BINGHAMTON UNIVERSITY

State University of New York

# EECE 301
# Signals & Systems
# Prof. Mark Fowler

## Note Set #33

- C-T Systems: Laplace Transform… "Power Tool" for system analysis

# Laplace Transform & C-T Systems

Like the Z Transform for the DT case… The Laplace Transform is a powerful tool for the analysis and design of CT LTI Systems

L-T is used to

Solve differential equations with non-zero initial conditions

We'll do this later

Characterize systems using the "Transfer Function"

Our initial focus is here

# Laplace Transform Definition

Given a C-T signal $x(t)$ $-\infty < t < \infty$ we've already seen how to use the CTFT:

$$CTFT: X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Unfortunately the CTFT doesn't "converge" for some signals… the LT mitigates this problem by including decay in the transform:

$$e^{-j\omega t} \quad vs. \quad e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t}e^{-j\omega t}$$

Controls decay of integrand

For the Laplace Transform we use: $s = \sigma + j\omega$. So… $s$ is just a complex variable that we almost always view in rectangular form

Recall that for ZT we kept $z$ in polar form!

$$CTFT: X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \Rightarrow LT: X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

There are 2 types of LT: → Two-sided (bilateral) / One-sided (unilateral)

**Two-Sided LT**

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \qquad \text{with} \quad s = \underbrace{\sigma + j\omega}_{\text{complex variable}}$$

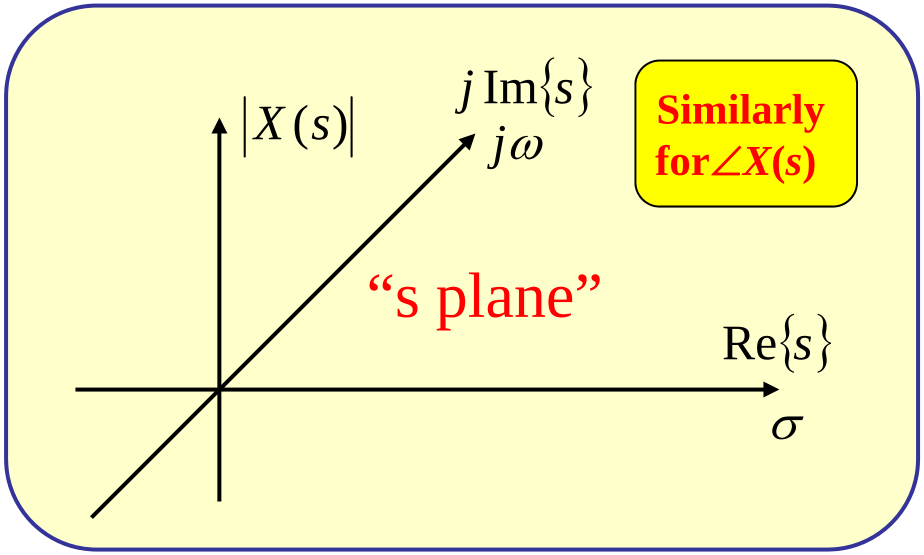
**One-Sided LT**

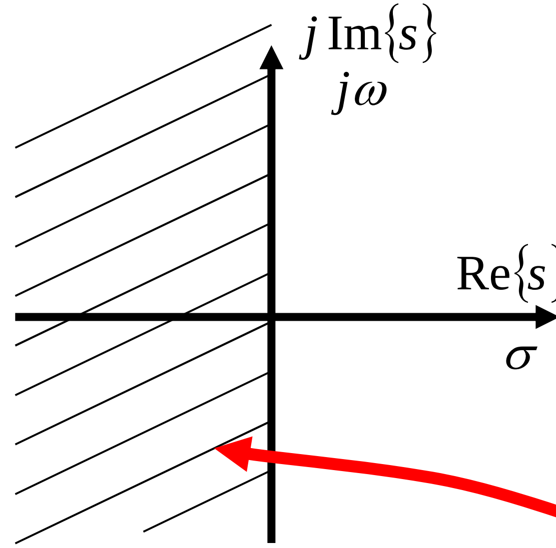
$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt \qquad \text{with} \quad s = \overbrace{\sigma + j\omega}^{\text{complex variable}}$$

