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OPTIMUM ARRAY PROCESSING AND REPRESENTATION OF NONSTATIONARY RANDOM SCATTERING

(Invited Paper)

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ABSTRACT

The purpose of this paper is to extend scattering function concepts to nonstationary and correlated scattering and use appropriate extensions to study optimum array processors in inhomogeneous Gaussian random noise fields. For this end we will review common functional representations of nonstationary stochastic processes and indicate how these functional representations can be used for analysis of nonstationary scattering. Then we will give expressions for optimum (maximum likelihood for Gaussian distributions) scalar and array processors in terms of Wigner distributions. Finally, we will use Moyal's formula to derive expressions for the detection indices in terms of Wigner distributions.

INTRODUCTION

Scattering functions and their equivalents have been used widely for characterization stochastic backscattering and multipath propagation media. [1,2] Utility of the scattering function approach for the design of optimum signals and signal processing systems has been demonstrated. [3,4] The scattering function approach in its usual applications assumes that the scattering process is a wide-sense stationary stochastic process and that the scattering process from different range intervals is uncorrelated. [1,2] In signal processing applications it is of particular interest to remove the wide-sense stationary scattering assumption and to extend the scattering function concept to nonstationary scattering processes. Since Wigner distributions are effective for characterization of nonstationary stochastic processes and more generally inhomogeneous random fields (multidimensional generalizations of nonstationary stochastic processes) it is of interest to characterize nonstationary scattering via Wigner distributions and their transforms. [5,6] Wigner distributions have many desirable properties and their application to characterization of nonstationary stochastic processes and transient signals is currently an active research topic. [7,8,9]

This paper will review common functional representations of nonstationary stochastic processes and indicate how these functional representations can be used for analysis of nonstationary scattering. Then we will give expressions for optimum (maximum likelihood for Gaussian distributions) [10] scalar and array processors in terms of Wigner distributions. Finally, we will use Moyal's formula to derive expressions for the detection indices in terms of Wigner distributions.[9]

FUNCTIONAL REPRESENTATION OF NONSTATIONARY PROCESSES AND INHOMOGENEOUS RANDOM FIELDS

Nonstationary processes require two parameters for their second order representation: two time parameters t,s for the nonstationary covariance function $R(t,s)$, time and frequency parameters t,λ for the Wigner distribution $W(t,\lambda)$, and two frequency parameters λ,μ for the two-dimensional spectral density $f(\lambda,\mu)$. 2-n parameters are required for analogous representations of inhomogeneous random fields in the n-dimensional parameter space. [5,6]

We will now discuss the interrelations between these three representations. The Wigner distribution (WD) fits between time-time and frequency-frequency representation.

Originally, Wigner used Wigner distribution as a phase-space description of quantum mechanical operators. [11] Its parameters were the conjugate variable of position and momentum. Ville recognized applicability of WD distribution to analysis of finite energy signals. [12] Application of WD to analysis of nonstationary processes is more recent. [5]

Since the primary goal of our work is investigation of optimum array processing structure, we present needed functional representations of inhomogeneous random fields. Random fields are random functions of four continuous parameters: time and three spatial coordinates. Nonstationary stochastic processes are one-dimensional special cases or inhomogeneous random fields. Functional representatives that arise in discrete array processing are also special cases of continuous parameter representation and are presented in this paper as examples.

Derivation of interrelations between various representations is facilitated by spectral representation of harmonizable inhomogeneous random fields:

$$x(t,\omega) = \frac{1}{(2\pi)^n} \int_{\Re^n} \exp[j\boldsymbol{\lambda} \cdot \boldsymbol{t}]\, dz(\boldsymbol{\lambda},\omega) \tag{1}$$

where

$$\boldsymbol{t} = [t, x_1, x_2, x_3]^T$$

$$\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3, \lambda_4]^T$$

$$\boldsymbol{\lambda} \cdot \boldsymbol{t} = [t\lambda_1, +x_1\lambda_2 + x_2\lambda_3 + x_3\lambda_4]$$

$$\omega \in \Omega \quad \text{(PROBABILITY SPACE)} \tag{2}$$

The covariance function of a harmonizable random field can be expressed in terms of 2n dimensional spectral representation and spectral density functions.

