EECE 301
Signals & Systems
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Note Set #33
• C-T Systems: Laplace Transform… “Power Tool” for system analysis
Laplace Transform & C-T Systems

Like the Z Transform for the DT case… The Laplace Transform is a powerful tool for the analysis and design of CT LTI Systems

L-T is used to

- Solve differential equations with non-zero initial conditions
  
  We’ll do this later

- Characterize systems using the “Transfer Function”
  
  Our initial focus is here
Laplace Transform Definition

Given a C-T signal $x(t)$ $-\infty < t < \infty$ we’ve already seen how to use the CTFT:

$$CTFT : X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Unfortunately the CTFT doesn’t “converge” for some signals… the LT mitigates this problem by including decay in the transform:

$$e^{-j\omega t} \text{ vs. } e^{-st} = e^{-(\sigma+j\omega)t} = e^{-\sigma t}e^{-j\omega t}$$

Controls decay of integrand

For the Laplace Transform we use: $s = \sigma + j\omega$. So… $s$ is just a complex variable that we almost always view in rectangular form

$$CTFT : X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \quad \text{LT : } X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$
There are 2 types of LT:

- **Two-sided (bilateral)**
- **One-sided (unilateral)**

**Two-Sided LT**

\[
X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt \quad \text{with} \quad s = \sigma + j\omega
\]

complex variable

**One-Sided LT**

\[
X(s) = \int_{0}^{\infty} x(t)e^{-st} \, dt \quad \text{with} \quad s = \sigma + j\omega
\]

One-sided LT defined this way → even if \( x(t) \neq 0, \ t < 0 \)
Note that $X(s)$ is:
\[
\begin{cases}
\text{a complex valued function} \\
\text{of a complex variable } s = \sigma + j\omega
\end{cases}
\]

\[|X(s)|, \quad j \text{Im}\{s\}, \quad j\omega, \quad j \text{Im}\{s\}, \quad j\omega\]

Similarly for $\angle X(s)$

\[\text{Re}\{s\}, \quad j \text{Im}\{s\}, \quad j\omega, \quad j \text{Im}\{s\}, \quad j\omega\]

“s plane”

Recall that for the ZT we often needed to keep track if $z$ values were inside the UC… so polar form of $z$ was convenient. Inside UC when $|z| < 1$.

For the LT we will often need to keep track if $s$ values are in the Left Half Plane (LHP)… so rectangular form of $s$ is convenient. In LHP if $\text{Re}\{s\} < 0$.

**Region of Convergence (ROC)**

Set of all $s$ values for which the integral in the LT definition converges.

Each signal has its own region of convergence.
Example of Finding a LT

Consider the signal \( x(t) = e^{-bt}u(t) \), \( b \in \mathbb{R} \)

This is a causal signal.

By definition of the LT:

\[
X(s) = \int_{0}^{\infty} e^{-bt} e^{-st} dt = \int_{0}^{\infty} e^{-(s+b)t} dt
\]

\[
X(s) = \left[-\frac{1}{s+b}\right]_{t=0}^{t=\infty} = \left(-\frac{1}{s+b}\right) \left[\lim_{t \to \infty} e^{-(s+b)t} - 1\right]
\]

Back when we studied the FT we had to limit \( b \) to being \( b > 0 \)… with the LT we don’t need to restrict that!!!

This is an easy integral to do!!

The limit is here by the definition of the integral

If this limit does not converge… then we say that the integral “does not exist”

So… we need to find out under what conditions this integral exists.

So… let’s look at the function inside this limit…
\[ e^{-(s+b)t} = e^{-(\sigma+b+j\omega)t} \]

**if \( \sigma + b > 0 \) \( \Rightarrow \sigma > -b \)**

**if \( \sigma + b < 0 \) \( \Rightarrow \sigma < -b \)**

**Has Two Main Behaviors**

Thus, \( \lim_{t \to \infty} e^{-(s+b)t} \) "exists" only for \( \sigma > -b \)

So, we can’t “find” this \( X(s) \) for values of \( s \) such that \( \text{Re}\{s\} \leq -b \)

**But for \( s \) with \( \text{Re}\{s\} > -b \) we have no trouble.** This set of \( s \) is ROC for this transform.  

Don’t worry too much about ROC… at this level it kind of takes care of itself

So for \( x(t) = e^{-bt}u(t) \) We have

\[ X(s) = \frac{1}{s+b} \quad \text{Re}\{s\} > -b \]

This result… and many others… is on the Table of Laplace Transforms
If $b > 0$ then $x(t)$ itself decays:
For $b > 0$, $-b$ is negative

And we have on the s-plane:

If $b < 0$ then $x(t)$ itself “explodes”:
For $b < 0$, $-b$ is positive

And we have on the s-plane:

This case can’t be handled by the FT… but by restricting our focus to values of $s$ in the ROC, the LT can handle it!!!
Connection between CTFT & LT

CTFT: \( X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \)

LT: \( X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \)

It appears that letting \( \sigma = 0 \) gives \( \text{LT} = \text{FT} \ldots \)

But this is only true if ROC includes the “\( j\omega \) axis”!!!