One-sided LT defined this way → even if $x(t) \neq 0$, $t < 0$

Note that $X(s)$ is: $\begin{cases} \text{a complex valued function} \\ \text{of a complex variable } s = \sigma + j\omega \end{cases}$

Must plot on a plane… the "s-plane"




Recall that for the ZT we often needed to keep track if $z$ values were inside the UC… so polar form of $z$ was convenient. Inside UC when $|z| < 1$.

For the LT we will often need to keep track if $s$ values are in the Left Half Plane (LHP)… so rectangular form of $s$ is convenient. In LHP if $\text{Re}\{s\} < 0$.

**<u>Region of Convergence (ROC)</u>**

Set of all $s$ values for which the integral in the LT definition converges.

Each signal has its own region of convergence.

## Example of Finding a LT

Consider the signal $x(t) = e^{-bt}u(t) \quad b \in \Re$

Back when we studied the FT we had to limit $b$ to being $b > 0$… with the LT we don't need to restrict that!!!

This is a causal signal.

By definition of the LT:

$$X(s) = \int_0^\infty e^{-bt}e^{-st}dt = \int_0^\infty e^{-(s+b)t}dt$$

This is an easy integral to do!!

The limit is here by the definition of the integral

$$X(s) = \frac{-1}{s+b}\left[e^{-(s+b)t}\right]_{t=0}^{t=\infty} = \frac{-1}{s+b}\left[\underbrace{\lim_{t\to\infty} e^{-(s+b)t}}_{\text{look at this}} - 1\right]$$

**look at this**

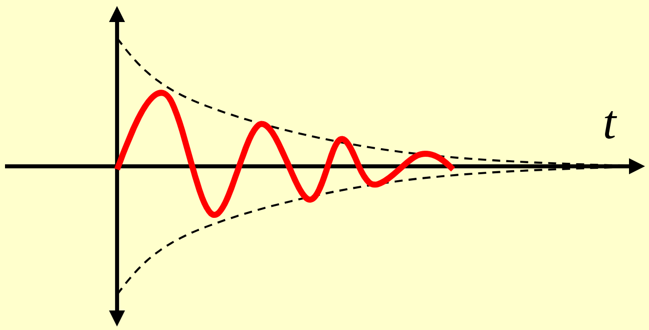
**If this limit does not converge… then we say that the integral "does not exist"**

**So… we need to find out under what conditions this integral exists.**

**So… let's look at the function inside this limit…**

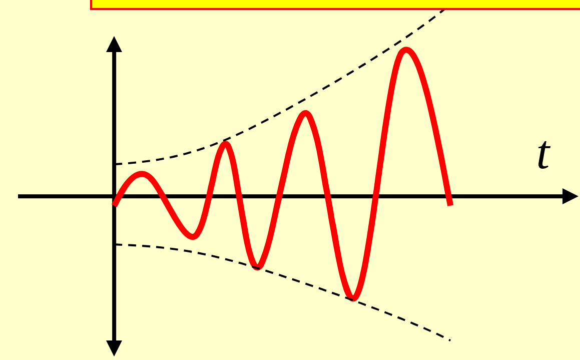
$$e^{-(s+b)t} = e^{-[(\sigma+b)+j\omega]t}$$

