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SIGNAL DETECTION USING GROUP TRANSFORMS

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ABSTRACT

It is shown that transforms arising from square integrable group representations can be used for the detection of signals in noise. This class of group transforms includes the Gabor transform and the wavelet transform. We use these transforms to map the reproducing kernel Hilbert space (RKHS) associated to a noise covariance into another RKHS; the RKHS formulation of the detection problem is then applied to this new space. Using the discrete form of the Gabor transform or the wavelet transform results in a discrete-parameter correlator structure. It is shown that the use of the wavelet transform for the detection of signals in the presence of $1/f$ noise results in a structurally simple form for the correlation receiver.

I. INTRODUCTION

There has been a recent explosion of interest in the Gabor transform and the wavelet transform for signal analysis in the time-frequency domain. These transforms arise from the theory of group representations [1] and are just two examples of a class of transforms we will call group transforms.

Because the Gabor and wavelet transforms offer many advantages over classical Fourier analysis, their application to signal processing problems is of interest. One such area is the detection of signals in noise. The Gabor transform has been shown to be useful for detecting transient signals in the presence of white noise [2]. A weighted correlation is performed between the Gabor coefficients of the received signal and the Gabor coefficients of a reference signal; the weighting matrix is the correlation matrix of the Gabor coefficients of the white noise. We extend these results to the case of nonstationary noise. Although the use of the wavelet transform for the detection of signals is mentioned in [2], no formulation is given. However, the wavelet transform has been used for the detection of signals [3], but this approach does not use a correlator structure. Instead, the presence of a signal is determined by searching for characteristic peaks in the wavelet transform of the received signal. We develop a wavelet-based correlation detector and demonstrate its usefulness.

The Gabor and wavelet transforms have a clear link to the theory of reproducing kernel Hilbert space (RKHS). In general, the range space of a group transform is an RKHS [1]. Since the solution to the detection problem can be simply expressed in terms of the inner product of an RKHS [4], [5], the link between group transforms and RKHSs can be exploited for the detection of signals in nonstationary noise. This results in a general method for which the result of [2] is a special case.

II. RKHS BACKGROUND

Roughly, an RKHS is a Hilbert space that contains an element that plays the role of the Dirac delta "function". Specifically, let $H(K)$ be a Hilbert space of functions defined on an arbitrary set I; $H(K)$ is called an RKHS if there is a function $K(\cdot,\cdot)$ defined on $I \times I$ such that, for each $t \in I$, $K(\cdot,t) \in H(K)$ and $< f, K(\cdot,t) >_{H(K)} = f(t), \forall f \in H(K)$. The function $K(\cdot,\cdot)$ is called the reproducing kernel of $H(K)$. It is well-known that to each covariance kernel $K(\cdot,\cdot)$ there corresponds a unique RKHS $H(K)$ for which $K(\cdot,\cdot)$ is the reproducing kernel [4].

The signal detection problem requires a decision to be made between the two hypotheses

$$\begin{aligned} H_1: \ r(t) &= \gamma(t) + n(t), \\ H_0: \ r(t) &= n(t), \end{aligned} \tag{1}$$

where $r(t)$ is the received signal observed over the bounded interval I, $\gamma(t)$ is the Gaussian signal to be detected, and $n(t)$ is a zero-mean Gaussian noise with covariance kernel $K(t,s)$ that is square-integrable on $I \times I$.

The classical solution consists of comparing a sufficient statistic Λ to a threshold. The sufficient statistic can be expressed as $\Lambda = < r, hr >_{H(K)}$ where h is the unique causal operator that is the solution of an RKHS Wiener-Hopf equation [5]. That is, the function hr is an estimate of the signal γ, making this formulation a generalization of the estimator-correlator. If $\gamma(t)$ in (1) is a known deterministic signal, then the estimator-correlator reduces to the correlator $\Lambda = < r, \gamma >_{H(K)}$ [4]. Because of the similarity of the inner product structure between this case and the stochastic case, there is no loss of generality to consider only the deterministic case in the remainder of the paper.

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III. DETECTION USING GROUP TRANSFORMS

In this section we define group transforms and develop a group transform-based correlation detector.