If ROC includes “\( j\omega \) axis” Then the FT is “embedded” in the LT
Get the FT by taking the LT and evaluating it only on the \( j\omega \) axis… i.e., take a “slice” of the LT on the \( j\omega \) axis (where \( \sigma = 0 \)).
Let’s Revisit the Example Above

\[ x(t) = e^{-bt}u(t) \quad \Leftrightarrow \quad X(s) = \frac{1}{s + b} \quad \text{Re}\{s\} > -b \]

If \( b > 0 \), then ROC includes the “j\(\omega\) axis”:

\[ \Rightarrow X(s)\bigg|_{s=j\omega} = \left[\frac{1}{s + b}\right]_{s=j\omega} = \left[\frac{1}{j\omega + b}\right] \]

Same as on FT table
Inverse LT

Like the FT…once you know $X(s)$ you can use the inverse LT to get $x(t)$

The definition of the inverse LT is:

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$

with $c$ chosen such that $s = c + j\omega$ is in ROC

This is a “complex line integral” in complex s-plane…

**HARD TO DO!!**

But…if $X(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \ldots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0}$

Then it’s easy to find $x(t)$ using partial fraction expansion and a table of LT pairs

- **Ratio of polynomials in $s$**
  - “Rational Function”

Done EXACTLY like for ZT…

BUT you don’t “divide by $s$”
Do PFE and find ILT of this:

\[ Y(s) = \frac{s + 1}{s^3 + \frac{3}{4} s^2 + \frac{1}{8} s} \]

Then use matlab’s residue to do a partial fraction expansion on \( Y(s) \)

[r,p,k]=residue([1 1],[1 0.75 0.125 0])

\begin{align*}
  r &= \begin{bmatrix} 4 \\ -12 \\ 8 \end{bmatrix} \\
  p &= \begin{bmatrix} -0.5000 \\ -0.2500 \\ 0 \end{bmatrix} \\
  k &= []
\end{align*}

For each term:

\[ \frac{r}{s - p} \]

\[ Y(s) = \frac{4}{s + \frac{1}{2}} - \frac{12}{s + \frac{1}{4}} + \frac{8}{s} \]

Now… each of these terms is on the LT table!!!

\[ y[n] = 4e^{-0.5t} u(t) - 12e^{-0.25t} u(t) + 8u(t) \]
A Few Properties of Bilateral LT

Because of the connection between FT & LT we expect these to be similar to the FT properties we already know!

**Linearity:**

\[ ax(t) + by(t) \leftrightarrow aX(s) + bY(s) \]

**Time Shift:**

Figures here for show causal signal (but result is general case)

\[ x(t - c) \leftrightarrow e^{-cs} X(s) \]

Compare to time shift for FT:

\[ e^{-j\omega c} \text{ vs. } e^{-cs} \]

Recall: \[ s = \sigma + j\omega \]
Time Differentiation: \[ \dot{x}(t) \leftrightarrow sX(s) \]

Integration:
\[ \int_{-\infty}^{t} x(\lambda) d\lambda \leftrightarrow \frac{1}{s} X(s) \]

Note: Differentiation \[\Rightarrow\] Multiply by s  
Integration \[\Rightarrow\] Divide by s

These two properties have a nice “opposite” relationship:

These two properties are crucial for linking the LT to the solution of Diff. Eq.

They are also crucial for thinking about “system block diagrams”
**System Property**

The output of a LTI CT system has LT $Y(s)$ given by $Y(s) = X(s)H(s)$.

So we have:

\[
x(t) \xrightarrow{X(s)} H(s) \xrightarrow{y(t)} \mathcal{L}^{-1} \{Y(s)\} \xrightarrow{Y(s)} = X(s)H(s)
\]

Note how similar this is to what we saw for CTFT:

\[
x(t) \xrightarrow{X(\omega)} H(\Omega) \xrightarrow{y(t)} \mathcal{F}^{-1} \{Y(\omega)\} \xrightarrow{Y(\omega)} = X(\omega)H(\omega)
\]

**Terminology**

- Frequency Response: $H(\omega)$
- Transfer Function: $H(s)$