EECE 301
Signals & Systems
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Discussion #1

• Complex Numbers and Complex-Valued Functions
• Reading Assignment: Appendix A of Kamen and Heck
Complex Numbers

Complex numbers arise as roots of polynomials.

Recall that the solution of differential equations involves finding roots of the “characteristic polynomial”

So…

Definition of imaginary # \( j \) and some resulting properties:

\[
\begin{align*}
j &= \sqrt{-1} \Rightarrow j^2 = -1 \\
\Rightarrow (-j)(j) &= 1 \\
\Rightarrow (-j)(-j) &= -1
\end{align*}
\]

Rectangular form of a complex number:

\[
z = a + jb \quad a = \text{Re}\{z\} \quad b = \text{Im}\{z\}
\]

The rules of addition and multiplication are straightforward:

\[
\begin{align*}
\text{Add} : \quad (a + jb) + (c + jd) &= (a + c) + j(b + d) \\
\text{Multiply} : \quad (a + jb)(c + jd) &= (ac - bd) + j(ad + bc)
\end{align*}
\]
**Polar Form**

\[ z = re^{j\theta} \quad r > 0 \]

Polar form… an alternate way to express a complex number…

Polar Form… good for multiplication and division

Note: you may have learned polar form as \( r\angle\theta \)… we will **NOT** use that here!!

The advantage of the \( re^{j\theta} \) is that when it is manipulated using rules of exponentials and it behaves properly according to the rules of complex #s:

\[
(a^x)(a^y) = a^{x+y} \quad a^x / a^y = a^{x-y}
\]

**Multiplying Using Polar Form**

\[
(r_1e^{j\theta_1})(r_2e^{j\theta_2}) = r_1r_2e^{j(\theta_1+\theta_2)}
\]

\[
z^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta}
\]

\[
z^{1/n} = r^{1/n} e^{j\theta/n}
\]

**Dividing Using Polar Form**

\[
\frac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}
\]

\[
1 = \frac{1}{r_2e^{j\theta_2}} = \frac{1}{r_2} e^{-j\theta_2}
\]
We need to be able convert between Rectangular and Polar Forms… this is made easy and obvious by looking at the geometry (and trigonometry) of complex #s:

**Geometry of Complex Numbers**

```
\[ z = a + jb \]
```

**Conversion Formulas**

\[
\begin{align*}
b &= r \sin \theta \\
a &= r \cos \theta \\
r &= \sqrt{a^2 + b^2} \\
\theta &= \tan^{-1}\left(\frac{b}{a}\right)
\end{align*}
\]
Q: Why the form \( re^{j\theta} \) for polar form?? Start with trigonometry:

\[ z = a + jb = r\cos \theta + j(r \sin \theta) = r[\cos \theta + j\sin \theta] \]

From Calc II:

\[
\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots
\]

\[
j\sin \theta = j\theta - j\frac{\theta^3}{3!} + j\frac{\theta^5}{5!} - j\frac{\theta^7}{7!} + \ldots
\]

\[
\cos \theta + j\sin \theta = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots
\]

Also From Calc II:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\]

\[
e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots
\]

\[
\Rightarrow \text{Since } \cos \theta + j\sin \theta \text{ has the same expansion as } e^{j\theta} \text{ we can say that:}
\]

\[
\cos \theta + j\sin \theta = e^{j\theta}
\]
Complex Exponentials vs. Sines and Cosines

Euler’s Equations:

\( e^{j\theta} = \cos(\theta) + j\sin(\theta) \)  
\( e^{-j\theta} = \cos(\theta) - j\sin(\theta) \)  
\( \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \)  
\( \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \)

Note:  
Eq. C = (Eq. A + Eq. B)/2  
D = (A − B)/2  
A = C + jD  
B = C − jD
Summary of Rectangular & Polar Forms

**Rect Form:**
\[ z = a + jb \]
Re\{z\} = \( a = r \cos \theta \)
Im\{z\} = \( b = r \sin \theta \)

**Polar Form:**
\[ z = re^{j\theta} \quad r \geq 0 \quad \theta \in (-\pi, \pi] \]
\[ |z| = r = \sqrt{a^2 + b^2} \]
\[ \angle z = \theta = \tan^{-1}\left(\frac{b}{a}\right) \]

**Warning:** Your *calculator* will give you the *wrong answer* whenever you have \( a < 0 \). In other words, for \( z \) values that lie in the II and III quadrants.

You can always fix this by either adding or subtracting \( \pi \).

Use common sense… looking at the signs of \( a \) and \( b \) will tell you what quadrant \( z \) is in… make sure your angle agrees with that!!!
Conjugate of $Z$

Denoted as $z^*$ or $\bar{z}$

$z = a + jb \implies z^* = a - jb$

$z = re^{j\theta} \implies z^* = re^{-j\theta}$

Properties of $z^*$

1. $z + z^* = 2 \text{Re}\{z\}$

2. $z \times z^* = (a + jb)(a - jb) = a^2 + b^2 = |z|^2$
Unit Circle: A set of complex numbers with magnitude of 1 ($|z|=1$)

All $z$ on the unit circle look like: $1e^{j\theta}$

Four special points on the unit circle:

- $1 = e^{j0}$
- $-1 = e^{j\pi}$
- $j = e^{j\pi/2}$
- $-j = e^{-j\pi/2}$

Know these!!!