if $\sigma + b > 0 \quad \Rightarrow \quad \sigma > -b$




**Has Two Main Behaviors**

if $\sigma + b < 0 \quad \Rightarrow \quad \sigma < -b$




Thus, $\lim_{t\to\infty} e^{-(s+b)t}$ "exists" only for $\sigma > -b$

So, we can't "find" this $X(s)$ for values of $s$ such that Re{$s$} ≤ -$b$

**But for $s$ with Re{$s$} > -$b$ we have no trouble.** This set of $s$ is ROC for this transform.

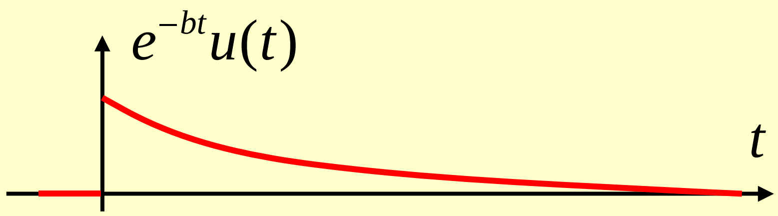
Don't worry too much about ROC… at this level it kind of takes care of itself

So for $x(t) = e^{-bt}u(t)$ We have $X(s) = \dfrac{1}{s+b} \qquad \mathrm{Re}\{s\} > -b$

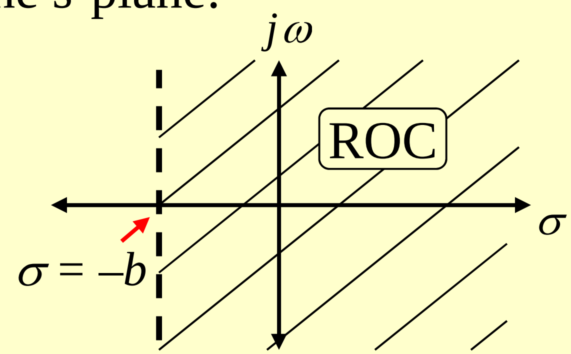
**This result… and many others… is on the Table of Laplace Transforms**

**If $b > 0$** then $x(t)$ itself decays:

For $b > 0$, $-b$ is negative



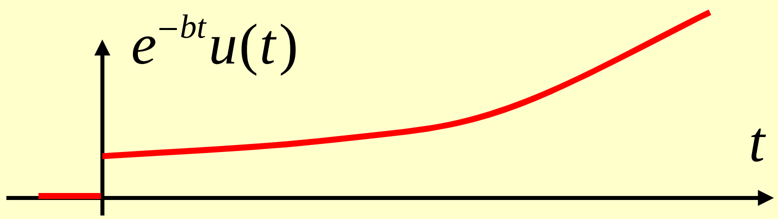

And we have on the s-plane:



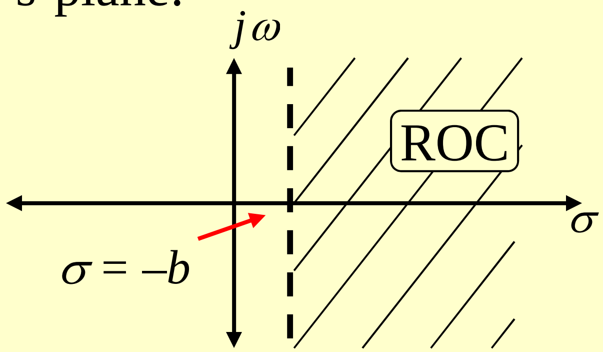

**This case can be handled by the FT… and can also be handled by the LT**

**If $b < 0$** then $x(t)$ itself "explodes":

For $b < 0$, $-b$ is positive




And we have on the s-plane:




**This case can't be handled by the FT… but by restricting our focus to values of *s* in the ROC, the LT can handle it!!!**

