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EECE 301
Signals & Systems
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Note Set #34

- C-T Transfer Function and Frequency Response

Finding the Transfer Function from Differential Eq.

Recall: we found a DT system's Transfer Function $H(z)$ by taking the ZT of the Difference Eq. For a CT system we do the same kind if thing by taking the LT of the Differential Eq:

$$a_N y^{(N)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = b_M x^{(M)}(t) + \ldots + b_1 \dot{x}(t) + b_0 x(t)$$

ZT

$$LT\left\{a_N y^{(N)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t)\right\} = LT\left\{b_M x^{(M)}(t) + \ldots + b_1 \dot{x}(t) + b_0 x(t)\right\}$$

Diff. Prop. & Algebra

$$Y(s)\left[a_N s^N + \cdots + a_1 s + a_0\right] = X(s)\left[b_M s^M + \ldots + b_1 s + b_0\right]$$

Algebra

$$Y(s) = \underbrace{\left[\frac{b_M s^M + \ldots + b_1 s + b_0}{a_N s^N + \cdots + a_1 s + a_0}\right]}_{H(s)} X(s)$$

So… can just write $H(s)$ by inspection of D.E. coefficients!

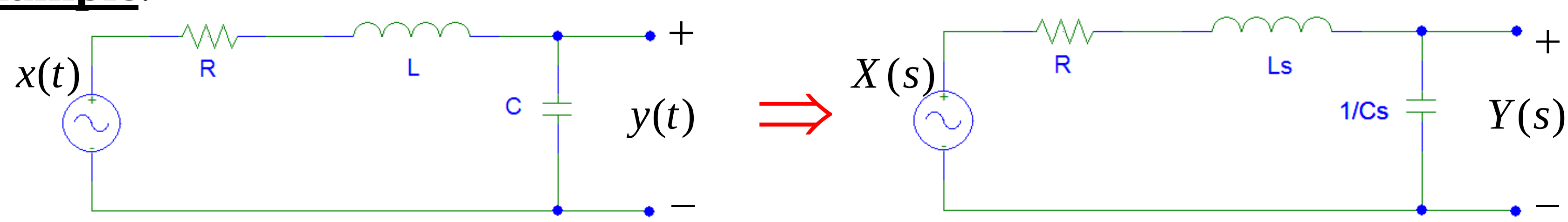
<u>Finding the Transfer Function from the Circuit</u>

Can find the freq. resp. of a circuit using impedance in terms of $j\omega$.

Now generalize: to find the TF… the complex variable s replaces $j\omega$:

- Use <u>s-variable impedances</u>: $Z_C(s) = 1/Cs \qquad Z_L(s) = Ls$
- Replace input source time function symbol $x(t)$ by LT symbol $X(s)$
- Analyze circuit to find $Y(s)$… **thing that multiplies $X(s)$ is T.F. $H(s)$**

<u>Example</u>:



For this circuit the easiest approach is to use the <u>Voltage Divider</u>

$$Y(s) = \left[\frac{1/Cs}{R + Ls + (1/Cs)}\right] X(s) \quad \Rightarrow \quad = \underbrace{\left[\frac{1/LC}{s^2 + (R/L)s + (1/LC)}\right]}_{H(s)} X(s)$$

A "standard" form: a ratio of two polynomials in s
- w/ unity coeff. on the highest power in the denom.

Poles and Zeros of Transfer Function

$$H(s) = \frac{b_M s^M + \ldots + b_1 s + b_0}{s^N + \cdots + a_1 s + a_0} = \frac{B(s)}{A(s)}$$

Can always normalize this coeff. to 1

Assume there are no common roots in the numerator $B(s)$ and denominator $A(s)$.
(If not, assume they've been cancelled and redefine $B(s)$ and $A(s)$ accordingly)

Poles of $H(s)$: The values on the complex s-plane where $|H(s)| \to \infty$

Zeros of $H(s)$: The values on the complex s-plane where $|H(s)| = 0$

The roots of the **denominator** polynomial $A(s)$ determine N poles.