$$e^{\pm jn\pi} = \begin{cases} 
-1, & n \text{ odd integer} \\
1, & n \text{ even integer} 
\end{cases}$$

$$e^{\pm jn2\pi} = 1 \text{ for all integers}$$

$$e^{jn\pi/2} = \begin{cases} 
 j, & n = 1, 5, 9, \ldots \\
-j, & n = 3, 7, 11, \ldots \\
1, & n = 0, 4, 8, \ldots \\
-1, & n = 2, 6, 10, \ldots 
\end{cases}$$
A sinusoid is completely defined by its three parameters:

- Amplitude $A$ (for EE’s typically in volts or amps or other physical unit)
- Frequency $\omega$ in radians per second
- Phase shift $\phi$ in radians

$T$ is the period of the sinusoid and is related to the frequency
Phase shift (often just shortened to phase) shows up explicitly in the equation but shows up in the plot as a time shift (because the plot is a function of time).

Q: What is the relationship between the plot-observed time shift and the equation-specified phase shift?

A: We can write the time shift of a function by replacing $t$ by $t + t_o$ (more on this later, but you should be able to verify that this is true!) Then we get:

$$f(t + t_o) = A \sin(\omega(t + t_o)) = A \sin(\omega t + \omega t_o)$$

So we get that: $\phi = \omega t_o$ (unit-wise this makes sense!!!)

Frequency can be expressed in two common units:
- Cyclic frequency: $f = 1/T$ in Hz (1 Hz = 1 cycle/second)
- Radian Frequency: $\omega = 2\pi/T$ (in radians/second)

From this we can see that these two frequency units have a simple conversion factor relationship (like all other unit conversions – e.g. feet and meters):

$$\omega = 2\pi f$$
In circuits you used “phasors” (we’ll call them “static” phasors here)… the point of using them is to make it EASY to analyze circuits that are driven by a single sinusoid. Here is an example to refresh your memory!!

Find output voltage of the following circuit:

\[ x(t) = 5 \cos\left(1000t + \frac{\pi}{4}\right) \]

\[ R = 1\Omega \]

\[ L = 2\text{mH} \]

\[ y(t) = ? \]

Use phasor and impedance ideas:

Impedance of Inductor: \[ Z_L = j\omega L = j2 \]

Phasor of Input: \[ \hat{x} = 5e^{j\frac{\pi}{4}} \]

Use voltage divider to find output:

\[ \hat{y} = \hat{x} \left[ \frac{j2}{1 + j2} \right] = 5e^{j\frac{\pi}{4}} \left[ 0.89e^{j0.46} \right] \]

Output phasor:

\[ \hat{y} = 4.45e^{j1.25} \]

Output signal:

\[ y(t) = 4.45 \cos(1000t + 1.25) \]
Note that in using “static” phasors there was no need to “carry around the frequency” … it gets suppressed in the static phasor

BUT… if you have multiple driving sinusoids (each at its own unique frequency) then you’ll need to keep that frequency in the phasor representation… that leads to:

Rotating Phasors

Keeping the frequency “part”

\[ A_o \cos(\omega_o t + \phi_o) \rightarrow A_o e^{j(\omega_o t + \phi_o)} \]

\[ = A_o e^{j\phi_o} e^{j\omega_o t} \]

“static” phasor part

\[ e^{j\omega_o t} = \cos(\omega_o t) + j \sin(\omega_o t) \]

\[ \text{Re}\{A_o e^{j\phi} e^{j\omega_o t}\} = A_o \cos(\omega_o t + \phi_o) \]
If \( \tilde{x}(t) = Ae^{+j[\omega_o t + \phi_o]} \)

\( (x(t) = A \cos(\omega_o t + \phi_o)) \)

What is: \( \tilde{x}^*(t) \)

\[
\begin{align*}
\tilde{x}^*(t) & = Ae^{-j[\omega_o t + \phi_o]} \\
& = Ae^{-j\phi_o} e^{-j\omega_o t}
\end{align*}
\]

\[
\frac{\tilde{x}(t) + \tilde{x}^*(t)}{2} = 2 \Re\{\tilde{x}(t)\}
\]

\[
= 2A \cos(\omega_o t + \phi_o)
\]

Because rotating phasors take the value of a complex number at each instant of time they must follow all the rules of complex numbers…

Especially: EULER’S EQUATIONS!!
Rotating Phasors

Euler’s Equations

\[ e^{j(\omega t + \theta)} = \cos(\omega t + \theta) + j \sin(\omega t + \theta) \]  \hspace{1cm} (A)

\[ e^{-j(\omega t + \theta)} = \cos(\omega t + \theta) - j \sin(\omega t + \theta) \]  \hspace{1cm} (B)

\[ \cos(\omega t + \theta) = \frac{e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}}{2} \]  \hspace{1cm} (C)

\[ \sin(\omega t + \theta) = \frac{e^{j(\omega t + \theta)} - e^{-j(\omega t + \theta)}}{2j} \]  \hspace{1cm} (D)

**Note:**

Eq. C = (Eq. A + Eq. B)/2  \hspace{1cm} D = (A - B)/2

A = C + jD  \hspace{1cm} B = C - jD
**$t = 0$**

Viewing rotating phasor on the complex plane

**$t = t_1 > 0$**

**$t = t_2 > t_1$**