## Connection between CTFT & LT

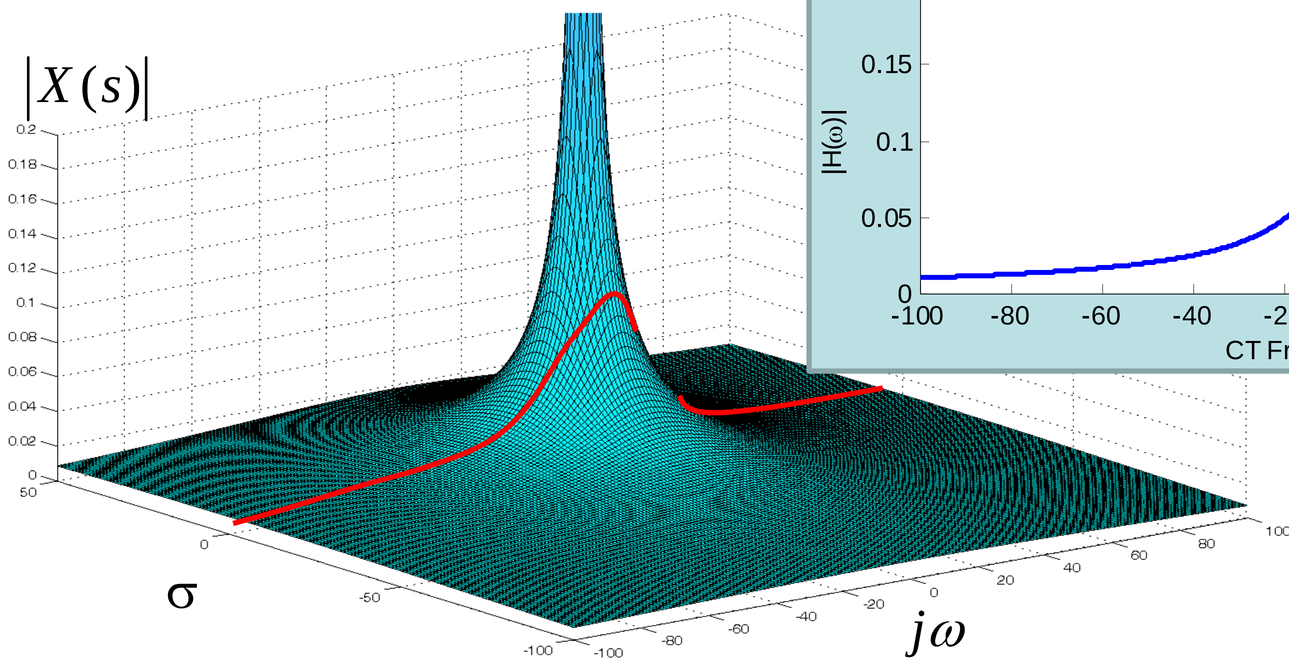
$$\text{CTFT: } X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \text{LT: } X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

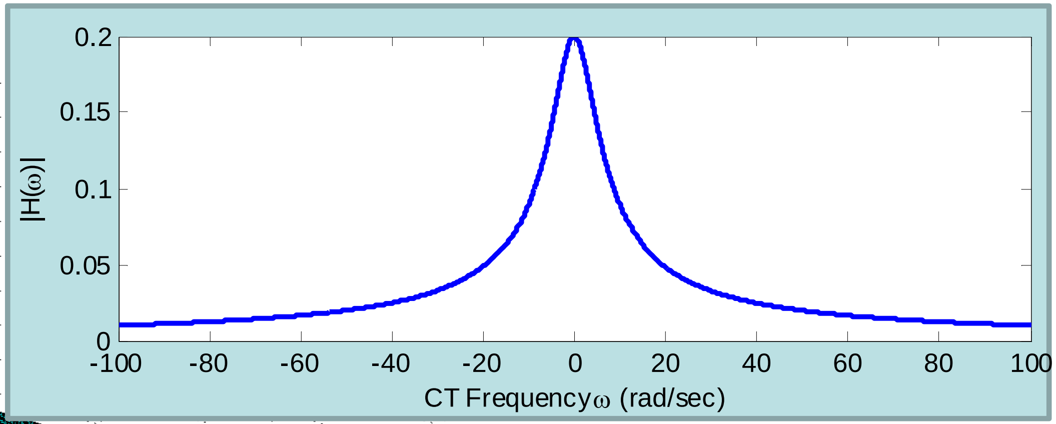
**It appears that letting $\sigma = 0$ gives LT = FT…**