The roots of the **Numerator** polynomial $B(s)$ determine M zeros.

$$H(s) = \frac{b_M (s - z_1)(s - z_2)\ldots(s - z_M)}{(s - p_1)(s - p_2)\ldots(s - p_N)}$$

Example: Finding Poles and Zeros

$$\dddot{y}(t)+6\ddot{y}(t)+10\dot{y}(t)+8y(t)=2\ddot{x}(t)+12\dot{x}(t)+20x(t)$$

$$H(s)=\frac{2s^2+12s+20}{s^3+6s^2+10s+8}=\frac{2(s+3-j)(s+3+j)}{(s+4)(s+1-j)(s+1+j)}$$

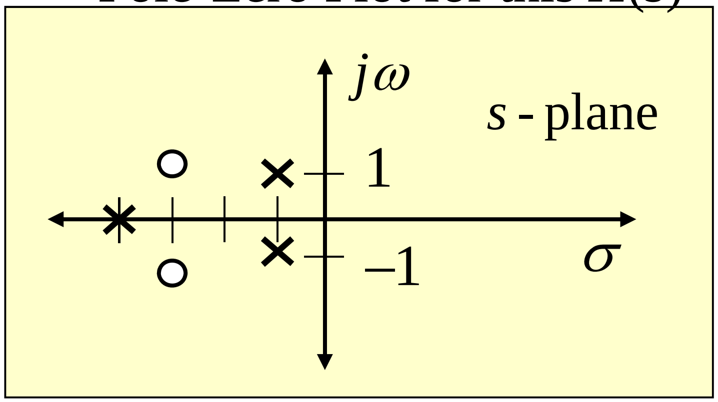
zeros: $s=-3\pm j$ (Conjugate Pair)

poles: $s=-4,\ \ s=-1\pm j$ (Conjugate Pair)

Using MATLAB:

```
>>  roots([2 12 20])
>> roots([1 6 10 8])
```

Pole-Zero Plot for this $H(s)$

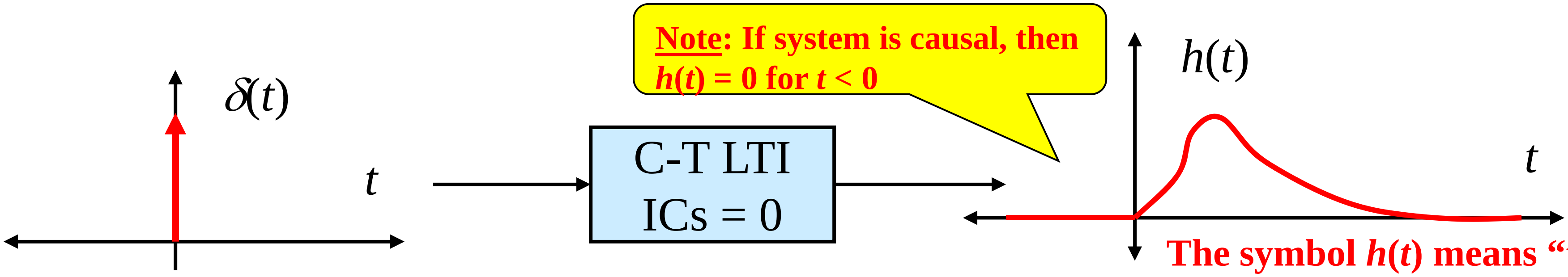


x denotes a pole

o denotes a zero

Impulse Response of System

Sometimes looking at how a system responds to the impulse function (i.e., delta function) $\delta(t)$ can tell much about a system. Hitting a system with $\delta(t)$ is lot like ringing a bell to hear how it sounds…



The symbol $h(t)$ means "the impulse response".

Noting that the LT of $\delta(t) = 1$ and using the properties of the transfer function and the LT transform:

$$h(t) = \mathscr{L}^{-1}\left\{H(s)\mathscr{L}\left\{\delta(t)\right\}\right\}$$

$$\boxed{h(t) = \mathscr{L}^{-1}\left\{H(s)\right\}}$$

$$\boxed{h(t) = \mathscr{F}^{-1}\left\{H(\omega)\right\}}$$

From PFE and Poles/Zeros we see that a TF like this: $H(s) = \dfrac{B(s)}{A(s)}$

…will have an impulse response with terms like this:

$$\boxed{h(t) = k_1 e^{p_1 t}u(t) + k_2 e^{p_2 t}u(t) + \cdots + k_N e^{p_N t}u(t)}$$

Some simplifying assumptions made here!

