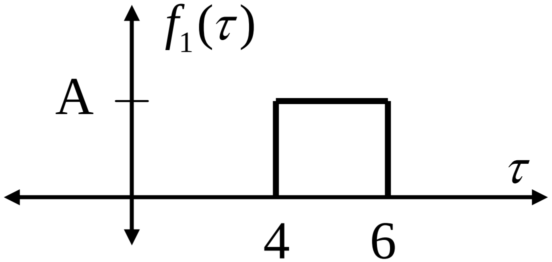
<u>RIII:</u> $\left.\begin{array}{l} t+4 \geq 4 \\ t+5 < 6 \end{array}\right\} \quad 0 \leq t < 1$

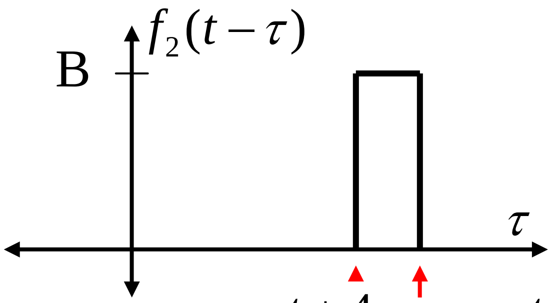
product:



$$c(t) = \int_{t+4}^{t+5} AB\,d\tau$$

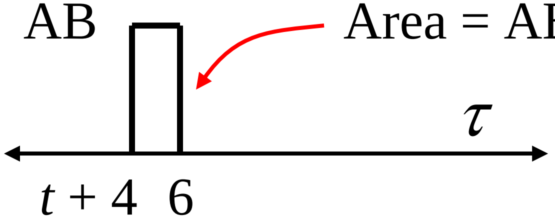
$$\boxed{c(t) = AB \quad for \quad 0 \leq t < 1}$$





<u>RIV:</u> $\left.\begin{array}{l} t+5 \geq 6 \\ t+4 < 6 \end{array}\right\}$ $1 \leq t < 2$

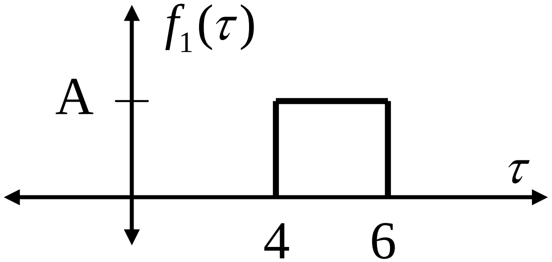
product:

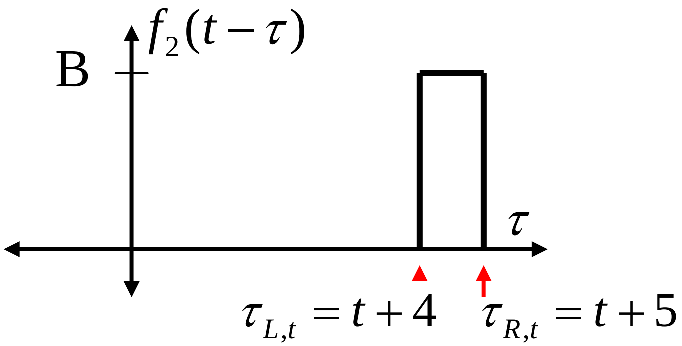


Area = AB(6 – (t + 4))

= 2AB - ABt

$$c(t) = \int_{t+4}^{6} AB\,d\tau = 2AB - ABt \quad for \;\; 1 \leq t < 2$$



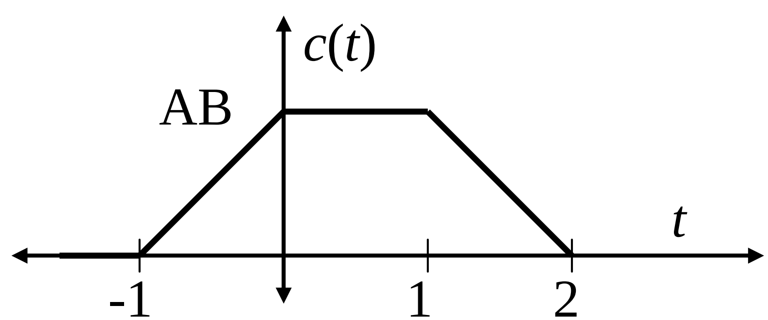


RV: $t + 4 \geq 6 \;\Rightarrow\; t \geq 2$ product = 0

$c(t) = 0 \;\; for \;\; t \geq 2$

Now… assemble the pieces:

$$c(t) = \begin{cases} 0, & t < -1 \\ ABt + AB, & 1 \leq t < 0 \\ AB, & 0 \leq t < 1 \\ 2AB - ABt, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases}$$

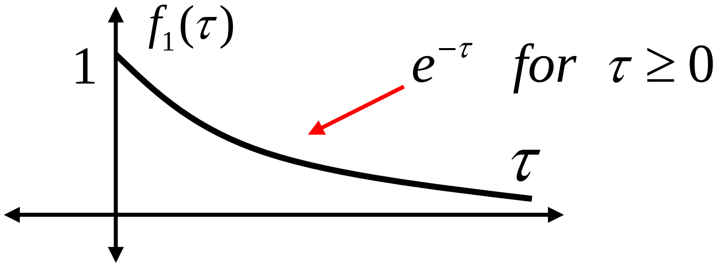


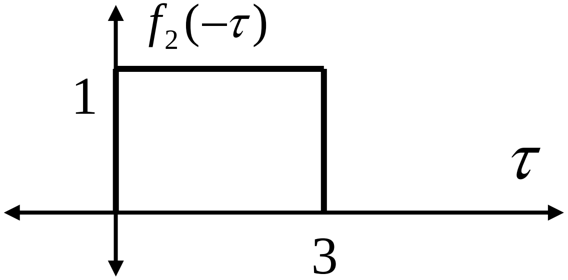
Example 4:

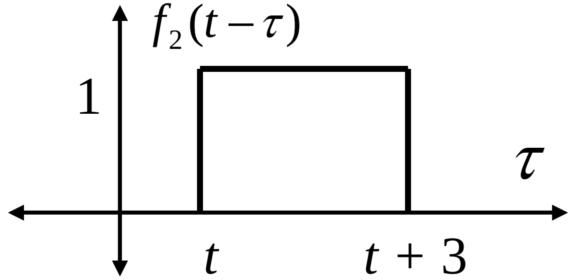
Given: $f_1(t) = e^{-t} * u(t)$

$f_2(t) = u(t+3) - u(t)$

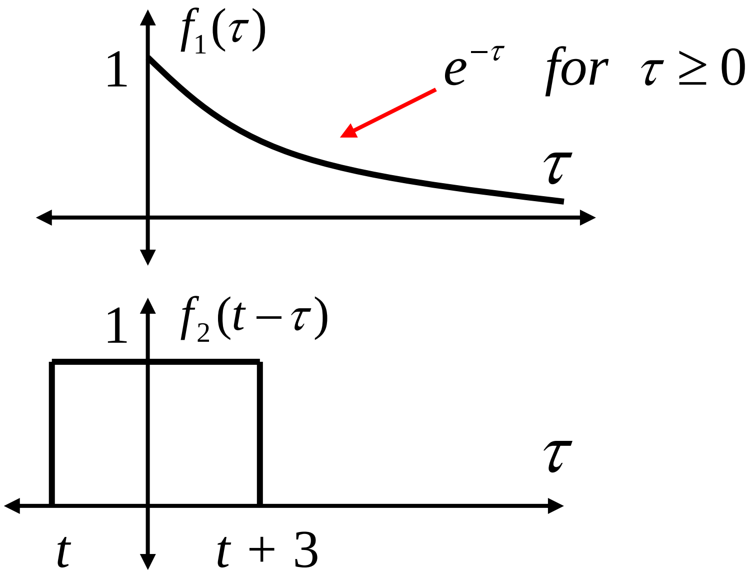
Find: $c(t) = f_1(t) * f_2(t)$





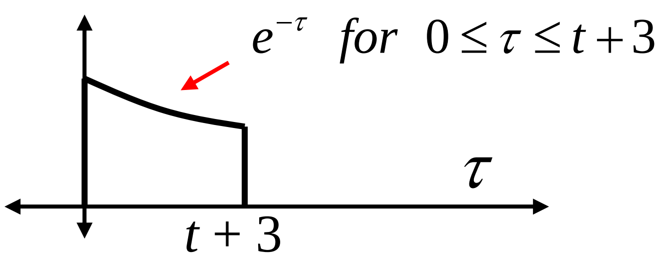


RI: $t + 3 < 0 \Rightarrow t < -3$ prod = 0 $\Rightarrow$ $c(t) = 0$ $t < -3$



RII: $t + 3 \geq 0$, $t < 0$ $\Big\}$ $-3 \leq t < 0$

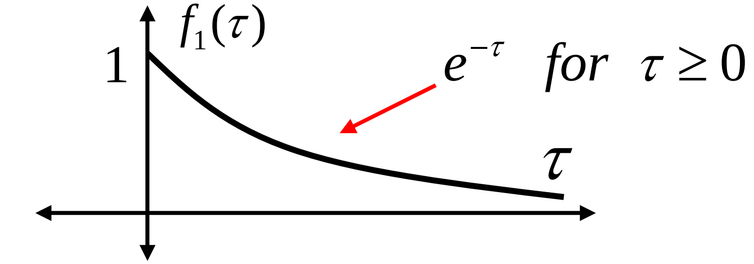
product:

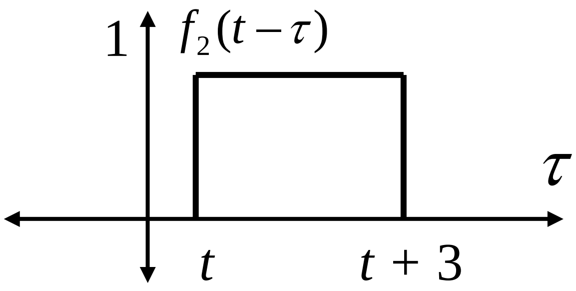


$$\Rightarrow c(t) = \int_0^{t+3} e^{-\tau} d\tau = -\left[e^{-\tau}\right]_0^{t+3}$$

$$= -\left[e^{-(t+3)} - 1\right] =$$

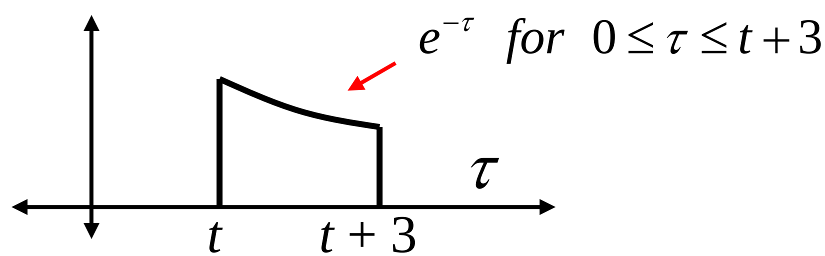
$$\boxed{1 - e^{-(t+3)} \quad for \;\; -3 \leq t < 0}$$





<u>RIII:</u> $t \geq 0$

product:



$$\Rightarrow c(t) = \int_t^{t+3} e^{-\tau} d\tau = -\left[e^{-\tau}\right]_t^{t+3} = e^{-t} - e^{-(t+3)}$$

$$= e^{-t} - e^{-3}e^{-t} = \boxed{\left(1 - e^{-3}\right)e^{-t} \quad for \ \ t \geq 0}$$

$$c(t)=\begin{cases}0, & t<-3\\ 1-e^{-(t+3)}, & -3\le t<0\\ (1-e^{-3})e^{-t}, & t\ge 0\end{cases}$$