Let G be a group with group operation $*$. A unitary representation U of G on a Hilbert space H is a mapping assigning to each $x \in G$ a unitary operator $U_x : H \rightarrow H$, such that for any $x, y \in G$, $U_{x*y} = U_x U_y$.

Given a unitary representation U of G, the associated group transform is an operator $\mathcal{U}$ defined on the Hilbert space H as follows: for $f \in H$,

$$(\mathcal{U}f)(x) = < f, U_x g >_H,$$

where x ranges over G, and a fixed nonzero $g \in H$ is chosen such that (i) g is admissible, that is,

$$\int_G |(\mathcal{U}g)(x)|^2 \, d\mu(x) < \infty,$$

where $\mu(x)$ is the left Haar measure on G, and (ii) g is cyclic, that is, $(\mathcal{U}f)(x) = 0, \forall x \in G$ iff $f \equiv 0$ [6]. We shall call such a g an analyzing function. The representation U is said to be square-integrable if every nonzero $g \in H$ is cyclic, and if there exists at least one nonzero admissible function.

For signal processing applications, H is usually taken to be $L^2(\boldsymbol{R})$, and

$$(\mathcal{U}f)(x) = \int_{-\infty}^{\infty} f(t) \overline{U_x g(t)} \, dt, \qquad (2)$$

where the overbar denotes complex conjugation. Then $\mathcal{U}$ maps $L^2(\boldsymbol{R})$ into $L^2(G, d\mu)$ and its range $R(\mathcal{U})$ is itself an RKHS with reproducing kernel

$$\tilde{K}(x,y) = < U_x g, U_y g >_{L^2},$$

where $x, y \in G$ [1]. This can be thought of as

$$\tilde{K}(x,y) = \left[\mathcal{U}_t \left[\overline{\mathcal{U}_s \delta(s-t)}\right]_y\right]_x, \qquad (3)$$

where the subscripts "s" and "y" denote "operation on $\delta(s-t)$ as a function of s and evaluation of the resulting function at y" (likewise for "t" and "x").

Since $H(K) \subset L^2(\boldsymbol{R})$ it is possible to use (2) to map the RKHS $H(K)$ into another RKHS as follows. Following the lead of (3), define the new reproducing kernel to be

$$\tilde{K}(x,y) = \left[\mathcal{U}_t \left[\overline{\mathcal{U}_s K(s,t)}\right]_y\right]_x. \qquad (4)$$

It is shown in [7] that $\mathcal{U} : H(K) \rightarrow H(\tilde{K})$ is isometric. Therefore, the sufficient statistic can be expressed as

$$\Lambda = < \mathcal{U}r, \mathcal{U}\gamma >_{H(\tilde{K})}. \qquad (5)$$

Note that $\tilde{K}(x,y)$ as given in (4) can be interpreted as the covariance of the group transform of the noise process. Thus we see that the sufficient statistic can be formulated as a correlator in the new space $H(\tilde{K})$: an RKHS correlation is performed between the transform $\mathcal{U}r$ of the received signal and the transform $\mathcal{U}\gamma$ of the signal γ.

The complexity of the computation of (5) depends on the structure of the reproducing kernel $\tilde{K}(\cdot,\cdot)$. As seen in (4), the choice of the group transform determines this structure. The usefulness of (5) will be illustrated in the next section, but first we introduce two group transforms that are gaining much attention from the signal processing community.

The Weyl-Heisenberg group $\mathcal{H}$ gives rise to the Gabor transform. The corresponding unitary representation $U_{\mathcal{H}}$ on $L^2(\boldsymbol{R})$ is defined by

$$U_{\mathcal{H}}(\omega,\tau)\, g(t) = e^{j\omega t}\, \overline{g(t-\tau)}.$$

Any function $g \in L^2(\boldsymbol{R})$ is a Weyl-Heisenberg analyzing function; therefore, $U_{\mathcal{H}}$ is a square-integrable representation on $L^2(\boldsymbol{R})$. The Gabor transform of a finite-energy signal f is then

$$Gf(\omega,\tau) = \int_{-\infty}^{\infty} f(t)\, g(t-\tau)\, e^{-j\omega t}\, dt, \qquad (6)$$

and, if we normalize g such that $\| g \|^2 = 1$, is an isometry from $L^2(\boldsymbol{R})$ into $L^2(\boldsymbol{R}^2)$ [6].