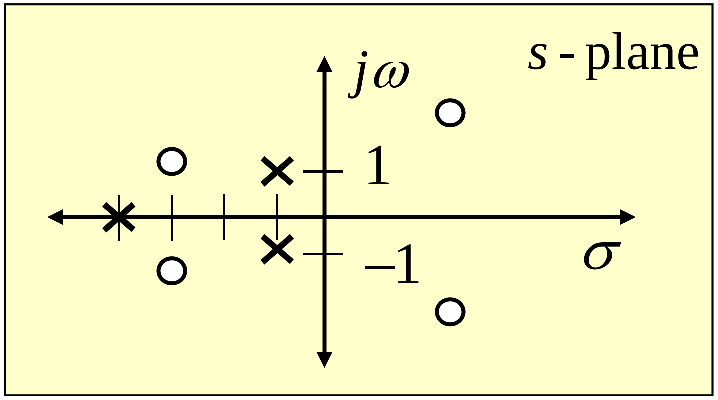
We almost always want this to decay (like a bell!): all poles Re{p_i} < 0

Stability of System

Definition: A system is said to be **stable** if its output will never grow without bound when any bounded input signal is applied… and that seems like a good thing!!!

Without going into all the details… a system with an impulse response that decays "fast enough" is said to be stable.

From our exploration of the effect of poles on the impulse response we say that:



For a Stable System

- Poles must be "in Left Half Plane"
- Zeros can be anywhere

RLC circuits are always stable

But…Once you start including linear amplifiers with gain > 1 this may not be true… especially if there is feedback involved

Relationship: Transfer Function and Freq. Resp.

Recall: CTFT = LT evaluated on $j\omega$ axis… if $j\omega$ axis is inside ROC

Fact: For causal systems $j\omega$ axis is inside ROC if all poles are in LHP

$$H(\omega) = H(s)\big|_{s=j\omega}$$

If all poles are in LHP

Like freqz for DT frequency response…the MATLAB command freqs can be used to compute the Frequency Response from the Transfer Function coefficients:

$$H(s) = \frac{b_M s^M + \ldots + b_1 s + b_0}{a_N s^N + \cdots + a_1 s + a_0}$$

>> num = $[b_M \ \ldots \ b_1 \ b_0]$ ← must put any zero b_i into the vector

>> den = $[a_N \ \ldots \ a_1 \ a_0]$ ← must put any zero a_i into the vector

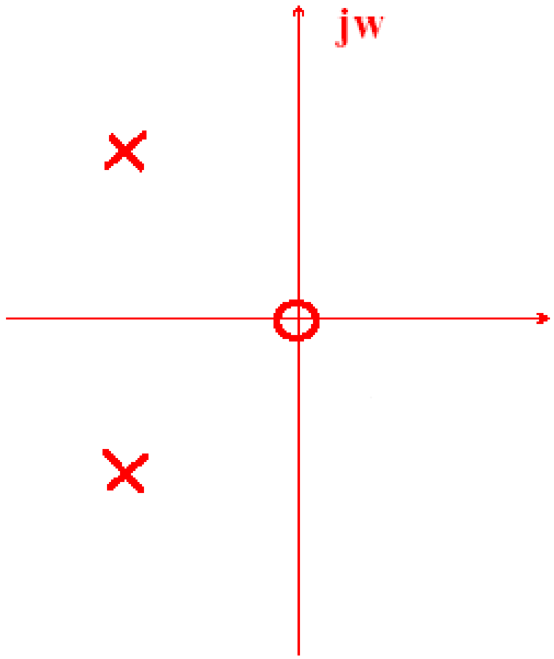
>> omega = -w_max: ?:w_max (Pick appropriate spacing)

>> H = freqs(num, denom, omega)

>> plot(omega, abs(H))

>> plot(omega, angle(H))

From the Pole-Zero Plots we can Visualize the TF function on the s-plane:



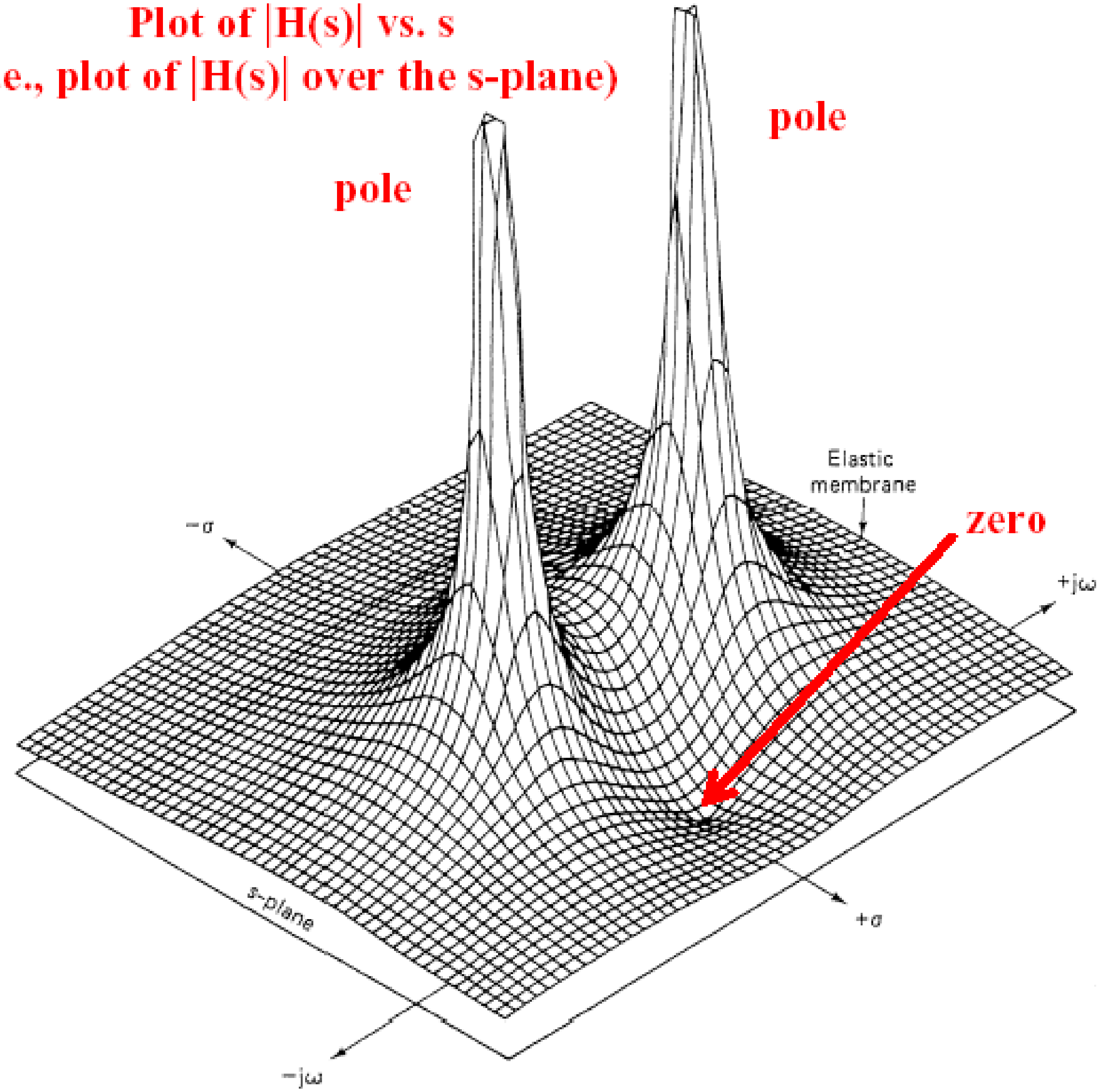


Fig. 9-4. Shape of elastic membrane for a pair of poles and a zero.

From our Visualization of the TF function on the s-plane we can see the Freq. Resp.:

To get the frequency response H(w) from the transfer function H(s) we replace s by jw... this is graphically equivalent to "cutting along the jw axis"