**But this is only true if ROC includes the "$j\omega$ axis"!!!**

**If ROC includes "$j\omega$ axis" Then the FT is "embedded" in the LT**

**Get the FT by taking the LT and evaluating it only on the $j\omega$ axis… i.e., take a "slice" of the LT on the $j\omega$ axis (where $\sigma = 0$).**

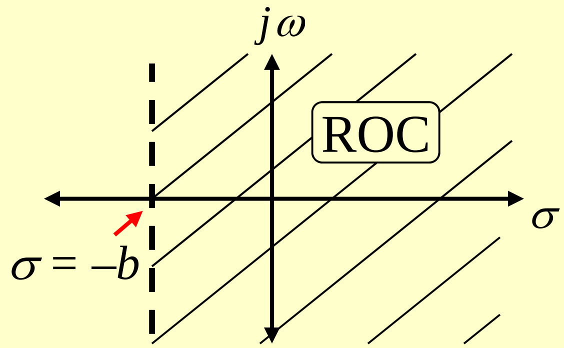


## Let's Revisit the Example Above

$$x(t) = e^{-bt}u(t) \quad \leftrightarrow \quad X(s) = \frac{1}{s+b} \quad \text{Re}\{s\} > -b$$

If $b > 0$, then ROC includes the "$j\omega$ axis":




$$\Rightarrow X(s)\Big|_{s=j\omega} = \left[\frac{1}{s+b}\right]_{s=j\omega} = \left[\frac{1}{j\omega+b}\right]$$

Same as on FT table

# Inverse LT

Like the FT…once you know $X(s)$ you can use the inverse LT to get $x(t)$

The definition of the inverse LT is:

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

with $c$ chosen such that $s = c + j\omega$ is in ROC

This is a "complex line integral" in complex s-plane…

**HARD TO DO!!**

But…if $X(s) = \dfrac{b_M s^M + b_{M-1}s^{M-1} + \ldots + b_1 s + b_0}{a_N s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + a_0}$

Ratio of polynomials in $s$

"Rational Function"

Then its easy to find $x(t)$ using partial fraction expansion and a table of LT pairs

**Done EXACTLY like for ZT… BUT you don't "divide by s"**

Do PFE and find ILT of this: $Y(s) = \frac{s+1}{s^3 + \frac{3}{4}s^2 + \frac{1}{8}s}$

Then use matlab's residue to do a partial fraction expansion on $Y(s)$

```
[r,p,k]=residue([1 1],[1 0.75 0.125 0])

r =             p =              k = []
  4               -0.5000
 -12              -0.2500
  8                0
```

For each term:

$$\frac{r}{s-p}$$

$$\Rightarrow \quad Y(s) = \frac{4}{s+\frac{1}{2}} + \frac{-12}{s+\frac{1}{4}} + \frac{8}{s} \quad \Rightarrow \quad Y(s) = \frac{4}{s+\frac{1}{2}} - \frac{12}{s+\frac{1}{4}} + \frac{8}{s}$$

Now… each of these terms is on the LT table!!!

