EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #34
• C-T Transfer Function and Frequency Response
Finding the Transfer Function from Differential Eq.

Recall: we found a DT system’s Transfer Function $H(z)$ by taking the ZT of the Difference Eq. For a CT system we do the same kind of thing by taking the LT of the Differential Eq:

$$a_N y^{(N)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = b_M x^{(M)}(t) + \cdots + b_1 \dot{x}(t) + b_0 x(t)$$

\[
\begin{align*}
\text{LT} \{ a_N y^{(N)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) \} &= \text{LT} \{ b_M x^{(M)}(t) + \cdots + b_1 \dot{x}(t) + b_0 x(t) \} \\
Y(s) \left[ a_N s^N + \cdots + a_1 s + a_0 \right] &= X(s) \left[ b_M s^M + \cdots + b_1 s + b_0 \right] \\
Y(s) = \frac{b_M s^M + \cdots + b_1 s + b_0}{a_N s^N + \cdots + a_1 s + a_0} H(s) &= X(s)
\end{align*}
\]

So… can just write $H(s)$ by inspection of D.E. coefficients!
Finding the Transfer Function from the Circuit

Can find the freq. resp. of a circuit using impedance in terms of \( j\omega \).

Now generalize: to find the TF… the complex variable \( s \) replaces \( j\omega \):

- Use \( s \)-variable impedances: \( Z_C(s) = 1/Cs \quad Z_L(s) = Ls \)
- Replace input source time function symbol \( x(t) \) by LT symbol \( X(s) \)
- Analyze circuit to find \( Y(s) \)... thing that multiplies \( X(s) \) is T.F. \( H(s) \)

**Example:**

\[
Y(s) = \left[ \frac{1/Cs}{R + Ls + (1/Cs)} \right] X(s) \quad \Rightarrow \quad H(s) = \left[ \frac{1/LC}{s^2 + (R/L)s + (1/LC)} \right] X(s)
\]

A “standard” form: a ratio of two polynomials in \( s \)
- w/ unity coeff. on the highest power in the denom.
Poles and Zeros of Transfer Function

\[ H(s) = \frac{b_M s^M + ... + b_1 s + b_0}{s^N + \cdots + a_1 s + a_0} = \frac{B(s)}{A(s)} \]

Assume there are no common roots in the numerator \(B(s)\) and denominator \(A(s)\).
(If not, assume they’ve been cancelled and redefine \(B(s)\) and \(A(s)\) accordingly)

**Poles of \(H(s)\):** The values on the complex s-plane where \(|H(s)| \to \infty\)

**Zeros of \(H(s)\):** The values on the complex s-plane where \(|H(s)| = 0\)

The roots of the **denominator** polynomial \(A(s)\) determine \(N\) poles.

The roots of the **Numerator** polynomial \(B(s)\) determine \(M\) zeros.

\[ H(s) = \frac{b_M(s-z_1)(s-z_2)\ldots(s-z_M)}{(s-p_1)(s-p_2)\ldots(s-p_N)} \]
Example: Finding Poles and Zeros

\[ \ddot{y}(t) + 6\dot{y}(t) + 10\dot{y}(t) + 8y(t) = 2\dot{x}(t) + 12\dot{x}(t) + 20x(t) \]

\[ H(s) = \frac{2s^2 + 12s + 20}{s^3 + 6s^2 + 10s + 8} = \frac{2(s + 3 - j)(s + 3 + j)}{(s + 4)(s + 1 - j)(s + 1 + j)} \]

zeros: \( s = -3 \pm j \)
poles: \( s = -4, \ s = -1 \pm j \)

Using MATLAB:
>>> roots([2 12 20])
>>> roots([1 6 10 8])

Pole-Zero Plot for this \( H(s) \)

x denotes a pole
o denotes a zero
**Impulse Response of System**

Sometimes looking at how a system responds to the impulse function (i.e., delta function) $\delta(t)$ can tell much about a system. Hitting a system with $\delta(t)$ is lot like ringing a bell to hear how it sounds…

Noting that the LT of $\delta(t) = 1$ and using the properties of the transfer function and the LT transform:

$$h(t) = \mathcal{L}^{-1}\{H(s)\mathcal{L}\{\delta(t)\}\}$$

From PFE and Poles/Zeros we see that a TF like this:

...will have an impulse response with terms like this:

$$h(t) = k_1e^{p_1t}u(t) + k_2e^{p_2t}u(t) + \cdots + k_Ne^{p_Nt}u(t)$$

We almost always want this to decay (like a bell!): all poles $\text{Re}\{p_i\} < 0$

**Note:** If system is causal, then $h(t) = 0$ for $t < 0$
Stability of System

Definition: A system is said to be **stable** if its output will never grow without bound when any bounded input signal is applied... and that seems like a good thing!!!

Without going into all the details... a system with an impulse response that decays “fast enough” is said to be stable.