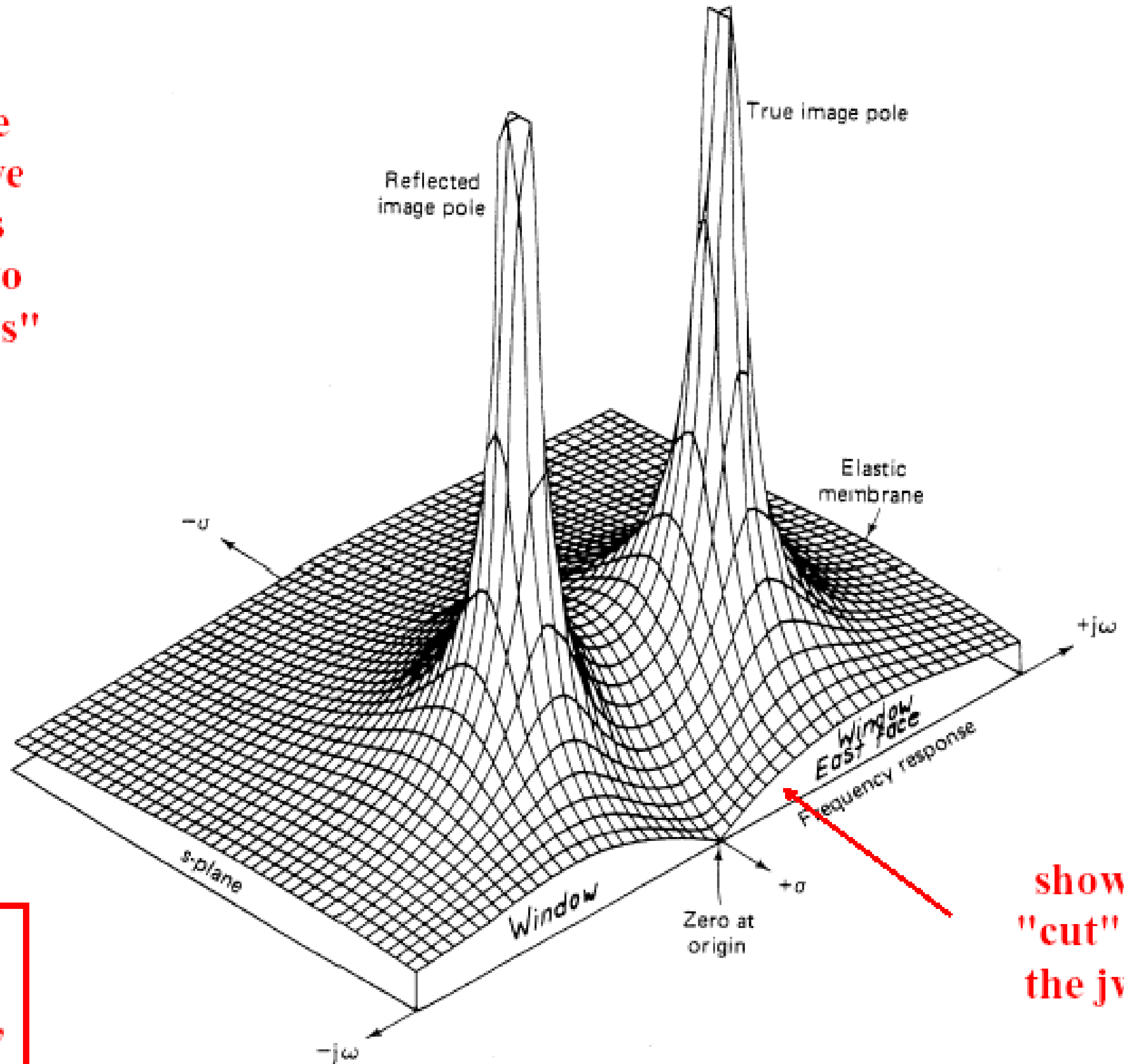


Fig. 9-5. Pole-zero diagram showing the east face.

shows the "cut" along the jw axis

Plots from my favorite book on Op Amps: "Operational Amplifier: Characteristics and Applications" by Robert G. Irvine, Prentice-Hall, 1981

Can also look at a pole-zero plot and see the effects on Freq. Resp.