$$\Rightarrow \quad y[n] = 4e^{-0.5t}u(t) - 12e^{-0.25t}u(t) + 8u(t)$$

# A Few Properties of Bilateral LT

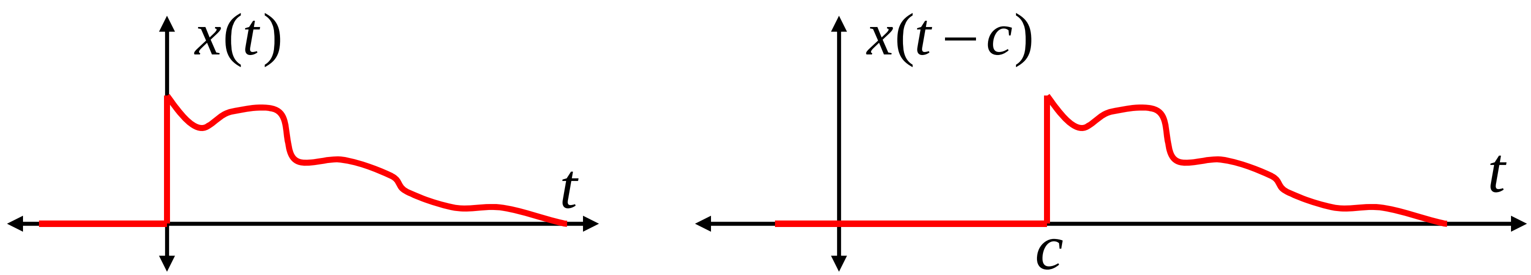
Because of the connection between FT & LT we expect these to be similar to the FT properties we already know!

**There are several other properties… they are listed on the Table of Laplace Transform Properties.**

**Linearity**: $ax(t) + by(t) \quad \leftrightarrow \quad aX(s) + bY(s)$

**Time Shift :**

Figures here for show causal signal (but result is general case)




$$x(t-c) \leftrightarrow e^{-cs}X(s)$$

Compare to time shift for FT: $e^{-j\omega c}$ vs. $e^{-cs}$

Recall: $s = \sigma + j\omega$

**Time Differentiation:**

$$\dot{x}(t) \leftrightarrow sX(s)$$

**Integration:**

$$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{s}X(s)$$

These two properties have a nice "opposite" relationship:

| | | | |
|---|---|---|---|
| **Note**: | Differentiation | $\Rightarrow$ | Multiply by s |
| | Integration | $\Rightarrow$ | Divide by s |

"opposites" "opposites"

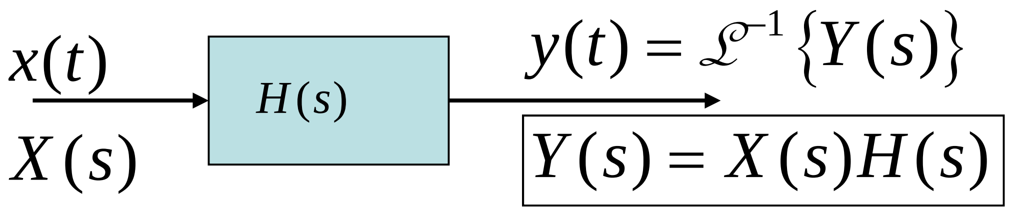
These two properties are crucial for linking the LT to the solution of Diff. Eq.

They are also crucial for thinking about "system block diagrams"

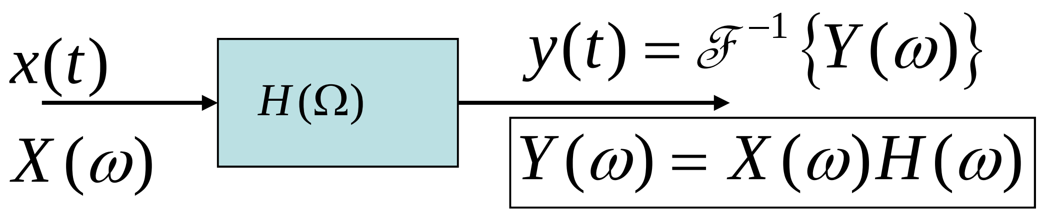
## System Property

The output of a LTI CT system has LT $Y(s)$ given by $\boxed{Y(s) = X(s)H(s)}$

So we have:




Note how similar this is to what we saw for CTFT:




**Terminology**

- Frequency Response: $H(\omega)$
- Transfer Function: $H(s)$