Similarly, the wavelet transform is associated with the affine group $\mathcal{A}$ for which the unitary representation $U_{\mathcal{A}}$ on $L^2(\boldsymbol{R})$ is defined as

$$U_{\mathcal{A}}(s,\tau)\, g(t) = e^{s/2}\, \overline{g(e^s t - \tau)}$$

for $s, \tau \in \boldsymbol{R}$. The wavelet transform of $f \in L^2(\boldsymbol{R})$ is then

$$Wf(s,\tau) = \int_{-\infty}^{\infty} f(t)\, e^{s/2}\, \overline{g(e^s t - \tau)}\, dt, \qquad (7)$$

where the analyzing function $g \in L^2(R)$ has Fourier transform satisfying

$$\int_{-\infty}^{\infty} \frac{|G(\omega)|^2}{|\omega|}\, d\omega < \infty, \qquad (8)$$

and

$$c_g \triangleq \int_0^{\infty} \frac{|G(\omega)|^2}{|\omega|}\, d\omega = \int_{-\infty}^{0} \frac{|G(\omega)|^2}{|\omega|}\, d\omega. \qquad (9)$$

Functions satisfying (8) are admissible and those satisfying (9) are cyclic [6]. Without loss of generality, we shall normalize g such that $c_g = 1$ so that the wavelet transform is an isometry into $L^2(\boldsymbol{R}^2)$.

There is redundancy in each of these representations; therefore, equivalent discrete versions arise by appropriately sampling (6) and (7). The discrete Gabor transform of a finite-energy signal f is

$$Gf(m,n) = \int_{-\infty}^{\infty} f(t)\, g(t - nT)\, e^{-jm\Omega t}\, dt.$$

By proper choice of the function g and the sampling parameters (Ω, T), the discrete Gabor transform can be made into an isometry [8]. The discrete wavelet transform of a finite-energy signal f is

$$Wf(m,n) = \int_{-\infty}^{\infty} f(t)\, S^{m/2}\, g(S^m t - nT)\, dt.$$

When no confusion can arise we write $Wf(m,n)$ as W_{mn}. It is possible to choose g and the sampling parameters (S, T) such that the discrete wavelet transform is an isometry [8].

The use of the discrete Gabor transform in (4) and (5) generalizes the result of [2] where the Gabor transform was used to detect overlapping transients in white noise. The wavelet transform is used to advantage in the next section.

IV. APPLICATION OF THE WAVELET TRANSFORM

In this section we give two examples that demonstrate the usefulness of the wavelet transform for signal detection.

Example 1: Fractional Brownian Motion

A fractional Brownian motion has a covariance given by

$$K_H(t,\tau) = V_H \left[|t|^{2H} + |\tau|^{2H} - |t-\tau|^{2H} \right],$$

where V_H is a constant depending on a parameter $H \in (0,1)$ [9]. We apply (4) to this covariance using an alternative form of the wavelet transform given by

$$\mathcal{W}f(s,\tau) = s^{-1/2} \int_{-\infty}^{\infty} f(t)\, g((t-\tau)/s)\, dt.$$

An easy generalization of the result in [9] shows the new reproducing kernel to be

$$\tilde{K}_H(s',\tau';s,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nu - s\lambda|^{2H}\; G(\nu,\lambda,s';\tau'-\tau)\, d\nu\, d\lambda,$$

where

$$G(\nu,\lambda,s';\tau'-\tau) \triangleq V_H\, (s/s')^{1/2}\, g((\nu-(\tau'-\tau))/s')\, \overline{g(\lambda)}.$$

Thus, $\tilde{K}_H(s',\tau';s,\tau)$ depends on $\tau'-\tau$. This shows that the wavelet transform of fractional Brownian motion is stationary in the time variable and makes the wavelet transform a particularly effective group transform for detection in the presence of fractional Brownian motion.

Although this result is interesting, a more useful result is given in the next example.