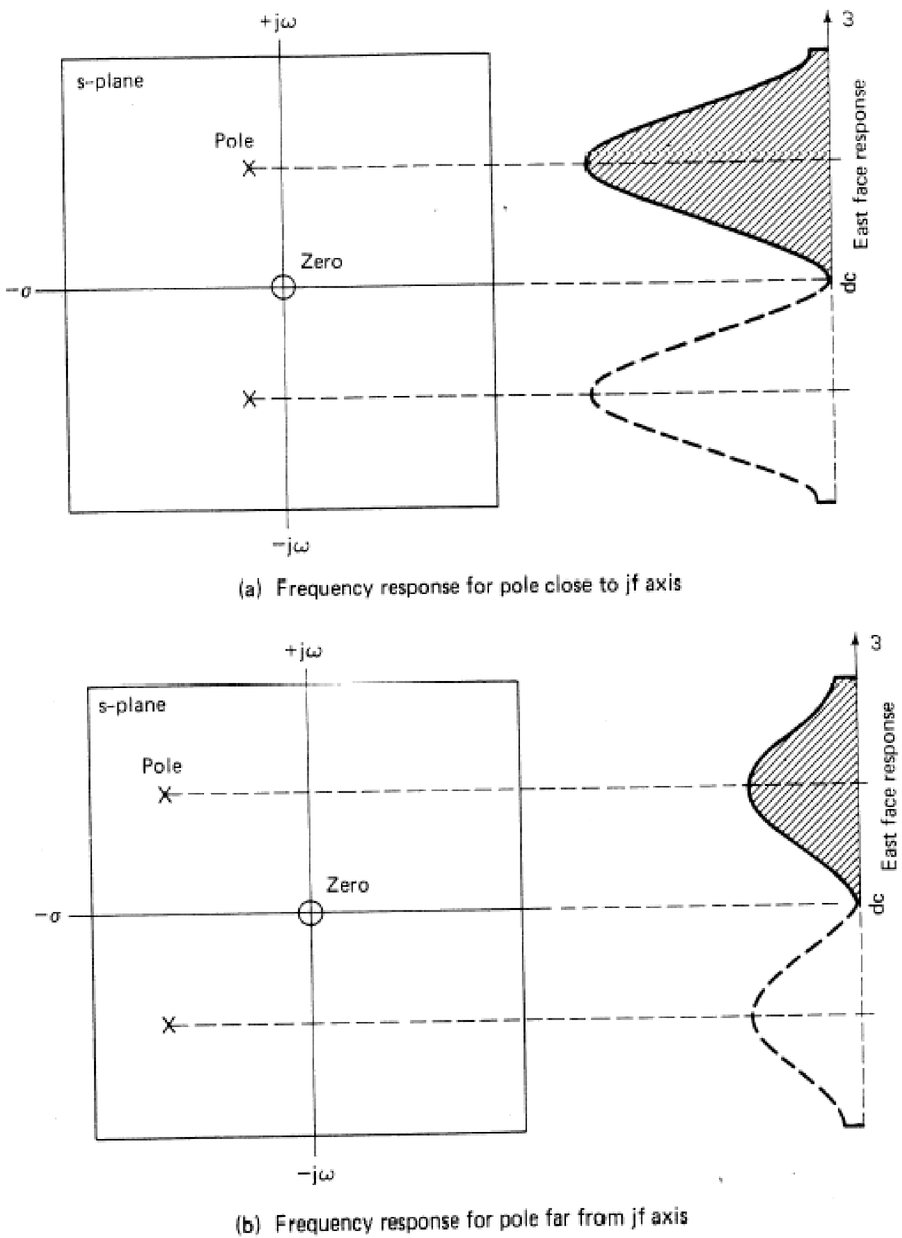


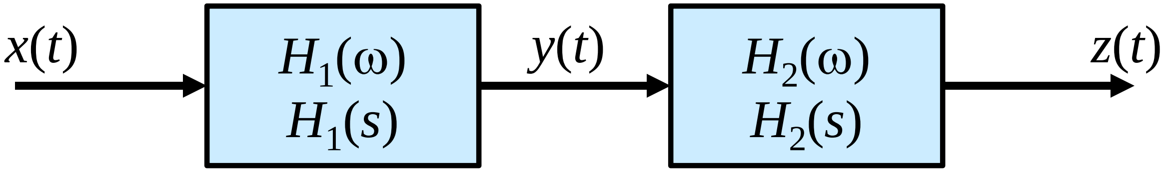
Fig. 9-6. Frequency response versus pole location.

As the pole moves closer to the $j\omega$ axis it has a stronger effect on the frequency response $H(\omega)$. Poles close to the $j\omega$ axis will yield sharper and taller bumps in the frequency response.

By being able to visualize what $|H(s)|$ will look like based on where the poles and zeros are, an engineer gains the ability to know what kind of transfer function is needed to achieve a desired frequency response… then through accumulated knowledge of electronic circuits (requires experience accumulated AFTER graduation) the engineer can devise a circuit that will achieve the desired effect.