$$\begin{aligned} R(t,s) &= E\{x(t,\omega)x^*(s,\omega)\} \\ &= \frac{1}{(2\pi)^{2n}} \int_{\Re^{2n}} \exp[j(t\cdot\lambda - s\cdot\mu)]\, d_{\lambda,\mu}F(\lambda,\mu) \\ &= \frac{1}{(2\pi)^{2n}} \int_{\Re^{2n}} \exp[j(t\cdot\lambda - s\cdot\mu)]\, f(\lambda,\mu)\, d\lambda, d\mu \end{aligned} \tag{3}$$

The WD of a harmonizable random field is

$$W(t,\lambda) = \int_{\Re^n} \exp[-j\lambda\cdot\tau]\, R(t+\tau/2, t-\tau/2)\, d\tau \tag{4}$$

Martin [5] has shown that the WD can be calculated from the 2n-dimensional spectral density by

$$W(t,\lambda) = \frac{1}{(2\pi)^n} \int_{\Re^n} \exp[jt\cdot\mu]\, f(\lambda+\mu/2, \lambda-\mu/2)\, d\mu \tag{5}$$

the 2n-dimensional spectral density by the inverse transform

$$f(\lambda+\mu/2, \lambda-\mu/2) = \int_{\Re^n} \exp[-jt\cdot\mu]\, W(t,\lambda)\, dt \tag{6}$$

and the covariance function by

$$R(t+\tau/2, t-\tau/2) = \frac{1}{(2\pi)^n} \int_{\Re^n} \exp[j\lambda\cdot\tau]\, W(t,\lambda)\, d\lambda \tag{7}$$

Some of the important properties of the WD are that WD is always real, and it preserves the time support of a signal x(t), and frequency support of the Fourier transform of X(λ). [8] The WD is related to the ambiguity function by a double Fourier transform. We define the complex ambiguity function for a random field by

$$A(\nu,\tau) = \int_{\Re^n} R(t+\tau/2, t-\tau/2)\, \exp[-j\nu\cdot t]\, dt \tag{8}$$

The complex ambiguity function and Wigner distribution are related by double Fourier transforms

$$A(\nu,\tau) = \frac{1}{(2\pi)^n} \int_{\Re^{2n}} W(t,\lambda)\, \exp[j(\lambda\cdot\tau - \nu\cdot t)]\, dt\, d\lambda \tag{9}$$

and

$$W(t,\lambda) = \frac{1}{(2\pi)^n} \int_{\Re^{2n}} A(\nu,\tau)\, \exp[-j(\lambda\cdot\tau - \nu\cdot t)]\, d\tau d\nu \tag{10}$$

These equations are the desired double transforms that relate WD and ambiguity functions. WD is real and can be thought of as a density function, where the complex ambiguity function can be thought of as a characteristic function of the density function. These relations are important in establishing connections between WD on one hand and the ambiguity function and scattering function theory on the other hand. [1]

It is evident from this discussion that all four representations of nonstationary processes are isomorphic. They only differ in practical convenience and in the insight they give. Their interrelations are displayed in Figure 1. Wigner distributions clearly display the properties of the nonstationary properties on the time-frequency plane. Two important relations for the Wigner distributions are the expression for the inner product

$$\ell = \langle r(t), g(t)\rangle = \frac{1}{(2\pi)^n} \int_{\Re^{2n}} W_{rg}(t,\lambda)\, dt\, d\lambda \tag{11}$$

and Moyal's formula [9]:

$$\langle f_1, f_2\rangle \langle g_1, g_2\rangle^* = \frac{1}{(2\pi)^n} \int_{\Re^{2n}} W_{f_1g_1}(t,\lambda) W^*_{f_2g_2}(t,\lambda)\, dt\, d\lambda \tag{12}$$

An important special case of Moyal's formula is obtained for $f_1 = f_2 = g_1 = g_2 = f$.

$$\frac{1}{(2\pi)^n} \int_{\Re^{2n}} W^2_{ff}(t,\lambda)\, dt\, d\lambda = \| f \|^4 \tag{13}$$