Example 2: Nearly $1/f$ Noise

Consider the nonstationary noise $n(t)$ defined [10] by the discrete wavelet expansion

$$n(t) = \sum_m \sum_n W_{mn}\, g_{mn}(t), \tag{10}$$

where W_{mn} are the random wavelet coefficients of the discrete wavelet transform, and $\{g_{mn}(t)\}$ is an orthonormal wavelet basis related to the mother wavelet $g(t)$ according to

$$g_{mn}(t) = 2^{m/2}\, g(2^m t - n). \tag{11}$$

Let the Fourier transform $G(\omega)$ of g be continuous at $\omega = 0$ and let $|G(\omega)|$ decay at least as fast as $1/\omega$. Let the random sequence W_{mn} be such that for arbitrary distinct pairs m' and m, $W_{m'n}$ and W_{mn} are uncorrelated sequences, and, for each fixed m, let the sequence W_{mn} be white with average power $2^{-\beta m}\sigma^2$. Then $n(t)$ is a nearly $1/f$ noise [10], i.e., it has measured spectra satisfying

$$\frac{k_1}{|\omega|^\beta} \le S(\omega) \le \frac{k_2}{|\omega|^\beta},$$

with $0 < k_1 \le k_2 < \infty$, and $\beta \in (0,2)$ a fixed parameter; this includes $1/f$ noise as a special case.

If the discrete form of the wavelet transform given in (10) and (11) is used for the detection of signals in the presence of nearly $1/f$ noise, then the reproducing kernel of (4) has the simple form

$$\tilde{K}(m,n;m',n') = \begin{cases} E\{W_{mn}\}\, E\{\overline{W_{m'n'}}\}, & m \ne m' \\ 2^{-\beta m}\sigma^2\, \delta_{n,n'}, & m = m'. \end{cases}$$

If the noise has zero mean then the mean of W_{mn} is also zero and we get the particularly simple form

$$\tilde{K}(m,n;m',n') = 2^{-\beta m}\sigma^2\, \delta_{n,n'}\delta_{m,m'}. \tag{12}$$

In practice it is necessary to limit the number of wavelet coefficients used to a finite number; we therefore consider only the range $M_1 \le m \le M_2$ and $N_1 \le n \le N_2$. Then $\tilde{K}^{-1}(m,n;m',n')$ exists, and the wavelet-based correlator is

$$\Lambda = \sum_{m=M_1}^{M_2}\sum_{n=N_1}^{N_2} Wr(m,n) \sum_{j=M_1}^{M_2}\sum_{k=N_1}^{N_2} \tilde{K}^{-1}(j,k;m,n)\,\overline{W\gamma(j,k)}.$$

Using (12), we can write this in a simpler form

$$\begin{aligned} \Lambda &= \sum_{m=M_1}^{M_2} 2^{\beta m}/\sigma^2 \sum_{n=N_1}^{N_2} Wr(m,n)\overline{W\gamma(m,n)} \\ &\triangleq \sum_{m=M_1}^{M_2} \left(2^{\beta m}/\sigma^2\right) \Lambda_m. \end{aligned}$$

This has a nice interpretation. At each scale m we form the statistic Λ_m by correlating $Wr(m,n)$ and $W\gamma(m,n)$ as if making a decision in unity variance white noise. The sufficient statistic Λ is then formed by fusing these "white-noise statistics" together via a linear combination with weights $2^{\beta m}/\sigma^2$.

V. CONCLUSION

We have shown that group transforms, such as the Gabor and wavelet transforms, can be used to transform the detection problem into an RKHS. The inner product of this RKHS can be used to express the sufficient statistic Λ, and can be interpreted as a correlator. In so doing, we have shown how to use the Gabor and wavelet transforms in the nonstationary noise case. An important advantage of this approach is that each of these transforms has an equivalent discrete version, which results in a discrete-parameter weighted correlator well-suited to implementation.

The use of the wavelet transform for detection was illustrated using two examples: (i) fractional Brownian Motion, and (ii) nearly $1/f$. For the latter case, the particularly simple form of the transformed reproducing kernel allowed the sufficient statistic to be formed using a linear combination of statistics computed at each scale as if the noise were white.

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