Cascade of Systems

Suppose you have a "cascade" of two systems like this:



$$Y(\omega) = H_1(\omega)X(\omega) \qquad Z(\omega) = H_2(\omega)Y(\omega)$$
$$Y(s) = H_1(s)X(s) \qquad Z(s) = H_2(s)Y(s)$$

$$Z(\omega) = H_2(\omega)H_1(\omega)X(\omega)$$
$$Z(s) = H_2(s)H_1(s)X(s)$$

Thus, the ***overall*** frequency response/transfer function is the product of those of each stage:

$$H_{total}(\omega) = H_1(\omega)H_2(\omega)$$
$$H_{total}(s) = H_1(s)H_2(s)$$

Obviously, this generalizes to a cascade of N systems:

$$H_{total}(\omega) = H_1(\omega)H_2(\omega)\cdots H_N(\omega)$$
$$H_{total}(s) = H_1(s)H_2(s)\cdots H_N(s)$$

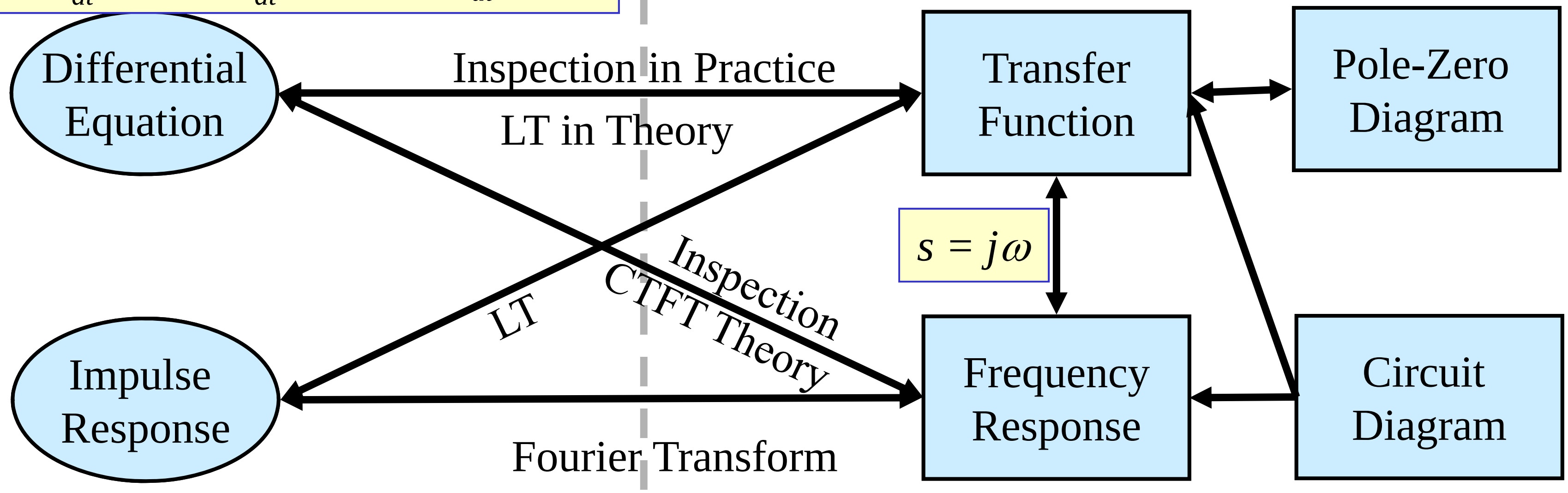
Continuous-Time System Relationships

Time Domain

$$\frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_M\frac{d^M x(t)}{dt^M} + b_{M-1}\frac{d^{M-1}x(t)}{dt^{M-1}} + \cdots + b_1\frac{dx(t)}{dt} + b_0 x(t)$$

Freq Domain

$$H(s) = \frac{b_M s^M + b_{M-1}s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}$$



$h(t)$

$$H(\omega) = H(s)\big|_{s=j\omega}$$

This Chart provides a "Roadmap" to the CT System Relationships!!!

In practice you may need to start your work in any spot on this diagram…

1. From the differential equation you can get:
   a. Transfer function, then the impulse response, the pole-zero plot, and if allowable you can get the frequency response
2. From the impulse response you can get:
   a. Transfer function, then the Diff. Eq., the pole-zero plot, and if allowable you can get the frequency response
3. From the Transfer Function you can get:
   a. Diff. Eq., the impulse response, the pole-zero plot, and if allowable you can get the frequency response
4. From the Frequency Response you can get:
   a. Transfer function, then the Diff. Eq., the pole-zero plot, and the impulse response
5. From the Pole-Zero Plot you can get:
   a. (up to a scaling factor) Transfer function, then the Diff. Eq., the impulse response, and possible the Frequency Response
6. From the Circuit Diagram you can get:
   a. Transfer function, then the Diff. Eq., the impulse response, and possibly the Frequency Response