The two-dimensional spectral density function can be used for quantitative determination of how nonstationary a harmonizable process is. [7] Distribution of $f(\lambda,\mu)$ in the $\lambda\mu$ plane is a measure of stationarity/non-stationarity of the process. Frequently the equal height contours of the central peak of $f(\lambda,\mu)$ is an approximate ellipse with its major axis along λ-μ line. A narrow ellipse indicates an almost wide-sense stationary process, where a broad ellipse indicates a highly nonstationary process. Ratio of major axis to the minor axis length is a quantitative measure of stationarity. Details of this technique have been discussed in the S. Carosso's M.S. thesis. [7] Furthermore, $f(\lambda,\mu)$ can be estimated from realizations of stochastic processing using computationally efficient FFT computations.

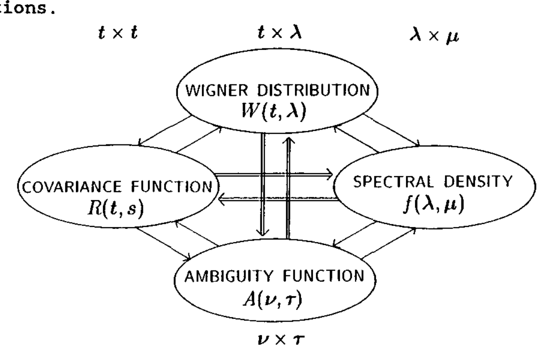


Figure 1. Bilinear representations of inhomogeneous random fields

REPRESENTATION OF NONSTATIONARY SCATTERING

The scattering function theory uses so-called wide sense stationary sense stationary uncorrelated scattering (WSSUS) assumption. Without WSSUS assumption a scalar scattering function requires four parameters for its representation. [1,2] For many signal processing applications it is of interest to avoid the WSSUS assumption. In this section we use linear compact operator theory [15] to investigate representation of nonstationary scattering by the functional representations that were presented in the previous section. The starting point for the characterization of nonstationary scattering is the time-varying impulse response representative of the scattering process:

$$y(t,\omega) = \int_{-\infty}^{\infty} H(t,s,\omega)\, f(s)\, ds = (Kf)(t) \tag{14}$$

where H(t,s,ω) is the time-varying random response at time t to the impulse applied at time s. [1] The integral operator K is usually not a self adjoint operator,but it is reasonable to assume that it is a compact operator (finite double norm). [15] Then the operator K can be expanded in terms of its singular value decomposition:

$$\begin{aligned} \sigma_k \phi_k &= K\Psi_k \\ \sigma_k \Psi_k &= K^{\dagger}\phi_k \end{aligned} \tag{15}$$

by

$$Kf = \sum_k \sigma_k(\omega)\, \langle f, \Psi_k \rangle\, \phi_k \tag{16}$$

where $K^{\dagger}$ is adjoint of K. The covariance of the scattered process can be represented by

$$\begin{aligned} R_{yy}(t,u) &= E\,\{y(t,\omega)\, y^*(u,\omega)\} \\ &= \sum_k \sum_\ell E\,\{\sigma_k(\omega)\,\sigma_\ell^*(\omega)\}\, \underbrace{\langle f, \Psi_k \rangle}_{f_u}\, \underbrace{\langle f, \Psi_\ell \rangle^*}_{f_\ell^*}\, \phi_k(f)\, \phi_\ell^*(u) \\ &= f^T H(t,u) f^* \end{aligned} \tag{17}$$

where

$$f^T = [f_1 \ldots f_k \ldots] \tag{18}$$

and the nonstationary scattering is represented by

$$H(t,u) = [E\,\{\sigma_k \sigma_\ell^*\}\, \phi_k(t)\, \phi_\ell^*(u)] \tag{19}$$