From our exploration of the effect of poles on the impulse response we say that:

- **For a Stable System**
  - Poles must be “in Left Half Plane”
  - Zeros can be anywhere

RC circuits are always stable

But…Once you start including linear amplifiers with gain > 1 this may not be true… especially if there is feedback involved
Relationship: Transfer Function and Freq. Resp.

Recall: CTFT = LT evaluated on \( j\omega \) axis… if \( j\omega \) axis is inside ROC

**Fact:** For causal systems \( j\omega \) axis is inside ROC if all poles are in LHP

\[
H(\omega) = H(s)\bigg|_{s=j\omega}
\]

If all poles are in LHP

Like freqz for DT frequency response…the MATLAB command freqs can be used to compute the Frequency Response from the Transfer Function coefficients:

\[
H(s) = \frac{b_M s^M + \ldots + b_1 s + b_0}{a_N s^N + \ldots + a_1 s + a_0}
\]

\[
\gg \text{num} = [b_M \ldots b_1 b_0]
\]

\[
\gg \text{den} = [a_N \ldots a_1 a_0]
\]

\[
\gg \omega = -\text{w}\_\text{max}: ?:\text{w}\_\text{max}
\]

\[
\gg H = \text{freqs(num, den, omega)}
\]

\[
\gg \text{plot(omega, abs(H))}
\]

\[
\gg \text{plot(omega, angle(H))}
\]
From the Pole-Zero Plots we can Visualize the TF function on the s-plane:

Plot of $|H(s)|$ vs. $s$
(i.e., plot of $|H(s)|$ over the s-plane)

Fig. 9-4. Shape of elastic membrane for a pair of poles and a zero.
From our Visualization of the TF function on the s-plane we can see the Freq. Resp.:

To get the frequency response $H(w)$ from the transfer function $H(s)$ we replace $s$ by $jw$... this is graphically equivalent to "cutting along the $jw$ axis"


Fig. 9-5. Pole-zero diagram showing the east face.
As the pole moves closer to the $j\omega$ axis it has a stronger effect on the frequency response $H(\omega)$. Poles close to the $j\omega$ axis will yield sharper and taller bumps in the frequency response.

By being able to visualize what $|H(s)|$ will look like based on where the poles and zeros are, an engineer gains the ability to know what kind of transfer function is needed to achieve a desired frequency response... then through accumulated knowledge of electronic circuits (requires experience accumulated AFTER graduation) the engineer can devise a circuit that will achieve the desired effect.
Cascade of Systems

Suppose you have a “cascade” of two systems like this:

\[
\begin{align*}
Y(\omega) &= H_1(\omega)X(\omega) \\
Y(s) &= H_1(s)X(s) \\
Z(\omega) &= H_2(\omega)Y(\omega) \\
Z(s) &= H_2(s)Y(s) \\
Z(\omega) &= H_2(\omega)H_1(\omega)X(\omega) \\
Z(s) &= H_2(s)H_1(s)X(s)
\end{align*}
\]

Thus, the overall frequency response/transfer function is the product of those of each stage:

\[
\begin{align*}
H_{total}(\omega) &= H_1(\omega)H_2(\omega) \\
H_{total}(s) &= H_1(s)H_2(s)
\end{align*}
\]

Obviously, this generalizes to a cascade of \(N\) systems:

\[
\begin{align*}
H_{total}(\omega) &= H_1(\omega)H_2(\omega) \cdots H_N(\omega) \\
H_{total}(s) &= H_1(s)H_2(s) \cdots H_N(s)
\end{align*}
\]
Continuous-Time System Relationships

**Time Domain**

\[
\frac{d^Ny(t)}{dt^N} + a_{N-1} \frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = \\
b_M \frac{d^Mx(t)}{dt^M} + b_{M-1} \frac{d^{M-1}x(t)}{dt^{M-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)
\]

**Freq Domain**

\[
H(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}
\]

This Chart provides a “Roadmap” to the CT System Relationships!!!
In practice you may need to start your work in any spot on this diagram…

1. From the differential equation you can get:
   a. Transfer function, then the impulse response, the pole-zero plot, and if allowable you can get the frequency response

2. From the impulse response you can get:
   a. Transfer function, then the Diff. Eq., the pole-zero plot, and if allowable you can get the frequency response

3. From the Transfer Function you can get:
   a. Diff. Eq., the impulse response, the pole-zero plot, and if allowable you can get the frequency response

4. From the Frequency Response you can get:
   a. Transfer function, then the Diff. Eq., the pole-zero plot, and the impulse response

5. From the Pole-Zero Plot you can get:
   a. (up to a scaling factor) Transfer function, then the Diff. Eq., the impulse response, and possible the Frequency Response

6. From the Circuit Diagram you can get:
   a. Transfer function, then the Diff. Eq., the impulse response, and possibly the Frequency Response