The Wigner distribution of the scattered process is:

$$\begin{aligned} W(t,\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{yy}\left(t + \frac{\tau}{2}, t - \frac{\tau}{2}\right) e^{-j\lambda\tau}\, d\tau \\ &= f^T W(t,\lambda) f^* \end{aligned} \tag{20}$$

where the elements of the matrix $W(t,\lambda)$ are

$$\begin{aligned} &E\,\{\sigma_k(\omega)\sigma_\ell(\omega)\}\; W_{\phi_k\phi_\ell}(t,\lambda) \\ &W_{\phi_k\phi_\ell}(t,\lambda) = \int_{-\infty}^{\infty} \phi_k\left(t + \frac{\tau}{2}\right) \phi_\ell^*\left(t - \frac{\tau}{2}\right) e^{-j\lambda\tau}\, d\tau \end{aligned} \tag{21}$$

Similarly, in terms of complex ambiguity function the representation of the scattered process is:

$$\begin{aligned} A(\nu,\tau) &= \int_{-\infty}^{\infty} R_{yy}\left(t + \frac{\tau}{2}, t - \frac{\tau}{2}\right) e^{-j\nu\tau}\, dt \\ &= f^T A(\nu,\tau) f^* \end{aligned} \tag{22}$$

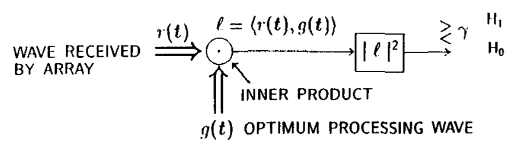
where elements of matrix $A(\nu,\tau)$ are

$$\begin{aligned} &E\,\{\sigma_k(\omega)\sigma_\ell(\omega)\}\; A_{\phi_k\phi_\ell}(\nu,\tau) \\ &A_{\phi_k\phi_\ell}(\nu,\tau) = \int_{-\infty}^{\infty} \phi_k\left(t + \frac{\tau}{2}\right) \phi_\ell^*\left(t - \frac{\tau}{2}\right) e^{-j\nu\tau}\, dt \end{aligned} \tag{23}$$

In all of these representation the scattering process is characterized by $E\,\{\sigma_k(\omega)\,\sigma_\ell^*(\omega)\}$ $\{\phi_k\}$ and $\{\Psi_k\}$. The set $\{f_k\}$ is calculated from known transmitted signal by

$$\begin{aligned} f_k = \langle f, \Psi_k \rangle &= \int_{-\infty}^{\infty} f(t)\, \Psi_k^*(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{f\Psi_k}(t,\lambda)\, dt\, d\lambda \end{aligned} \tag{24}$$

Effective measurement and characterization of inhomogeneous scattering is an important open research problem. We have outlined a possible approach to this problem.



$$\begin{aligned} r(t) &= K_1 f(t) + K_0 f(t) + n(t) \quad &: H_1 \\ r(t) &= K_0 f(t) + n(t) \quad &: H_0 \\ t &= [t, x_1, x_2, x_3]^T \end{aligned}$$

Figure 2. Array processor

OPTIMUM ARRAY PROCESSOR IN TERMS OF WIGNER DISTRIBUTION

The array processor for maximum likelihood detection of doubly spread (range and Doppler spread) Gaussian scatterers in the Gaussian noise is shown in Figure 2. Operator K_1 represents desired scattering and operator K_0 represents undesired scattering such as clutter or reverberation. Optimum receiver computes the inner product:

$$\ell = \langle r(t), g(t) \rangle = \int_{\Re^n} r(t) g^*(t) dt = \frac{1}{(2\pi)^n} \int_{\Re^{2n}} W_{rg}(t, \lambda)\, dt\, d\lambda \quad (25)$$

This relation has an interesting interpretation. Receiver computes the integral of the cross Wigner distribution over all time, space and frequencies on which the $r(\underline{t})$ and $g(\underline{t})$ exist. [9] Since Wigner distribution conserves the time and space support of signals and frequency support of their Fourier transforms integration needs to be only over appropriate support regions. [8,9] Thus, the computation of the log likelihood function ℓ uses all the "information" on time/space frequency plane of the Wigner distribution.

In the case of a discrete array of points sensors that are located at the spatial coordinates $\underline{x}_i$, the processor computes:

$$\begin{aligned} \langle r(t), \tilde{g}(t) \rangle &= \langle r(t), \sum_i^M g \begin{pmatrix} t \\ x_1 \end{pmatrix} \delta(x - x_i) \rangle \\ &= \int_{-\infty}^{\infty} \sum_i^M r_i(t) g_i^* \, dt \\ &= \int_{-\infty}^{\infty} R(t) \cdot g(t)\, dt \\ &= \frac{1}{2\pi} \sum_i^M \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W r_i g_i (t, \lambda)\, dt\, d\lambda \\ &= \sum_i^M \ell_i \end{aligned} \quad (26)$$

Note that the derivation of above array processor structure required no specific assumptions as to the array geometry or plane wave propagation. Summation of the inner products computed by each sensor processor can be thought of as a method for optimal fusion of outputs of arbitrarily distributed sensor processors. As it can be seen from an example the array processor structure for plane wave propagation can be simplified.

The probability of detection P_D of zero mean Gaussian signal in uncorrelated (signal and noise are mutually uncorrelated) Gaussian noise is given by performance measure Δ and false alarm probability P_F by

$$P_D = P_F^{1/(1+\Delta)} \quad (27)$$

where the array performance measure Δ (SIR) is given by

$$\Delta = \frac{E\{|\ell|^2/H_1\} - E\{|\ell|^2/H_0\}}{E\{|\ell|^2/H_0\}} \quad (28)$$

Terms in equation (28) can be computed by the Moyal's formula

$$\begin{aligned} |\ell|^2 &= \sum_{ij} \ell_i \ell_j = \sum_{ij} \langle r_i(t), g_i(t) \rangle \langle r_j(t), g_j(t) \rangle^* \\ &= \frac{1}{2\pi} \sum_{ij} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{r_i r_j}(t, \lambda)\, W^*_{g_i g_j}(t, \lambda)\, dt\, d\lambda \end{aligned} \quad (29)$$

To obtain insight into performance of the array processor let us consider a simple plane wave propagation example. Let

$$\begin{aligned} &H_0 : r_i(t) = n_i(t) \\ &H_1 : r_i(t) = b(\omega)\, s(t - \tau_i) + n_i \qquad E\{b(\omega)\} = 0 \\ &g_i(t) = s(t - \tau_i) \qquad E\{|b(\omega)|^2\} = \overline{E} \end{aligned} \quad (30)$$

Then the performance measure becomes

$$\Delta = \frac{M^2 \overline{E} \int \int_{-\infty}^{\infty} W_{ss}^2(t, \lambda)\, dt\, d\lambda \Longrightarrow M^2 \overline{E} \, \| s \|^4 \, 2\pi}{\sum_{ij} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\lambda(\tau_j - \tau_i)} E\{W_{n_i n_j}(t, \lambda)\}\, W_{ss}\left(t + \frac{\tau_i - \tau_j}{2}, \lambda\right) dt\, d\lambda} \quad (31)$$

Now let the noise be a sum of white noise that is uncorrelated between sensors and a correlated colored noise:

$$n(t) = n_0(t) + c(t) \quad (32)$$

where

$$R_{nn}(t, s) = N_0 I \, \delta(t - s) + R_{cc}(t, s) \quad (33)$$

then the Wigner distribution of noise is

$$W_{nn}(t, \lambda) = N_0 I + W_{cc}(t, \lambda) \quad (34)$$

Now the numerator of Δ becomes

$$2\pi N_0 M \, \| s \|^2 + \sum_{i,j} e^{j\lambda(\tau_j - \tau_i)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{c_i c_j}(t, \lambda)\, W_{ss}\left(t + \frac{\tau_i - \tau_j}{2}, \lambda\right) dt\, d\lambda \quad (35)$$

If the noise term $\underline{c}(t)$ is generated by the undesired backscattering of the transmitted signal, then the Equation (35) is analogous to the noise and interference terms of Δ that are obtained for WSSUS scattering. [1-4] If the colored and correlated (sensor to sensor) noise component vanishes then, as expected, the performance measure becomes the usual signal to noise ratio at the array output.

$$\Delta = M \frac{\overline{E} \, \| s \|^2}{N_0} \quad (36)$$

Extension of the above results to optimum array processor and signal design is our present research topic.

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